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# Solution of $S^3$ Problem of Deformation Mechanics for a Definite Condition and Resulting Modifications of Important Failure Theories

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***Abstract*****—**Analysis of stresses for an infinitesimal tetrahedron leads to a situation where we obtain a cubic equation consisting of three stress invariants. This cubic equation, when solved for a definite condition, gives the principal stresses directly without requiring any cumbersome and time-consuming trial and error methods or iterative numerical procedures. Since the failure criterion of different materials are generally expressed as functions of principal stresses, an attempt has been made in this study to incorporate the solutions of the cubic equation in the form of principal stresses, obtained for a definite condition, into some of the established failure theories to determine their modified descriptions. It has been observed that the failure theories can be represented using the quadratic stress invariant and the orientation of the principal plane.

***Keywords*****—**Cubic equation, stress invariant, trigonometric, explicit solution, principal stress, failure criterion.

## I. Introduction

ANALYSIS of an infinitesimal tetrahedron leads to a cubic equation consisting of three stress invariants, which are linear, quadratic and cubic. Solution of such equation gives the principal stresses and the orientation of the principal planes. However previous researchers state [1], [2] that solution of such cubic equation requires time consuming trial and error method or cumbersome numerical techniques as there is no direct method for that. Some researchers, however, had derived some explicit method for the evaluation of roots of such cubic equations by converting them into cubic equations consisting of deviatoric invariants and using a trigonometric identity [3], [4]. In this present study an approach has been presented for obtaining explicit solutions for the cubic equation for a definite condition and determining the principal stresses and the orientation of the principal planes. The solutions are then applied to various failure theories to obtain modified descriptions of those theories.

## II. Cubic Equation of Stress Invariants

Let us consider an infinitesimal tetrahedron OABC having an oblique plane ABC (direction cosines $l=\cos\alpha_1, m=\cos\alpha_2, n=\cos\alpha_3$ (Fig. 2)), which intersects Cartesian X, Y & Z axes at A, B & C respectively (Fig. 1) and having $S_x, S_y \& S_z$ the three stress components along the 3 Cartesian directions on the oblique plane. The consideration of equilibrium in the X, Y & Z directions yields:

$$\begin{Bmatrix} S_x \\ S_y \\ S_z \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}.$$

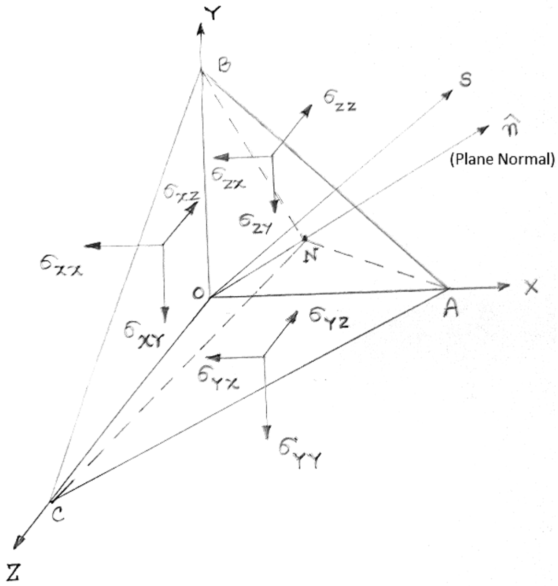


Fig. 1 Stresses on an infinitesimal tetrahedron

Now, if the resultant stress coincides with the normal stress i.e. the direction magnitudes of both become same, the oblique plane becomes the principal plane resulting $S_x, S_y \& S_z$ to be equal to $S\times l, S\times m$ & $S\times n$ respectively and from that we obtain $\begin{bmatrix} \sigma_{xx}-S & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy}-S & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}-S \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = 0$. Since $l=m=n=0$ is not possible as $l^2+m^2+n^2=1$, we can clearly say that, $\begin{vmatrix} \sigma_{xx}-S & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy}-S & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}-S \end{vmatrix} = 0$. Expanding the determinant we can obtain Cubic Equation of Invariants:

