# 4-Transitivity and 6-Figures in Finite Klingenberg Planes of Parameters $\left(p^{2 k-1}, p\right)$ 

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#### Abstract

In this paper, we carry over some of the results which are valid on a certain class of Moufang-Klingenberg planes $\mathbf{M}(\mathcal{A})$ coordinatized by an local alternative ring $\mathcal{A}:=\mathbf{A}(\varepsilon)=\mathbf{A}+\mathbf{A} \varepsilon$ of dual numbers to finite projective Klingenberg plane $\mathbf{M}(\mathcal{A})$ obtained by taking local ring $\mathbf{Z}_{q}$ (where prime power $q=p^{k}$ ) instead of $\mathbf{A}$. So, we show that the collineation group of $\mathbf{M}(\mathcal{A})$ acts transitively on 4 -gons, and that any 6 -figure corresponds to only one inversible $m \in \mathcal{A}$.


Keywords-finite Klingenberg plane, projective collineation, 4transitivity, 6-figures.

## I. Introduction

Projective Klingenberg and Hjelmslev planes (more briefly: PK-planes and PH-planes, resp.) are generalizations of ordinary projective planes. These structures were introduced by Klingenberg in [14], [15]. As for finite PK-planes, these structures introduced by Drake and Lenz in [8] have been studied in detail by Bacon in [2].

In our previous paper [6] we studied a certain class (which we will denote by $\mathbf{M}(\mathcal{A})$ ) of Moufang-Klingenberg (briefly, MK ) planes coordinatized by an local alternative ring

$$
\mathcal{A}:=\mathbf{A}(\varepsilon)=\mathbf{A}+\mathbf{A} \varepsilon
$$

of dual numbers (an alternative ring $\mathbf{A}, \varepsilon \notin \mathbf{A}$ and $\varepsilon^{2}=0$ ) introduced by Blunck in [5]. We showed that its collineation group is transitive on quadrangles and the coordinatization of these Moufang-Klingenberg planes is independent of the choice of the coordinatization quadrangle. By extending the concepts of 6 -figure to these Moufang - Klingenberg planes, we examined some properties of 6 -figures.

In the present paper we deal with finite PK-plane $\mathbf{M}(\mathcal{A})$ obtained by taking local ring $\mathbf{Z}_{q}$ (where $q$ is a prime power) instead of $\mathbf{A}$. So, we will carry the results that are well-known for MK-planes from [6] $\mathbf{M}(\mathcal{A})$ to the finite PK-plane $\mathbf{M}(\mathcal{A})$.

## II. Preliminaries

Let $\mathbf{M}=(\mathbf{P}, \mathbf{L}, \in, \sim)$ consist of an incidence structure $(\mathbf{P}, \mathbf{L}, \in)$ (points, lines, incidence) and an equivalence relation ' $\sim$ ' (neighbour relation) on $\mathbf{P}$ and on $\mathbf{L}$. Then $\mathbf{M}$ is called a projective Klingenberg plane (PK-plane), if it satisfies the following axioms:
(PK1) If $P, Q$ are two non-neighbour points, then there is a unique line $P Q$ through $P$ and $Q$.
(PK2) If $g, h$ are two non-neighbour lines, then there is a unique point $g \wedge h$ on both $g$ and $h$.

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(PK3) There is a projective plane $\mathbf{M}^{*}=\left(\mathbf{P}^{*}, \mathbf{L}^{*}, \in\right)$ and incidence structure epimorphism $\Psi: \mathbf{M} \rightarrow \mathbf{M}^{*}$, such that the conditions

$$
\Psi(P)=\Psi(Q) \Leftrightarrow P \sim Q, \Psi(g)=\Psi(h) \Leftrightarrow g \sim h
$$

hold for all $P, Q \in \mathbf{P}, g, h \in \mathbf{L}$.
PK-plane $\mathbf{M}$ is called a projective Hjelmslev plane $(\mathrm{PH}-$ plane) If $\mathbf{M}$ furthermore provides the following axioms:
(PH1) If $P, Q$ are two neighbour points, then there are at least two lines through $P$ and $Q$.
(PH2) If $g, h$ are two neighbour lines, then there are at least two points on both $g$ and $h$.

