

A Comparison of Some Splines-Based Methods for the One-dimensional Heat Equation

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Abstract—In this paper, collocation based cubic B-spline and extended cubic uniform B-spline method are considered for solving one-dimensional heat equation with a nonlocal initial condition. Finite difference and θ -weighted scheme is used for time and space discretization respectively. The stability of the method is analyzed by the Von Neumann method. Accuracy of the methods is illustrated with an example. The numerical results are obtained and compared with the analytical solutions.

Index Terms—Heat equation, Collocation based, Cubic B-spline, Extended cubic uniform B-spline.

I. INTRODUCTION

HEAT equation is a simple second-order partial differential equation that describes the variation temperature in a given region over a period of time. In this work, suppose the heat flows through a thin rod which is perfectly insulated along its length except at the two ends, its position in the rod is denoted as x where $0 \leq x \leq 1$ and the length of the rod is represented as L , as illustrated in Fig. 1.

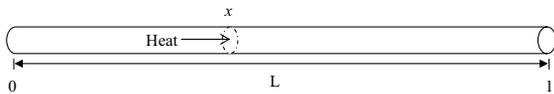


Fig. 1: Heat flows through an insulated rod at position x .

Let $U(x, t)$ represents the temperature at the point x in the rod at time t . Then, the partial differential equation which is used to model the one-dimensional temperature evolution can be written as

$$U_t = \alpha U_{xx}, \quad 0 \leq x \leq 1, \quad t > 0 \quad (1)$$

where the positive constant α represents the thermal diffusivity of the rod. This varies depending on the thermal conductivity of the material composing the rod, density and the specific heat of the rod. The higher thermal diffusivity, the faster the substance adjusts its temperature to that of its surrounding. Here, the initial temperature distribution of the rod is given as

$$U(x, 0) = f(x), \quad x \in [0, 1] \quad (2)$$

and the heat flows at the end of rod, $x = 1$ is in the condition of

$$\frac{\partial U}{\partial x}(1, t) = g(t), \quad t \in (0, T] \quad (3)$$

with the nonlocal boundary condition

$$\int_0^b U(x, t) dx = m(t), \quad 0 \leq b \leq 1, \quad t > 0 \quad (4)$$

Here, $f(x)$, $g(t)$, $m(t)$ are known functions and b is a constant. If $b = 1$, eq. (4) leads to

$$\int_0^1 U(x, t) dx = m(t) \quad (5)$$

and it can be differentiated to give

$$\begin{aligned} m'(t) &= \frac{d}{dt} m(t) = \frac{d}{dt} \int_0^1 U(x, t) dx = \int_0^1 U_t dx \\ &= \int_0^1 \alpha U_{xx} dx = \alpha U_x(1, t) - \alpha U_x(0, t) \end{aligned} \quad (6)$$

which can be rewritten as

$$U_x(1, t) - U_x(0, t) = \frac{m'}{\alpha} \quad (7)$$

This only holds if U and m are differentiable.

Heat equation mainly in one-dimension had been studied by many authors as in references therein [1], [8], [10], [11]. An comparative study between the traditional separation of variables method and Adomian method for heat equation had been examined by Gorguis and Benny Chan [5]. Dehghan [4] considered the use of second-order finite difference scheme to solve the two-dimensional heat equation. After that, Mohebbi and Dehghan [8] presented a fourth-order compact finite difference approximation and cubic C^1 -spline collocation method for the solution with fourth-order accuracy in both space and time variables, $O(h^4, k^4)$. In literature [7], Kumar concluded that spline gives a simple and practical method to solve the boundary problems compared to finite difference method.

On the other hand, an extension of B-spline function, namely extended cubic uniform B-spline had been proposed by Han and Liu in literature [6]. The advantage of using extended B-spline is that it possesses a free parameter, λ , to control the shape parameter. This, thus, provides the motivation for our work on investigating the accuracy and the efficiency between cubic B-spline and extended cubic uniform

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B-splines for solving the one-dimensional heat equation.