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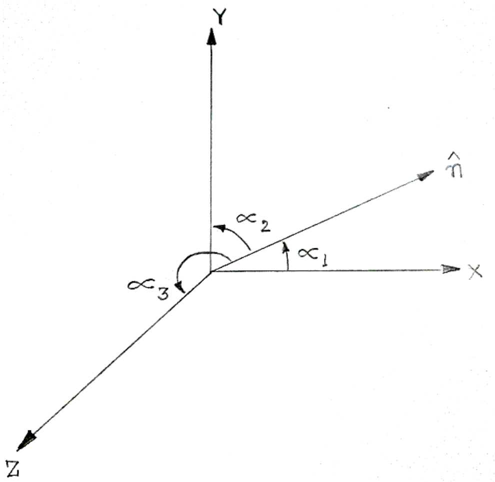


Fig. 2 Directional angles of a Plane Normal

$$S^3 - I_1 \times S^2 - I_2 \times S - I_3 = 0 \quad (1)$$

(also known as the characteristic equation of stress tensor) where,

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \text{ (Linear Invariant)}$$

$$I_2 = \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2 - \sigma_{xx} \times \sigma_{yy} - \sigma_{yy} \times \sigma_{zz} - \sigma_{zz} \times \sigma_{xx}$$
(Quadratic Invariant)

$$I_3 = 2 \times \sigma_{xy} \times \sigma_{yz} \times \sigma_{zx} + \sigma_{xx} \times \sigma_{yy} \times \sigma_{zz} - \sigma_{xx} \times \sigma_{yz}^2 - \sigma_{yy} \times \sigma_{xz}^2 - \sigma_{zz} \times \sigma_{xy}^2 \text{ (Cubic Invariant)}$$

## III. Solution of the Cubic Equation

The solution of the cubic equation ( 1) can be obtained explicitly for a definite condition $\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 0$ i.e. when the linear invariant is zero ($I_1 = 0$) [5]. The resulting cubic equation, after putting $S = r \times \sin\theta$ in $S^3 - I_2 \times S - I_3 = 0$, when compared to the trigonometric identity, $\sin^3\theta - \left(\frac{3}{4}\right) \times \sin\theta + \left(\frac{1}{4}\right) \times \sin 3\theta = 0$ gives the principal stresses as:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix} = \left(\frac{2}{\sqrt{3}}\right) \times \sqrt{I_2} \times \begin{Bmatrix} \sin\theta_1 \\ \sin\theta_2 \\ \sin\theta_3 \end{Bmatrix} \quad (2)$$

where,

$$\theta_1 = \left(\frac{1}{3}\right) \times \sin^{-1}(k) + \left(\frac{2\pi}{3}\right) = \theta + \left(\frac{2\pi}{3}\right)$$

$$\theta_2 = \left(\frac{1}{3}\right) \times \sin^{-1}(k) = \theta$$

$$\theta_3 = \left(\frac{1}{3}\right) \times \sin^{-1}(k) + \left(\frac{4\pi}{3}\right) = \theta - \left(\frac{2\pi}{3}\right).$$

This solution is valid when the value of $3\theta$ is in the range $\pm\frac{\pi}{2}$ i.e. $\frac{-\pi}{2} \le 3\theta \le \frac{\pi}{2}$ or $\frac{-\pi}{6} \le \theta \le \frac{\pi}{6}$.

## IV. Some Failure Theories and Their Modifications

A theory of failure consists essentially of a nucleating relationship in the form $F(\sigma_1, \sigma_2, \sigma_3, k_1, k_2, k_3, ...) = 0$ so constructed that a state of stress, defined by principal stresses $\sigma_1, \sigma_2$ and $\sigma_3$ that produce it, consistent with the material properties $k_1, k_2, k_3$ etc., lead to physical failure either by fracture or by yielding. Since failure/yielding should depend on the microstructure i.e. the orientation of the axes $x_1, x_2, x_3$, we can express the yield criteria in terms of principal stresses in the form $F(\sigma_1, \sigma_2, \sigma_3, k_i, n_i) = 0$ where $n_i$ represents the principal directions, which give the orientation of the principal stresses relative to the material directions $x_1, x_2, x_3$. In case of isotropic materials the failure criteria is independent of any material directions and can be expressed in the simple form as $F(\sigma_1, \sigma_2, \sigma_3, \sigma_Y) = 0$. Alternatively, since the three principal invariants of stress are independent of material orientation, one can write $F(I_1, I_2, I_3, \sigma_Y) = 0$. Over the years various researchers have proposed different failure theories for both brittle materials (Mohr – Coulumb [6], Drucker – Prager [7]) like soil and concrete and ductile materials (Guest – Tresca [6], Von Mises [8], Nadai [9]) like different metals. Some of the important failure theories are discussed here along with their modified forms obtained from the application of new expressions of the principal stresses (2) described earlier in this paper. It has been found that all the failure criterion can be expressed in a simple and convenient form as $F(I_2, \theta) = 0$.