A Moufang-Klingenberg plane (MK-plane) is a PK-plane $\mathbf{M}$ that generalizes a Moufang plane, and for which $\mathbf{M}^{*}$ is a Moufang plane (for the details see [1]).

A point $P \in \mathbf{P}$ is called near a line $g \in \mathbf{L}$ iff there exists a line $h$ such that $P \in h$ for some line $h \sim g$.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a collineation of $\mathbf{M}$.

Now we give the definition of an $n$-gon, which is meaningful when $n \geq 3$ : An n-tuple of pairwise non-neighbour points is called an (ordered) n-gon if no three of its elements are on neighbour lines [6].

An alternative ring (field) $\mathbf{R}$ is a not necessarily associative ring (field) that satisfies the alternative laws $a(a b)=$ $a^{2} b,(b a) a=b a^{2}, \forall a, b \in \mathbf{R}$. An alternative $\operatorname{ring} \mathbf{R}$ with identity element 1 is called local if the set $\mathbf{I}$ of its non-unit elements is an ideal.

We summarize some basic concepts about the coordinatization of MK-planes from [3].

Let $\mathbf{R}$ be a local alternative ring. Then

$$
\mathbf{M}(\mathbf{R})=(\mathbf{P}, \mathbf{L}, \in, \sim)
$$

is the incidence structure with neighbor relation defined as follows:

$$
\begin{aligned}
\mathbf{P}= & \{(x, y, 1): x, y \in \mathbf{R}\} \cup\{(1, y, z): y \in \mathbf{R}, z \in \mathbf{I}\} \\
& \cup\{(w, 1, z): w, z \in \mathbf{I}\} \\
\mathbf{L}= & \{[m, 1, p]: m, p \in \mathbf{R}\} \cup\{[1, n, p]: p \in \mathbf{R}, n \in \mathbf{I}\} \\
& \cup\{[q, n, 1]: q, n \in \mathbf{I}\}
\end{aligned}
$$

$$
\begin{aligned}
{[m, 1, p]=} & \{(x, x m+p, 1): x \in \mathbf{R}\} \\
& \cup\{(1, z p+m, z): z \in \mathbf{I}\} \\
{[1, n, p]=} & \{(y n+p, y, 1): y \in \mathbf{R}\} \\
& \cup\{(z p+n, 1, z): z \in \mathbf{I}\} \\
{[q, n, 1]=} & \{(1, y, y n+q): y \in \mathbf{R}\} \\
& \cup\{(w, 1, w q+n): w \in \mathbf{I}\}
\end{aligned}
$$

and also

$$
\begin{aligned}
P & =\left(x_{1}, x_{2}, x_{3}\right) \sim\left(y_{1}, y_{2}, y_{3}\right)=Q \\
& \left.\Leftrightarrow x_{i}-y_{i} \in \mathbf{I}(i=1,2,3)\right), \forall P, Q \in \mathbf{P} \\
g & =\left[x_{1}, x_{2}, x_{3}\right] \sim\left[y_{1}, y_{2}, y_{3}\right]=h \\
& \left.\Leftrightarrow x_{i}-y_{i} \in \mathbf{I}(i=1,2,3)\right), \forall g, h \in \mathbf{L} .
\end{aligned}
$$

Baker et al. [1] use $(O=(0,0,1), U=(1,0,0), V=$ $(0,1,0), E=(1,1,1))$ as a coordinatization 4-gon. We stick to this notation throughout this paper. For more detailed information about the coordinatization see [1] and [3].

Now it is time to give the following theorem from [1].
Theorem 2.1: $\mathbf{M}(\mathbf{R})$ is an MK-plane, and each MK-plane is isomorphic to some $\mathbf{M}(\mathbf{R})$.

