International Journal of Information, Control and Computer Sciences
ISSN: 2517-9942
Vol:15, No:1, 2021

# Two-Stage Approach for Solving the Multi-Objective Optimization Problem on Combinatorial Configurations

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***Abstract*—**The statement of the multi-objective optimization problem on combinatorial configurations is formulated, and the approach to its solution is proposed. The problem is of interest as a combinatorial optimization one with many criteria, which is a model of many applied tasks. The approach to solving the multi-objective optimization problem on combinatorial configurations consists of two stages; the first is the reduction of the multi-objective problem to the single criterion based on existing multi-objective optimization methods, the second stage solves the directly replaced single criterion combinatorial optimization problem by the horizontal combinatorial method. This approach provides the optimal solution to the multi-objective optimization problem on combinatorial configurations, taking into account additional restrictions for a finite number of steps.

***Keywords*—**Discrete set, linear combinatorial optimization, multi-objective optimization, multipermutation, Pareto solutions, partial permutation set, permutation, structural graph.

## I. Introduction

THE optimization problems of several functions arise in the study of many theoretical and practical problems. Any applied task of optimal design of complex economic and technical systems, schemes, technological devices, structures, scheduling, planning and management of production activities, etc. requires the construction of a mathematical model taking into account many criteria and limitations [1]-[6]. This is a multi-objective optimization problem. Research in the field of multi-purpose optimization is currently intensively stimulated by practical needs and the development of computer information technologies [7]-[9].

The main properties of multi-objective optimization problems – the presence of many criteria, significant a restriction, variables of different scales, and algorithmically defined functions – make traditional methods impossible to use. The way out of this situation is the use of adaptive stochastic algorithms that successfully overcome these difficulties. When investigating the problem of multi-objective often one of them is selected as the dominant one. All other criteria are taken as limitations, and optimization is carried out according to the dominant criterion. When combining many functions into a vector criterion, we get a standard optimization problem. However, an adequate mathematical model of real problems includes several objective functions, as well as some additional restrictions, which makes its solution much more complicated. Given the discreteness of many optimization problems, we obtain multi-objective problems on discrete sets. In particular, many problems of planning, management, design and placement are modeled using multi-purpose tasks, the solutions of which are combinatorial in nature, for example, permutations, partial permutations, combinations, compositions, partitions, as well as their composite sets. In this case, the search for the optimal solution is carried out on the corresponding combinatorial set or its own subset [10].

When mapping combinatorial sets into an arithmetic Euclidean space, they acquire special properties [11]-[18]. Euclidean combinatorial sets are vertex-spaced sets of space $R^n$ and in most cases coincide with the vertices of their convex cover [11], [12]. Optimization methods for linear, quadratic, and convex functions for various classes of vertex-spaced sets were considered in [19]-[28], and in a multi-objective setting in [29]-[33]. Combinatorial sets are closely related to the concept of combinatorial configuration. The study of combinatorial configurations and their properties is the subject of [11], [12], [22], [34], [35].

In [19], [29]-[33] methods for solving multi-criterion optimization problems on discrete sets are considered. The traditional methods of multi-objective optimization can be divided into three key approaches. The first is related to the idea of ranking the criteria according to the importance and sequence of further optimization of each criterion individually with the assignment of an allowable value for changing the value of the criterion obtained in the previous step. The second approach consists in isolating the main criterion from all the criteria, and then optimizing it and translating the rest into restrictions. The third approach is the scalarization of a vector criterion into one generalized criterion. The main problem that concerns most traditional methods is the need to run the algorithm several times to obtain a representative approximation of the set of effective points (the number of iterations is equal to the power of the proposed approximation of the Pareto set). Thus, the problem of constructing new methods for solving multi-objective problems on combinatorial sets and the simultaneous inclusion of many criteria is relevant.