In this paper, one-dimensional heat equation would be solved by collocation method of cubic B-spline and extended cubic uniform B-spline. For the numerical procedure, θ -weighted scheme would be applied to the space derivative at two adjacent time levels while forward finite difference approach would be used for discretizing the derivative of time. Then, B-spline functions would be applied to the resulting linear system and the approximations could be obtained by solving the system through Thomas algorithm. For the stability analysis, Von Neumann approach would be used to prove the unconditionally stable property of the method. Last but not least, numerical results would be presented to demonstrate the efficiency of the method.

II. CUBIC B-SPLINE METHOD

Definition: Consider a partition π of $[a,b]$ is equally divided by the knots x_i into n subinterval $[x_i, x_{i+1}]$, where $i = 0, 1, \dots, n - 1$, which means $a = x_0 < x_1 < \dots < x_n = b$ on $[a,b]$. Let $S_3(\pi)$ be the space of twice continuously differentiable piecewise third-degree cubic polynomials on π and suppose the cubic B-spline basis is the basis for $S_3(\pi)$. Therefore, the cubic B-spline function is defined by the relationship [3]

$$B_{3,i}(x) = \frac{1}{6h^3} \begin{cases} (x - x_i)^3, & x \in [x_i, x_{i+1}] \\ h^3 + 3h^2(x - x_{i+1}) + 3h(x - x_{i+1})^2 - 3(x - x_{i+1})^3, & x \in [x_{i+1}, x_{i+2}] \\ h^3 + 3h^2(x_{i+3} - x) + 3h(x_{i+3} - x)^2 - 3(x_{i+3} - x)^3, & x \in [x_{i+2}, x_{i+3}] \\ (x_{i+4} - x)^3, & x \in [x_{i+3}, x_{i+4}] \end{cases} \quad (8)$$

III. EXTENDED CUBIC UNIFORM B-SPLINE METHOD

Definition: Suppose $\lambda \in \mathbb{R}$, then the blending function of extended cubic uniform B-spline with degree 4 is defined as follows [12]:

$$Eb_{3,i}(x) = \frac{1}{24h^4} \begin{cases} 4h(1 - \lambda)(x - x_i)^3 + 3\lambda(x - x_i)^4, & x \in [x_i, x_{i+1}] \\ (4 - \lambda)h^4 + 12h^3(x - x_{i+1}) + 6h^2(2 + \lambda)(x - x_{i+1})^2 - 12h(x - x_{i+1})^3 - 3\lambda(x - x_{i+1})^4, & x \in [x_{i+1}, x_{i+2}] \\ (4 - \lambda)h^4 + 12h^3(x_{i+3} - x) + 6h^2(2 + \lambda)(x_{i+3} - x)^2 - 12h(x_{i+3} - x)^3 - 3\lambda(x_{i+3} - x)^4, & x \in [x_{i+2}, x_{i+3}] \\ 4h(1 - \lambda)(x_{i+4} - x)^3 + 3\lambda(x_{i+4} - x)^4, & x \in [x_{i+3}, x_{i+4}] \end{cases} \quad (9)$$

where x is variable, λ is a parameter, and $-8 \leq \lambda \leq 1$.

It should be noted that when $\lambda = 0$, the basis function will be reduced to the basis function of cubic uniform B-spline. Also, it can be found that extended cubic uniform B-spline possesses the same properties as B-spline, such as convex full property, symmetry, and geometric invariability [12].

IV. COLLOCATION METHOD

Assumed that the approximation U_i^k to the exact solution $U(x, t)$ at the point (x_i, t_k) is expressed as [9]

$$U_i^k = \sum_{j=-3}^{n-1} C_j^k B_{3,j}(x) \quad (10)$$

where C_j^k are time dependent quantities to be determined and $B_{3,j}(x)$ are cubic B-spline functions. For the variation of U_i^k over the subinterval $[x_i, x_{i+1}]$, the approximation can be simplified into

$$U_i^k = \sum_{j=i-3}^{i-1} C_j^k B_{3,j}(x) \quad (11)$$

The approximations of the solutions of eq. (1) at the point (x_i, t_{k+1}) can be obtained by applying the θ -weighted scheme ($0 \leq \theta \leq 1$) to the space derivative at two adjacent time levels to get the equation [2]

$$(U_t)_i^k + (1 - \theta) \left(-\alpha (U_{xx})_i^k \right) + \theta \left(-\alpha (U_{xx})_i^{k+1} \right) = 0 \quad (12)$$

where the superscripts k and $k + 1$ are successive time levels and $k = 0, 1, 2, \dots$. Rewrite the equation

$$(U_t)_i^k - \alpha(1 - \theta)(U_{xx})_i^k - \alpha\theta(U_{xx})_i^{k+1} = 0 \quad (13)$$