Let $\mathbf{A}$ be an alternative field and $\varepsilon \notin \mathbf{A}$. Consider $\mathcal{A}:=\mathbf{A}(\varepsilon)=\mathbf{A}+\mathbf{A} \varepsilon$ with componentwise addition and multiplication as follows:

$$
\left(a_{1}+a_{2} \varepsilon\right)\left(b_{1}+b_{2} \varepsilon\right)=a_{1} b_{1}+\left(a_{1} b_{2}+a_{2} b_{1}\right) \varepsilon,
$$

where $a_{i}, b_{i} \in \mathbf{A}, i=1,2$. Then $\mathcal{A}$ is an alternative ring with ideal $\mathbf{I}=\mathbf{A} \varepsilon$ of non-units. For more detailed information about $\mathcal{A}$ see the papers of [4], [5].

Theorem 2.2: If $\mathbf{R}$ is a (not necessarily commutative) local ring then $\mathbf{M}(\mathbf{R})$ is a PK-plane (cf. [15] or [9, Theorem 4.1]).

Drake and Lenz [8, Proposition 2.5] or [12, Theorem 1.2] observed that the following corollary is true for PK-planes. This corollary is a generalization of results which are given for PH-planes by Kleinfeld [13, Theorem 1] and Lüneburg [16, Satz 2.11].

Corollary 2.3: Let $\mathbf{M}(\mathbf{R})$ be PK-plane. Then there are natural numbers $t$ and $r$ which are called the parametres of $\mathbf{M}(\mathbf{R})$ and they are uniquely determined by incidence structure of a finite PK-plane [8, Proposition 2.7], with

1) every point (line) has $t^{2}$ neighbours;
2) given a point $P$ and a line $l$ with $P \in l$, there exist exactly $t$ points on $l$ which are neighbours to $P$ and exactly $t$ lines through $P$ which are neighbours to $l$;
3) Let $r$ be order of the projective plane $\mathbf{M}^{*}$. If $t \neq 1$ we have $r \leq t$ (then $\mathbf{M}$ is called proper; we have $t=1$ iff $\mathbf{M}$ is an ordinary projective plane)
4) every point (line) is incident with $t(r+1)$ lines (points);
$|\mathbf{P}|=|\mathbf{L}|=t^{2}\left(r^{2}+r+1\right)$.

Now consider ring $\mathbf{Z}_{q}$ where prime power $q=p^{k}$. We can state the elements of $\mathbf{Z}_{q}$ as $\mathbf{Z}_{q}=U^{\prime} \cup I$ where $U^{\prime}$ is the set of units of $\mathbf{Z}_{q}$ and $I$ is the set of non-units of $\mathbf{Z}_{q}$. Here it is clear that

$$
I=\left\{0 p, 1 p, 2 p, \cdots,\left(p^{k-1}-1\right) p\right\}
$$

and so $|I|=p^{k-1}$. Let $\varepsilon \notin \mathbf{Z}_{q}$. Then $\mathcal{A}:=\mathbf{Z}_{q}+\mathbf{Z}_{q} \varepsilon$ with componentwise addition and multiplication above is a local ring with ideal $\mathbf{I}:=I+\mathbf{Z}_{q} \varepsilon$ of non-units, $|\mathbf{I}|=\left(p^{k-1}\right) p^{k}$. Note that the set of units of $\mathcal{A}$ is $\mathbf{U}:=U^{\prime}+\mathbf{Z}_{q} \varepsilon$ and

$$
|\mathbf{U}|=\left(p^{k}-p^{k-1}\right) p^{k}=(p-1) p^{2 k-1} .
$$

Since $\mathcal{A}$ is a proper local ring and $\mathcal{A} / \mathbf{I}=\mathbf{Z}_{p}, \Psi$ induces an incidence structure epimorphism from finite PK-plane $\mathbf{M}(\mathcal{A})$ onto the Desarguesian projective plane (with order $p$ ) coordinatized by the field $\mathbf{Z}_{p}$ [9, page 169, above Theorem 4.1]. Because of this, $\mathbf{M}(\mathcal{A})$ is called as Desarguesian PK-plane.

So, we have the following
Corollary 2.4: For finite PK-plane $\mathbf{M}(\mathcal{A})$, the parameters $t$ and $r$ in Corollary 2.3 are equal to $p^{2 k-1}$ and $p$, respectively.