The purpose of the article is to present an approach to solving multi-objective optimization problems on

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combinatorial configurations. The problem is solved in two stages. The first one is to reduce the problem multi-objective to single criterion based on convolution methods. The second stage allows for solving the obtained single criterion problem on combinatorial set by the method of normalizing matrices.

The article is organized as follows - in the second part, the multicriteria optimization problem on combinatorial configurations is defined; - in the next section, the properties of combinatorial configurations and their graphs are described. The fourth part is devoted to the description of the approach to solving the problem of multi-objective optimization on the permutation set. The approach provides an optimal solution to a multi-objective optimization problem on a permutation set, taking into account additional restrictions for a certain number of steps. To demonstrate the work of the approach, the fifth part of the article presents numerical experiments characterizing its finiteness and effectiveness, as well as an analysis of their results.

## II. Statement of the Problem of Multi-Objective Optimization on Combinatorial Configurations

### *A. Euclidean Combinatorial Configurations*

Let the set $B=\{b_1,b_2,...,b_m\}$ be given, and let $A=\{a_1,a_2,..,a_n\}$ be a finite set, and let $\chi: B \rightarrow A$ is the mapping that corresponds to each element $b \in B$ a single element $a \in A$, i.e. $a=\chi(b)$. Define the configuration according to [11] as a mapping $\chi: B \rightarrow A$ that satisfies some set of constraints $\Lambda$. For most cases, one can unify the set $B$, i.e. the elements of the set can be replaced by their ordinal numbers. By setting the bijective mapping between $B$ and $J_m$, $J_m=\{1,2,...,\mathrm{m}\}$, we obtain the transformation of the mapping into

$$\phi: J_m \rightarrow A. \tag{1}$$

The combinatorial configuration can be represented by a tuple [11]

$$\langle \phi, A, \Lambda \rangle, \tag{2}$$

where $\phi$ – the representation of the form (1), which satisfies the set of constraints $\Lambda$, $A$ – the resulting set, the elements of which are strictly ordered.

We consider the set as the result set when forming the configuration (2) [35]

$$A^*=\{\alpha_1,\alpha_2,..,\alpha_n\}, \tag{3}$$

which is a set of vectors of the same dimension of space $R^k$, and as $\Lambda$ – consider the set of corresponding constraints that determine the required configuration.

We will put a vector in unambiguous correspondence to each configuration $\pi=\left[\alpha_{j_1},\alpha_{j_2},...,\alpha_{j_m}\right]$

$$x=(x_1,x_2,...,x_N) \in R^N,\ N=k \cdot m \tag{4}$$

whose components are an ordered set of elements of a multiset

$$\widetilde{A}(x)=\left\{a_{1j_1},a_{1j_2},...,a_{1j_m},a_{2j_1},a_{2j_2},...,a_{2j_m},a_{kj_1},a_{kj_2},...,a_{kj_m}\right\} \tag{5}$$

thus setting the bijective mapping such that

$$x=\psi(\pi),\ \pi=\psi^{-1}(x) \tag{6}$$

Euclidean combinatorial configuration (e-configuration) is called mapping

$$\psi:(\phi, A^*, \Theta) \rightarrow R^N \tag{7}$$

where $\phi: J_m \rightarrow A^*$, $A^*$ – the resulting set of the form (3), $\Theta$ – a system of constraints on the mappings $\phi, \psi$.

The defined of Euclidean combinatorial configuration is an image of the combinatorial configuration (2) in the arithmetic Euclidean space $R^N$ at given mappings $\phi, \psi$ and determines the vector $x$ form (4). $\widetilde{A}(x)$ is called the inducing multiset of the Euclidean combinatorial configuration. Let

$$E=\left\{x \in R^N : x - \textit{Euclidean combinatorial configurations}\ (7)\right\}. \tag{8}$$

A set $X=E^q_{mk} \subseteq R^m$ of the form (8) is called a set of e-configurations of permutations, if the inducing multiset of all its elements is a proper subset of the multiset inducing the set, i.e. $\forall x \in E \quad \tilde{A}(x) \subset \tilde{A}$.