### *A. Rankine – Lame – Navier Theory*

This theory states that the yield criterion is reached when the combined stress results in principal stresses that attain the ultimate strength (for brittle materials) or yield strength (for ductile materials) in the uniaxial state of stress. This theory specifies [10] very basic requirement and states that whenever the largest of the principal stresses $\sigma_1, \sigma_2, \sigma_3$ equals the strength of the material $\sigma_Y$, failure may occur. The criteria may therefore be expressed as $\sigma_1 \le \sigma_Y$, or in the modified form as:

$$\left(\frac{2}{\sqrt{3}}\right) \times \sqrt{I_2} \times \sin\left(\theta + \left(\frac{2\pi}{3}\right)\right) - \sigma_Y = 0$$

### *B. St. Venant's Principal Strain Theory*

As per this theory, a material will fail under combined stress

– state if the maximum unit linear strain ($\varepsilon_{\max}$) exceeds the allowable unit linear strain assumed for uniaxial tension. This theory does not conform to experimental data. As per this theory –

$$\varepsilon_{\max} = \varepsilon_1 = \frac{1}{E} \times [\sigma_1 - \mu \times (\sigma_2 + \sigma_3)] \le \frac{\sigma_Y}{E}.$$

If we now incorporate the values of $\sigma_1, \sigma_2, \sigma_3$ from (2) into the condition of failure as per the St. Venant's Principal Strain Theory the resulting expression comes out to be:

$$\left(\sqrt{\frac{I_2}{3}}\right) \times \left(\sqrt{3} \times \cos\theta - \sin\theta\right) \times (1 + \mu) - \sigma_Y = 0.$$

*C. Guest – Tresca Theory*

This theory is also known as the maximum shear stress theory and states that a material will fail under combined stress if the maximum shear stress exceeds the limiting shearing stress established from tests with uniaxial tension states. Tresca proposed this theory after carrying out an experimental programme on the extrusion of metals and therefore this theory is commonly used for defining the yield criteria for isotropic metallic materials and is a pressure independent criterion. Therefore, the theory can be expressed as:

$$\frac{\sigma_1 - \sigma_3}{2} \le \frac{\sigma_Y}{2}.$$

If we now put the values of $\sigma_1, \sigma_2, \sigma_3$ from (2) into the Tresca Criteria the resulting expression comes out to be:

$$2 \times \sqrt{I_2} \times \cos\theta - \sigma_Y = 0.$$

TABLE I
SUMMARY OF DIFFERENT FAILURE THEORIES AND THEIR MODIFICATIONS

| Failure theory | Original Descriptions | Modified Descriptions |
|---|---|---|
| Rankine – Lame – Navier | $\sigma_1 \le \sigma_Y$ | $\left(\frac{2}{\sqrt{3}}\right) \times \sqrt{I_2} \times \sin\left(\theta + \left(\frac{2\pi}{3}\right)\right) - \sigma_Y = 0$ |
| St. Venant's Principal Strain | $\frac{1}{E} \times [\sigma_1 - \mu \times (\sigma_2 + \sigma_3)] \le \frac{\sigma_Y}{E}$ | $\left(\sqrt{\frac{I_2}{3}}\right) \times \left(\sqrt{3} \times \cos\theta - \sin\theta\right) \times (1 + \mu) - \sigma_Y = 0$ |
| Guest – Tresca | $\frac{\sigma_1 - \sigma_3}{2} \le \frac{\sigma_Y}{2}$ | $2 \times \sqrt{I_2} \times \cos\theta - \sigma_Y = 0$ |
| Hencky – Huber – Von Mises | $\sqrt{3 \times J_2} \le \sigma_Y$ | $\sqrt{3 \times I_2} - \sigma_Y = 0$ |
| Nadai | $\tau_{OCT} \le \frac{\sqrt{2}}{3} \times \sigma_Y$ | $\sqrt{3 \times I_2} - \sigma_Y = 0$ |
| Mohr – Coulumb | $\tau \le c - \sigma \times \tan\phi$ | $\sqrt{I_2} \times \cos\theta - \sqrt{\frac{I_2}{3}} \times \sin\theta \times \sin\phi - c \times \cos\phi = 0$ |
| Drucker - Prager | $\alpha \times I_1 + \sqrt{3 \times J_2} \le \sigma_Y$ | $\sqrt{3 \times I_2} - \sigma_Y = 0$ |