A local ring $\mathbf{R}$ is called a Hjelmslev ring (briefly, H-ring) if it satisfies the following two conditions:
(HR1) I consists of two-sided zero divisor.
(HR2) For $a, b \in \mathbf{I}$, one has $a \in b \mathbf{R}$ or $b \in a \mathbf{R}$, and also $a \in \mathbf{R} b$ or $b \in \mathbf{R} a$.

By the last definition, we can say that $\mathcal{A}$ is not a H-ring. For example, for elements $a=3+3 \varepsilon$ and $b=\varepsilon$ of the ideal $\mathbf{I}$ of local ring $\mathcal{A}=\mathbf{Z}_{3^{2}}+\mathbf{Z}_{3^{2}}(\varepsilon)$, (HR2) is not valid.

From now on we restrict ourselves to PK-plane $\mathbf{M}(\mathcal{A})=$ $(\mathbf{P}, \mathbf{L}, \in, \sim)$ coordinatized by the local ring $\mathcal{A}:=\mathbf{Z}_{q}+\mathbf{Z}_{q} \varepsilon$, with neighbour relation defined above.

## III. 4-Transitivity and 6-Figures in $\mathrm{M}(\mathcal{A})$.

In the final section, first of all, from [6] we start by giving some collineations on $\mathbf{M}(\mathcal{A})$ where $w, z, q, n \in \mathbf{I}$ as follows:
For any $a, b \in \mathcal{A}$, the collineation $\mathrm{T}_{a, b}$ transforms points and lines as follows:

$$
\begin{aligned}
(x, y, 1) & \rightarrow(x+a, y+b, 1) \\
(1, y, z) & \rightarrow(1, y+z(b-a y), z) \\
(w, 1, z) & \rightarrow(w+z a, 1, z)
\end{aligned}
$$

and

$$
\begin{aligned}
{[m, 1, k] } & \rightarrow[m, 1, k+b-a m] \\
{[1, n, p] } & \rightarrow[1, n, p+a-b n] \\
{[q, n, 1] } & \rightarrow[q, n, 1] .
\end{aligned}
$$

For any $\alpha, \beta \notin \mathbf{I}$, the collineation $\mathbf{S}_{\alpha, \beta}$ (here, it is enough to give $\mathrm{S}_{\alpha, \beta}$ instead of the collineations $\mathrm{L}_{a}$ and $\mathrm{F}_{a}$ in [6]) transforms points and lines as follows:

$$
\begin{aligned}
(x, y, 1) & \rightarrow(\beta x, \alpha y, 1) \\
(1, y, z) & \rightarrow\left(1, \alpha \beta^{-1} y, \beta^{-1} z\right) \\
(w, 1, z) & \rightarrow\left(\alpha^{-1} \beta w, 1, \alpha^{-1} z\right)
\end{aligned}
$$

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and

$$
\begin{aligned}
{[m, 1, k] } & \rightarrow\left[\alpha \beta^{-1} m, 1, \alpha k\right] \\
{[1, n, p] } & \rightarrow\left[1, \alpha^{-1} \beta n, \beta p\right] \\
{[q, n, 1] } & \rightarrow\left[\beta^{-1} q, \alpha^{-1} n, 1\right] .
\end{aligned}
$$

The collineation $\mathrm{I}_{1}$ transforms points and lines as follows:

$$
\begin{aligned}
(x, y, 1) & \rightarrow\left(x^{-1}, x^{-1} y, 1\right) \quad \text { if } \quad x \notin \mathbf{I} \\
(x, y, 1) & \rightarrow(1, y, x) \quad \text { if } \quad x \in \mathbf{I} \\
(1, y, z) & \rightarrow(z, y, 1) \\
(w, 1, z) & \rightarrow(z, 1, w)
\end{aligned}
$$

and

$$
\left.\begin{array}{rl}
{[m, 1, k]} & \rightarrow[k, 1, m] \\
{[1, n, p]} & \rightarrow[p, n, 1] \quad \text { if } \quad p \in \mathbf{I} \\
{[1, n, p]} & \rightarrow\left[1,-n p^{-1}, p^{-1}\right] \quad \text { if }
\end{array} \quad p \notin \mathbf{I}\right)
$$