### *B. Multi-Objective Optimization Problem on Combinatorial Configurations (MOPCC)*

Let $X=E^q_{mk} \subseteq R^m$ and $D \subseteq X$ be the set of admissible values of e-configurations, which is distinguished from X by a system of additional constraints.

We suppose that $f_i: \mathrm{X} \rightarrow \mathrm{R}^1, i \in J_n$ the functions that are components of the optimality criterion $F(x)=(f_1(x),...,f_n(x))$ are given. We have the task of finding the optimal solution:

$$F(x)=(f_1(x),...,f_n(x)) \rightarrow extr, \quad x \in D \subseteq X. \tag{8}$$

Problem (8) is a problem of vector Euclidean combinatorial optimization.

Let all components of the vector criterion be linear functions, i.e.

$$f_i(x)=\langle c_{ij}x_j \rangle,\ i \in J_n,\ j \in J_m, \tag{9}$$

and $D$ is distinguished from $X$ by linear constraints. Then the problem MOPCC takes the form: find the set of optimal values of functions

$$f_i(x)=\langle c_{ij}x_j\rangle \to extr,\ i\in J_n,\ j\in J_m \qquad (10)$$
$$x\in D\subseteq X$$

where it is formed by restrictions of a kind

$$\langle a_{ij}x_j\rangle \le b_i,\ i\in J_k,\ j\in J_m \qquad (11)$$

Problems (8), (10), (11) will be a multi-objective problem of linear Euclidean combinatorial optimization (MOPLECO).

*C. Solutions Set of MOPLECO*

The solution of MOPLECO is $X^*$ – the effective solutions set. In the transition from combinatorial to Euclidean combinatorial optimization we obtain the corresponding sets $I(F,X)$ – ideal, $P(F,X)$ – Pareto-optimal, $Sl(F,X)$ – poorly efficient and $Sm(F,X)$ – strictly efficient solutions that correspond to the following sets and will look like:

$$I(F,X)=\{x^*\in X: F(x^*)=F^*\}$$
$$P(F,X)=\{x\in X:\nexists x'\in X: F(x')\ge F(x),\ F(x')\ne F(x)\}$$
$$Sl(F,X)=\{x\in X:\nexists x'\in X: F(x')>F(x)\}$$
$$Sm(F,X)=\{x\in X:\nexists x'\in X\setminus\{x\}: F(x')\ge F(x)\}$$

Then the set of solutions will look like

$$\Xi_X=\{I(F,X),\mathrm{P}(F,X),Sl(F,X),Sm(F,X)\}$$

When solving a problem in the general case, it is necessary to find some set $X^*\in\Xi_X$. We denote the required sets by $X_I^*=I(F,X)$, $X_P^*=P(F,X)$, $X_{Sl}^*=Sl(F,X)$, $X_{Sm}^*=Sm(F,X)$. In what follows, we will look for a set $X_P^*$ of Pareto-optimal solutions in the problems. As mentioned above, according to the requirements of this problem, it is often sufficient to find part of the set of optimal solutions or one solution that belongs to this set. When solving such problems, it is important to take into account the specifics of the combinatorial set. The use of their connection with graph theory is promising. In [34] the graphs of combinatorial polyhedra were studied, in [18], [22], [34] the representation of combinatorial sets in the form of grid graphs and structural graphs is described. This direction is promising for the development of new methods for solving problems (8), (10), (11) and is used in this work.