Now, discretizing the time derivative by a first order accurate forward difference scheme and rearrange the equation to obtain

$$U_i^{k+1} - \alpha\theta\Delta t(U_{xx})_i^{k+1} = U_i^k + \alpha(1-\theta)\Delta t(U_{xx})_i^k \quad (14)$$

where Δt is the time step. Note that when $\theta = 0$, the system gives an explicit scheme, $\theta = 1$ gives a fully implicit scheme and $\theta = 0.5$ gives a mixed scheme of Crank-Nicolson. Here, Crank-Nicolson approach is used. Hence, eq. (14) takes the form

$$U_i^{k+1} - 0.5\alpha\Delta t(U_{xx})_i^{k+1} = U_i^k + 0.5\alpha\Delta t(U_{xx})_i^k \quad (15)$$

for $i = 0, 1, \dots, n$ at each level of time. Therefore, a linear system of order $(n + 1)$ is obtained with $(n + 3)$ unknowns $\mathbf{C}^k = (C_{-3}^k, C_{-2}^k, \dots, C_{n-1}^k)$ at the level time $t = t_k$. To solve the system, two additional linear equations are needed. Thus, eq. (11) is applied to the derivative of the boundary conditions (3) and (7) to obtain

$$U_x(1, t_{k+1}) = g(t_{k+1}), \quad (16a)$$

$$U_x(1, t_{k+1}) - U_x(0, t_{k+1}) = \frac{m'(t_{k+1})}{\alpha}$$

$$\text{or } U_x(0, t_{k+1}) = U_x(1, t_{k+1}) - \frac{m'(t_{k+1})}{\alpha} = g(t_{k+1}) - \frac{m'(t_{k+1})}{\alpha} \quad (16b)$$

Above eqs. (15) and (16a)–(16b) leads to a $(n + 3) \times (n + 3)$ tridiagonal matrix system, which can be solved by the Thomas algorithm. Once the initial vector \mathbf{C}^0 have been calculated from the initial conditions, the approximation solution U_i^k at each level of time t_k , can be determined by the vector \mathbf{C}^k which is found by solving the recurrence relation repeatedly.

The initial vector \mathbf{C}^0 can be obtained from the initial condition and boundary values of the derivatives of the initial condition as the following expressions:

- 1) $U_x(x_i, 0) = f'(x_i), \quad i = 0$
- 2) $U(x_i, 0) = f(x_i), \quad i = 0, 1, \dots, n$
- 3) $U_x(x_i, 0) = f'(x_i), \quad i = n.$

This yields a $(n + 3) \times (n + 3)$ matrix system where the solution can be found by Thomas algorithms.

V. STABILITY ANALYSIS

Von Neumann stability method is applied for analyzing the stability of the proposed scheme. Consider the trial solution (one Fourier mode out of the full solution) at a given point x_m

$$C_m^k = \delta^k \exp(i\eta m h) \quad (17)$$

where $i = \sqrt{-1}$ and η is the mode number. By substituting the eq. (11) into (14) and rearrange the equation, it leads to

$$p_1 C_{m-3}^{k+1} + p_2 C_{m-2}^{k+1} + p_3 C_{m-1}^{k+1} = p_4 C_{m-3}^k + p_5 C_{m-2}^k + p_6 C_{m-1}^k \quad (18)$$

where

$$\begin{aligned} p_1 &= \frac{1}{6} - \frac{\theta\Delta t\alpha}{h^2} & p_4 &= \frac{1}{6} + \frac{(1-\theta)\Delta t\alpha}{h^2} \\ p_2 &= \frac{4}{6} + \frac{2\theta\Delta t\alpha}{h^2} & p_5 &= \frac{4}{6} - \frac{2(1-\theta)\Delta t\alpha}{h^2} \\ p_3 &= \frac{1}{6} - \frac{\theta\Delta t\alpha}{h^2} & p_6 &= \frac{1}{6} + \frac{(1-\theta)\Delta t\alpha}{h^2} \end{aligned}$$