The collineation $F$ transforms points and lines as follows:

$$
\begin{aligned}
(x, y, 1) & \rightarrow(y, x, 1) \\
(1, y, z) & \rightarrow(y, 1, z) \quad \text { if } \quad y \in \mathbf{I} \\
(1, y, z) & \rightarrow\left(1, y^{-1}, y^{-1} z\right) \quad \text { if } \quad y \notin \mathbf{I} \\
(w, 1, z) & \rightarrow(1, w, z)
\end{aligned}
$$

and

$$
\begin{aligned}
{[m, 1, k] } & \rightarrow[1, m, k] \quad \text { if } \quad m \in \mathbf{I} \\
{[m, 1, k] } & \rightarrow\left[m^{-1}, 1,-k m^{-1}\right] \quad \text { if } \quad m \notin \mathbf{I} \\
{[1, n, p] } & \rightarrow[n, 1, p] \\
{[q, n, 1] } & \rightarrow[n, q, 1] .
\end{aligned}
$$

For any $s \in \mathcal{A}$, the collineation $\mathrm{G}_{s}$ transforms points and lines as follows:

$$
\begin{aligned}
(x, y, 1) & \rightarrow(x, y-x s, 1) \\
(1, y, z) & \rightarrow(1, y-s, z) \\
(w, 1, z) & \rightarrow(w, 1, z)
\end{aligned}
$$

and

$$
\begin{aligned}
{[m, 1, k] } & \rightarrow[m-s, 1, k] \\
{[1, n, p] } & \rightarrow[1, n, p+p s n] \\
{[q, n, 1] } & \rightarrow[q+s n, n, 1] .
\end{aligned}
$$

The collineation $\mathrm{I}_{2}$ transforms points and lines as follows:

$$
\begin{aligned}
(x, y, 1) & \rightarrow\left(y^{-1} x, y^{-1}, 1\right) \quad \text { if } \quad y \notin \mathbf{I} \\
(x, y, 1) & \rightarrow\left(1, x^{-1}, x^{-1} y\right) \quad \text { if } y \in \mathbf{I} \wedge x \notin \mathbf{I} \\
(x, y, 1) & \rightarrow(x, 1, y) \quad \text { if } \quad y \in \mathbf{I} \wedge x \in \mathbf{I} \\
(1, y, z) & \rightarrow\left(y^{-1}, y^{-1} z, 1\right) \quad \text { if } \quad y \notin \mathbf{I} \\
(1, y, z) & \rightarrow(1, z, y) \quad \text { if } \quad y \in \mathbf{I} \\
(w, 1, z) & \longrightarrow(w, z, 1)
\end{aligned}
$$

and

$$
\begin{aligned}
{[m, 1, k] } & \rightarrow\left[-m k^{-1}, 1, k^{-1}\right] \quad \text { if } \quad k \notin \mathbf{I} \\
{[m, 1, k] } & \rightarrow\left[1,-k m^{-1}, m^{-1}\right] \quad \text { if } \quad k \in \mathbf{I} \wedge m \notin \mathbf{I} \\
{[m, 1, k] } & \rightarrow[m, k, 1] \quad \text { if } \quad k \in \mathbf{I} \wedge m \in \mathbf{I} \\
{[1, n, p] } & \rightarrow\left[p^{-1}, 1,-n p^{-1}\right] \quad \text { if } \quad p \notin \mathbf{I} \\
{[1, n, p] } & \rightarrow[1, p, n] \quad \text { if } \quad p \in \mathbf{I} \\
{[q, n, 1] } & \rightarrow[q, 1, n] .
\end{aligned}
$$

So, we can give the following theorem without proof. For, its proof is same to Theorem 2 of [6]. Furthermore, this theorem is proved by Lemma 4.15 in [11].

Theorem 3.1: The group $\mathcal{G}$ of collineations of $\mathbf{M}(\mathcal{A})$ acts transitively on 3-gons.

Now, we can state the analogue of the result given by [2, Proposition 5.2.10 in Vol.I]. For the case of uniform H-rings (for the definition of uniform see [10]), the result is also in [7, Theorem 17]. Here, it is possible to give the proof of the following theorem, as more shorthly than the proof of Theorem 3 in [6].