## III. Properties of Combinatorial Configurations and Their Graphs

We consider the representation of a set of Euclidean combinatorial configurations of placements in the form of a structural graph. Let $X'$ be a subset of a set of combinatorial configurations $X$ whose elements have the form $x=(x_1,x_2,...,x_m)$. $X'$ are formed on a certain basis, for example, on the fixed coordinates. Suppose a linear function is given, the coefficients of which are ordered in ascending order, i.e.

$$f(x)=\langle c_i x_i\rangle,\ i\in J_m,\ c_1\le c_2\le ...\le c_m. \qquad (12)$$

We describe the concept of a structural graph. Let $G^{X'}(\tilde V,\tilde U)$ be a graph of a subset of combinatorial configurations $X$, and the set $X'$ is structured in such a way $X'=X_1'\cup X_2'\cup ...\cup X_\lambda'$ that each subset $X_i',\ i\in J_\lambda$ corresponds to vertices with $h$ fixed coordinates and is represented by two vertices $x_i^0$ and $x_i^{st}$ such that the conditions for the function of form (12) hold:

$$f(x_i^0)=\max_{x\in X_i'} f(x), \qquad (13)$$
$$f(x_i^{st})=\min_{x\in X_i'} f(x),$$

and the edges of the graph $G^{X'}(\tilde V,\tilde U)$ are those that connect the corresponding vertices $x_i^0$, $x_i^{st}$, and the vertices formed by successive transpositions of fixed coordinates, then such a graph will be called a structural graph of the set of Euclidean combinatorial configurations and we will denote $G_S^X(\tilde V,\tilde U,h)$. Determinants for the construction of a structural graph are its vertices, which are defined as leakage and runoff.

**Definition 1.** The vertices $x_i^0, x_i^{st}\in X_i',\ i\in J_\lambda$ of the structural graph $G_S^X(\tilde V,\tilde U,h)$ for which conditions (13) are satisfied are called the vertices of leakage and runoff, respectively, and the quantity $\lambda$ is the level of the structural graph.

**Statement 1.** The number of levels of a structural graph $G_S^X(\tilde V,\tilde U,1)$ is equal to the number of different elements of the generating set $A$ for combinatorial configurations, i.e. the power of its basis $S(A)$.

Next, we consider the Euclidean combinatorial configuration of permutation and its properties and the properties of graphs of the set of permutation.

**Example 1.** Suppose we have a set of Euclidean combinatorial configurations of permutation of dimension 4 from the set $A=\{1,2,3,4,5,6\}$. Let's put $h=2$, that is we will fix last two coordinates. We obtain the following set of coordinate pairs:

$$\{(6,5),(6,4),(6,3),(6,2),(6,1),(5,6),(5,4),(5,3),$$
$$(5,2),(5,1),(4,6),(4,5),(4,3),(4,2),(4,1),(3,6),$$
$$(3,5),(3,4),(3,2),(3,1),(2,6),(2,5),(2,4),(2,3),$$
$$(2,1),(1,6),(1,5),(1,4),(1,3),(1,2)\}$$

Thus, we obtain thirty levels of the structural graph and the corresponding number of pairs of vertices $x_i^0$ and $x_i^{st}$. We denote them selectively in Fig. 1.

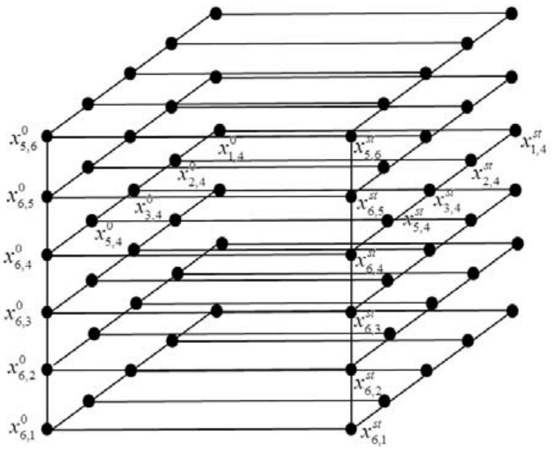


Fig. 1 The structural graph $G_S^X\left(\tilde{V},\tilde{U},2\right)$

**Statement 2.** Suppose a function $f(x)$ of the form (12) is given on the structural graph of the set e-configurations $G_S^X\left(\tilde{V},\tilde{U},h\right)$ and a value $B$ is given then one of the following conditions is satisfied:

a) if $f\left(x_i^0\right)\le B$, then $\forall x\in X_i':f\left(x\right)\le B$;

b) if $f\left(x_i^{st}\right)\ge B$, then $\forall x\in X_i':f\left(x\right)\ge B$;