Inserting the trial solution (17) into eq. (18) and simplifying the equation, it gives

$$\delta = \frac{\frac{1}{3}(2 + \cos \eta h) - \frac{2\alpha(1-\theta)\Delta t}{h^2}(1 - \cos \eta h)}{\frac{1}{3}(2 + \cos \eta h) + \frac{2\alpha\theta\Delta t}{h^2}(1 - \cos \eta h)} \quad (19)$$

The scheme is stable if and only if the amplification factor $|\delta| \leq 1$. As $\theta = 0.5$ is used in the proposed scheme, thus substitute the θ value into eq. (19), it gives

$$\delta = \frac{\frac{1}{3}(2 + \cos \eta h) - \frac{\alpha\Delta t}{h^2}(1 - \cos \eta h)}{\frac{1}{3}(2 + \cos \eta h) + \frac{\alpha\Delta t}{h^2}(1 - \cos \eta h)} \leq 1 \quad (20)$$

Therefore, it is obvious to notice that the equation always gives $|\delta| \leq 1$ and this had been proved that the presented numerical scheme for the one-dimensional heat equation is unconditionally stable.

VI. NUMERICAL RESULT

In this section, an example of one-dimensional heat equation problem which is discussed in the literature [1], is examined by the cubic B-spline and also extended cubic uniform B-spline collocation method. Consider the following heat equation

$$U_t = \alpha U_{xx} \quad 0 \leq x \leq 1, \quad t \geq 0$$

subject to the conditions

$$\begin{aligned} U(x, 0) &= \cos\left(\frac{\pi}{2}x\right) \\ \frac{\partial U}{\partial x}(1, t) &= -\frac{\pi}{2} \exp\left(-\frac{\pi^2}{4}t\right) \end{aligned}$$

and the nonlocal condition

$$\int_0^b U(x, t) dx = \frac{2}{\pi} \exp\left(-\frac{\pi^2}{4}t\right)$$

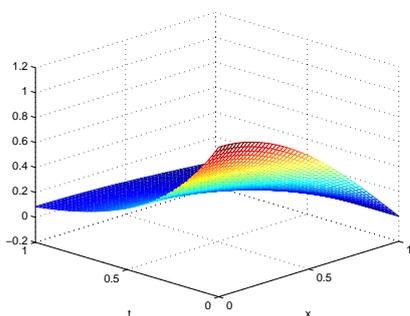
Here, $\alpha = 1$ and $b = 1$ is considered. The exact solution of the problem is known to be

$$U(x, t) = \exp\left(-\frac{\pi^2}{4}t\right) \cos\left(\frac{\pi}{2}x\right)$$

Table I shows the approximations and the exact solutions at the chosen point, x_i when final time $T = 1$. Numerical results obtained from the extended cubic uniform B-spline is depicted in Figure 2.

TABLE I: Numerical results for B-Spline and extended B-Spline when $h = \Delta t = 0.05$ and $\lambda = -0.0036$

x_i	Exact	yB_i	yEb_i
0	0.08480497	0.083746074	0.08481692
0.05	0.08454355	0.083484881	0.08455549
0.1	0.08376088	0.082702908	0.08377282
0.15	0.08246180	0.081404968	0.08247373
0.2	0.08065432	0.079599049	0.08066624
0.3	0.07556178	0.074510789	0.07557365
0.4	0.06860866	0.067563197	0.06862048
0.5	0.05996617	0.058927041	0.05997792
0.6	0.04984711	0.048814596	0.04985880
0.7	0.03850065	0.037474418	0.03851228
0.8	0.02620618	0.025185245	0.02621775
0.9	0.01326642	0.012249168	0.01327797
1	0.00000000	-0.001015944	0.00001153

**Fig. 2:** Space-time graph with $h = \Delta t = 0.05$ and $\lambda = -0.0036$.

VII. CONCLUSION

Numerical methods for solving one-dimensional heat equation with a nonclassical parabolic initial condition had been described in the previous section. These two methods had been tested on an example and the obtained as well as the exact solutions are tabulated. The results show that extended cubic uniform B-spline collocation method with an appropriate λ value would give a better results for solving one-dimensional heat equation than the cubic B-spline collocation method.

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