Theorem 3.2: $\mathcal{G}$ acts transitively on 4-gons of $\mathbf{M}(\mathcal{A})$.
Proof: Let $(P, Q, R, S)$ be a 4-gon in $\mathbf{M}(\mathcal{A})$. It suffices to show that the points $P, Q, R, S$ can be transformed by an element of $\mathcal{G}$ to $U, V,(1,1,1), O$, respectively. From Theorem 3.1, there exists a collineation $\sigma$ which transforms $P, Q, R$ to $U, V,(0,1,1)$, respectively. Let $E$ denote the intersection point of the lines $Q R$ and $P S$. Then, since $\sigma(E)$ is nonneighbour to the points $\sigma(P), \sigma(Q), \sigma(R)$, it has the form $(0, b, 1)$, where $b-1 \notin \mathbf{I}$, and so $\sigma(S)$ has the form $(a, b, 1)$, where $a \notin \mathbf{I}$. Therefore $\sigma$ transforms $P, Q, R, S$ to

$$
(1,0,0),(0,1,0),(0,1,1),(a, b, 1),
$$

respectively. Then the mapping $\mathrm{T}_{-a,-b}$ transforms these points to

$$
(1,0,0),(0,1,0),(-a, 1-b, 1),(0,0,1),
$$

respectively and $\mathrm{S}_{(1-b)^{-1},-a^{-1}}$ transforms these points to

$$
(1,0,0),(0,1,0),(1,1,1),(0,0,1)
$$

respectively.
The following corollary is an obvious result of the last theorem:

Corollary 3.3: The coordinatization of $\mathbf{M}(\mathcal{A})$ is independent of the choice of the coordinatization base.

From now on, we carry over some concepts related to 6figures to the $\mathbf{M}(\mathcal{A})$, in view of the paper of [6].

A 6-figure is a sequence of six non-neighbour points $\left(A B C, A_{1} B_{1} C_{1}\right)$ such that $(A, B, C)$ is 3 -gon, and $A_{1} \in$

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$B C, B_{1} \in C A, C_{1} \in A B$. The points $A, B, C, A_{1}, B_{1}, C_{1}$ are called vertices of this 6-figure. The 6-figures $\left(A B C, A_{1} B_{1} C_{1}\right)$ and ( $D E F, D_{1} E_{1} F_{1}$ ) are equivalent if there exists a collineation of $\mathbf{M}(\mathcal{A})$ which transforms $A, B, C, A_{1}, B_{1}$, $C_{1}$ to $D, E, F, D_{1}, E_{1}, F_{1}$ respectively. Now, we give a theorem from [6].

Theorem 3.4: Let $\mu=\left(A B C, A_{1} B_{1} C_{1}\right)$ be a 6 -figure in $\mathbf{M}(\mathcal{A})$. Then, there is an $m \in \mathbf{U}$ such that $\mu$ is equivalent to $(U V O,(0,1,1)(1,0,1)(1, m, 0))$ where $U=(1,0,0), V=$ $(0,1,0), O=(0,0,1)$ are elements of the coordinatization basis of $\mathbf{M}(\mathcal{A})$.

We again give a theorem from [6]. Note that the proof of this theorem is more shorter.

## Theorem 3.5: The 6 -figures

$$
\left(A B C, A_{1} B_{1} C_{1}\right),\left(B C A, B_{1} C_{1} A_{1}\right),\left(C A B, C_{1} A_{1} B_{1}\right)
$$

are equivalent.
Proof: By Theorem 3.4 we may without loss of generality take $\left(U V O, U_{1} V_{1} O_{1}\right)$ instead of $\left(A B C, A_{1} B_{1} C_{1}\right)$, where

$$
U_{1}=(0,1,1), V_{1}=(1,0,1), O_{1}=(1, m, 0)
$$

with $m \in \mathbf{U}$. The collineation

$$
h:=S_{m, 1} \circ I_{2} \circ I_{1}
$$

transforms $\left(U V O, U_{1} V_{1} O_{1}\right)$ to ( $V O U, V_{1} O_{1} U_{1}$ ) and also $\left(V O U, V_{1} O_{1} U_{1}\right)$ to $\left(O U V, O_{1} U_{1} V_{1}\right)$.

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