*c)* if $f\left(x_i^0\right)\ge B\ge f\left(x_i^{st}\right)$, then $\exists X'_{i,\ge B}\subset X'$, $X'_{i,\ge B}=\left\{x\in X_i':f\left(x\right)\ge B\right\}$, $\exists X'_{\le B}\subset X'$, $X'_{i,\le B}=\left\{x\in X_i':f\left(x\right)\le B\right\}$.

It follows from Statement 2 that by estimating the levels of the structural graph of the set of Euclidean combinatorial configurations, it is possible to determine the further direction of motion along the graph in depth. So then we define a structurally oriented graph $\vec{G}_S^X\left(\tilde{V},\tilde{U},h\right)$ (Fig. 2). That is, to estimate the value of a linear function, we must first consider the structural graph $\vec{G}_S^X\left(\tilde{V},\tilde{U},1\right)$, determining the value of the function at its vertices. Sequential consideration of structural graphs of sets of Euclidean combinatorial configurations $\vec{G}_S^X\left(\tilde{V},\tilde{U},1\right)$, $\vec{G}_S^X\left(\tilde{V},\tilde{U},2\right)$, $\vec{G}_S^X\left(\tilde{V},\tilde{U},3\right)\dots$ is called immersion in a structural graph $\vec{G}_S^X$.

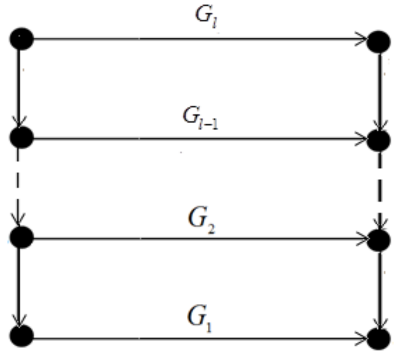


Fig. 2 The structural oriented graph $\vec{G}_S^X\left(\tilde{V},\tilde{U},h\right)$

## IV. Two-Stage Approach for Solving of MOPLECO

The two-stage approach to solving the multi-objective optimization problem on combinatorial configurations consists in using the combinatorial horizontal method.

An additional procedure has been introduced to bypass the structural ornate graph of a multitude of Euclidean combinational configuration permutations. The procedure is based on the horizontal method, and can be described by such steps.

### *A. The Horizontal Method*

Step0. Input the coefficients of the linear constraints of the problem $g(x)\ge b$, arranging the coefficients of a given linear function $g(x)=\sum_{i=1}^{m}c_ix_i$ in ascending order, then the function takes the form:

$$g(x)=\tilde{c}_1x_1+\dots+\tilde{c}_nx_n,\ \tilde{c}_1\le\tilde{c}_2\le\dots\le\tilde{c}_m.$$

Step1. Construct a structural graph of the set X of Euclidean combinatorial configurations of permutations $G_S^X\left(\tilde{V},\tilde{U},\tilde{h}\right)$, $\tilde{h}=1$, i.e. with one fixed coordinate.

Step2. The number of vertices of the graph $G_S^X\left(\tilde{V},\tilde{U},\tilde{h}\right)$ is determined by the formula $\left|\tilde{V}\right|=2\cdot m'!/\left(m'-\tilde{h}\right)!$, where $m'$ is the number of different elements of the base of the set $X$.

Step3. Calculate the values $g(x)$ at the vertices of the graph $G_S^X\left(\tilde{V},\tilde{U},\tilde{h}\right)$, which according to the definition of the structural graph will be equal to the minimum and maximum at each level of the graph $G_S^X\left(\tilde{V},\tilde{U},1\right)$.

Step4. Determine the levels of the graph containing the desired vertices that satisfy conditions $g(x)\ge b$, using statement 2. Increase the number of fixed coordinates, i.e. $\tilde{h}=\tilde{h}+1$. If $\tilde{h}\le m$, form a set of structural graphs that are under the graphs of the general graph $\vec{G}_S^{X_1}\left(\tilde{V},\tilde{U},\tilde{h}\right)$, $\vec{G}_S^{X_2}\left(\tilde{V},\tilde{U},\tilde{h}\right)$,…, $\vec{G}_S^{X_l}\left(\tilde{V},\tilde{U},\tilde{h}\right)$ (immersion in the graph), go to step 3 and repeat the study for each of the $l$ graphs, where $l$ is the number of levels that require further research.

Step5. Using the properties of the structural graph, form a set $D$ of such Euclidean combinatorial configurations for which the condition holds $g(x)\ge b$.

### *B. Approach to Solving of MOPLECO*

Since the problem is multi-criteria, it is necessary to use vector optimization methods. The most common, but not very accurate, are the methods of linear convolution. Also methods of an ideal point, satisfied requirements, and consecutive concessions are used [5].

We consider the method of consecutive concessions according to [5]. To simplify the description of the method, put $f_i \to \min$, $i \in J_m$.. First, the optimization criteria are sorted in descending order of importance $f_1 \succ f_2 \succ ... \succ f_m$. Ranking of criteria leads to the lexicographic solution of the problem. A vector $\vec{x} \in X$ is a lexicographical solution if the condition is satisfied for all $\vec{x}' \in X$.

Next, the optimal value for the first criterion is determined: $f_1^* = \min f_1(x), x \in D$. The value of the allowable concession is determined $(f_1^* + \Delta f_1)$, inequality is added to the system of restrictions $f_1(x) \le f_1^* + \Delta f_1$ and the optimal value of the second priority criterion is found. When the procedure is done for each of m criteria and the specified concessions are not exaggerated, the solution is considered optimal.

When combining the horizontal method with the method of consecutive concessions, we obtain a new method that takes into account the properties of the combinatorial set and the features of multi-criteria optimization. We describe it in this sequence of steps.

### *C. Steps for Solving of MOPLECO*

Step1. Input the data: elements of the generative set $A$ of Euclidean combinatorial configurations of permutations, optimality criteria $f_i\ (x) \to \min$, $i \in J_m$, linear constraints of the problem $g_i\ (x) \ge b_i$, $i \in J_k$.

Step2. For each of the $k$ constraints we construct a structurally oriented graph and perform its study according to the procedure of the horizontal method described above, obtaining sets $D_i, i \in J_k$.

Step3. Find the set $D^* = D_1 \cap D_2 \cap ... \cap D_k$.

Step4. Sort the criteria by priority $f_1 \succ f_2 \succ ... \succ f_m$.

Step5. Put $i = 1$, $D_1^* = D^*$.

Step6. Find $f_i^* = \min f_1(x), x \in D_i^*$. Determine the value of the allowable concession for the current criteria $(f_i^* + \Delta f_i)$.

Step7. For inequality $f_i(x) \le f_i^* + \Delta f_i$, apply the procedure of the horizontal method, obtaining a set $D_i'$.

Step8. Put $i = i+1$, $D_i^* = D_i' \cap D_{i-1}^*$. If $i \le k$ – move on to step 6, otherwise – the obtained solution for $f_k(x)$ is the solution of the problem, because it satisfies all the previous admissible concessions.

## V. Conclusion

The paper defines a multi-objective problem on Euclidean combinatorial configurations of permutation. It describes the properties of combinatorial configurations and their graphs. Two-stage approach for solving of multi-objective problem of linear Euclidean combinatorial optimization is proposed. Using this method allows you to reduce the number of vertices needed to find the solution of the problem. This is achieved by sequential analysis of structural graphs by the horizontal method. This method can be used for other sets of combinatorial configurations.

## References

[1] Ehrgott M. Multicriteria optimization. Berlin; New York: Springer. 2005. P.323

[2] Ehrgott M., Gandibleux X. Multiobjective combinatorial optimization – theory, methodology, and applications. In: Ehrgott, M. and Gandibleux, X. (eds.) Multiple criteria optimization: state of the art annotated bibliographic surveys. Berlin; Springer US, 2003. p. 369–444. https://doi.org/10.1007/0-306-48107-3_8.

[3] Ehrgott M., Wiecek M. Mutiobjective programming. In Multiple Criteria Decision Analysis: State of the Art Surveys, number 78 in International Series in Operations Research & Management Science, Springer New York, 2005. P. 667–708.

[4] Studniarski M., Koliechkina K., Dvernaya E., Solving a Combinatorial Multiobjective Optimization Problem by Genetic Algorithm. In.: Kulczycki P., Kowalski P., Łukasik Sz.(eds). Contemporary Computational Science. 3rd Conference on Information Technology, Systems Research and Computational Physics, Contemporary Computational Science, 2018, P.188-200.

[5] Semenova N.V., Kolechkin L.M. Vector Discrete Optimization Problems on Combinatorial Configurations: Research and Solution Methods (in Ukrainian). Kyiv: Nauk. Dumka, 2009. 266 p.

[6] Koliechkina, L., Pichugina, O.: Multiobjective Optimization on Permutations with Applications. DEStech Transactions on Computer Science and Engineering. 2018. P. 61–75.

[7] Koliechkina, L., Pichugina, O., Yakovlev, S.: A Graph-Theoretic Approach to Multiobjective Permutation-Based Optimization. In: Ja´cimovi´c, M., Khachay, M., Malkova, V., and Posypkin, M. (eds.) Optimization and Applications. Springer International Publishing, Cham 2020. , P. 383–400.

[8] Semenova N.V., Kolechkina L.N., Nagornaya A.N. Solution and investigation of vector problems of combinatorial optimization on a set of permutations. Journal of Automation and Information Sciences. 2008. 40, N 12. P. 67–80.

[9] Houda Abadlia, Nadia Smairi, Khaled Ghedira An Enhanced Particle Swarm Optimization Algorithm for Multiobjective Problems. World Academy of Science, Engineering and Technology International Journal of Computer and Information Engineering Vol:12, No:11, 2018

[10] Colbourn C.J., Dinitz J.H. Handbook of Combinatorial Designs, second edition. CRC Press, 2010. 784 p.

[11] Stoyan, Y.G., Yakovlev, S.V., Pichugina, O.S.: The Euclidean combinatorial configurations: a monograph. Constanta, Kharkiv. 2017, 268p.

[12] Stoyan, Y.G., Yakovlev, S.V. Theory and Methods of Euclidian Combinatorial Optimization: Current Status and Prospects. Cybernetics and Systems Analysis. 2020. 56, N 3. P. 366–379.

[13] Korte B., Vygen J. Combinatorial optimization: theory and algorithms. Heidelberg; New York: Springer, 2018. 698 p.

[14] Pardalos P.M., Du D-Z., Graham R.L. Handbook of combinatorial optimization. New York: Springer, 2013. 3409 p.

[15] Papadimitriou C.H., Steiglitz K. Combinatorial optimization: algorithms and complexity. Mineola: Dover Publications, 2013. 528 p.

[16] Sergienko I.V., Shilo V.P. Modern approaches to solving complex discrete optimization problems. Journal of Automation and Information Sciences. 2016. 48, N 1. P. 15–24.

[17] Hulianytskyi L., Riasna I. Formalization and classification of combinatorial optimization problems. Optimization methods and applications. S. Butenko et al. (eds.). New York: Springer, 2017. P. 239–250.

[18] Koliechkina L., Pichugina O. A Horizontal Method of Localizing Values of a Linear Function in Permutation-Based Optimization. In: Le Thi, H.A., Le, H.M., and Pham Dinh, T. (eds.) Optimization of Complex Systems: Theory, Models, Algorithms and Applications. Springer, Cham. 2019. P. 355–364.

[19] Koliechkina L. N., Dvirna O.A., Nagornaya A.N. Modified coordinate method to solve multicriteria optimization problems on combinatorial configurations. Cybernetics and Systems Analysis. 2014. 59, N 4. P.

620–626.
[20] Pichugina O.: Placement problems in chip design: Modeling and optimization. In: 2017 4th International Scientific-Practical Conference Problems of Infocommunications. Science and Technology (PIC S T). P. 465–473 (2017).
[21] Yakovlev S. Convex extensions in combinatorial optimization and their applications. Optimization Methods and Applications. 2017. 130. P. 567–584.
[22] Yemelichev V.A., Koval¨ev M.M., Kravtsov M.K. Polytopes, graphs and optimisation. Cambridge University Press, Cambridge. 1984. 243p.
[23] Donets G.P., Koliechkina L.N., Nahirna A.N. A Method to Solve Conditional Optimization Problems with Quadratic Objective Functions on the Set of Permutations. Cybernetics and Systems Analysis. 2020. Vol. 56, N 2. P. 278-288.
[24] Koliechkina L.N., Nahirna A.N. Solutions of the Combinatorial Problem with a Quadratic Fractional Objective Function on the Set of Permutations. Cybernetics and Systems Analysis. 2020. Vol. 56, N 3. P. 455-465.
[25] Yakovlev S.V., Pichugina O.S. Properties of combinatorial optimization problems over polyhedral-spherical sets. Cybernetics and Systems Analysis. 2018. 54, N 1. P. 99–109.
[26] Pichugina O., Yakovlev S. Optimization on polyhedral-spherical sets: theory and applications. In 2017 IEEE First Ukraine Conference on Electrical and Computer Engeneering (UKRCON 2017). Proceedings. Kyiv, 2017. P. 1167–1175.
[27] Stoyan Yu.G., Sokolovskii V.Z., Yakovlev S.V. Method of balancing rotating discretely distributed masses. Energomashinostroenie. 1982. N 2. P. 4–5.
[28] Yakovlev S.V., Grebennik I.V. Localization of solutions of some problems of nonlinear integer optimization. Cybernetics and Systems Analysis. 1993. 29, N 5. P. 419–426.
[29] Semenova N.V., Kolechkina L.N., Nagirna A.N. An approach to solving discrete vector optimization problems over a combinatorial set of permutations. Cybernetics and Systems Analysis. 2008. 44, N 3. P. 441–451.
[30] Semenova N.V., Kolechkina L.N. A polyhedral approach to solving multicriterion combinatorial optimization problems over sets of polyarrangements. Cybernetics and Systems Analysis. 2009. 45, N 3. P. 438–445.
[31] Semenova N.V., Kolechkina L.N., Nagornaya A.N. On approach to solving vector problems with fractionally linear functions of the criteria on the combinatorial set of arrangements. Journal of Automation and Information Sciences. 2010. 42, N 2. P. 67–80.
[32] Koliechkina L.N., Dvirna O.A. Solving extremum problems with linear fractional objective functions on the combinatorial configuration of permutations under multicriteriality. Cybernetics and Systems Analysis. 2017. 53, N 4. P. 590–599.
[33] Koliechkina L.N., Nagornaya A.N., Semenov V.V. Method of solving problem of conditional optimization on combinatorial set of arrangements. Journal of Automation and Information Sciences. 2019. 51, N.8. P. 31-42.
[34] Donetsk G.P., Kolechkin L.M. Extreme problems on combinatorial configurations (in Ukrainian). Poltava: PUET, 2011. 362 p.
[35] Yakovlev S.V. Formalization of spatial configuration optimization problems with a special function class. Cybernetics and Systems Analysis. 2019. 55, N 4. P. 512–523.

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