*D. Hencky – Huber – Von Mises Theory*

This theory is also known as shear distortion strain energy theory and states that the material will fail when the distortional strain energy exceeds the strain energy under uniaxial tension. This theory is very much suitable for isotropic, homogeneous and ductile materials and is therefore used for the modeling of plasticity in metals. Von Mises criterion is a pressure independent criterion and also used to predict the ductile fracture of concrete under high hydrostatic pressure. The conventional form of Von Mises criteria for failure is $\sqrt{3 \times J_2} \le \sigma_Y$ where $J_2 = I_2 + \frac{1}{3} \times I_1^2$ (the quadratic deviatoric invariant).

In the present case, since $I_1 = 0$, the modified expression is stated as $\sqrt{3 \times I_2} - \sigma_Y = 0$.

*E. Nadai's Theory*

This theory is also very much useful for modeling the plasticity in metals and is a pressure independent criterion like Von Mises and Tresca criteria. This theory is different form of Von Mises theory, which states that a material failure will occur when the octahedral shear stress ($\tau_{OCT} = \sqrt{\frac{2}{3} J_2}$) reaches a critical value given as $\frac{\sqrt{2}}{3} \sigma_Y$. The expression can be written as $\tau_{OCT} \le \frac{\sqrt{2}}{3} \times \sigma_Y$.

The modified expression comes out to be the same as the previous one i.e. that of Von Mises expression $\sqrt{3\times I_2}-\sigma_Y=0$.

*F. Mohr – Coulumb Theory*

This theory is mostly used for predicting the failure of brittle materials like soil and concrete. According to this theory, failure is assumed to occur when the shear stress, $\tau$ on any plane at a point in a concrete material reaches a value that depends linearly upon the normal stress $\sigma$ on the same plane. The general expression of this theory is given as $\tau \le c-\sigma\times\tan\phi$, in which $\tau$ is the shearing stress, $\sigma$ is the normal stress, $c$ is cohesion and $\phi$ is the angle of friction.

Here, $\tau=\left(\frac{\sigma_1-\sigma_3}{2}\right)\times\cos\phi$ and $\sigma=\left(\frac{\sigma_1+\sigma_3}{2}\right)+\left(\frac{\sigma_1-\sigma_3}{2}\right)\times\sin\phi$.

Use of expressions from (2) results into the following modified form:

$$\sqrt{I_2}\times\cos\theta-\sqrt{\frac{I_2}{3}}\times\sin\theta\times\sin\phi-c\times\cos\phi=0.$$

In situations, where, the angle of internal friction i.e. $\phi$ is zero and $c$ is the yield stress the criterion reduces to the similar form as derived in the case of Guest – Tresca criteria.

*G. Drucker – Prager Theory*

An approximation of the Coulumb law was expressed by Drucker – Prager as a simple modification of the Von Mises criteria whereby a hydrostatic dependent first invariant $I_1$ was introduced in the Von Mises equation.

$$\alpha\times I_1+\sqrt{3\times J_2}\le\sigma_Y.$$

This Drucker – Prager yield criterion is a hydrostatic pressure dependent yield criterion and had been introduced to model the plastic behaviour of soil although this model is sometimes used for modeling plastic behaviour of concrete also.

Since, in the present case $I_1=0$, the modified expression comes out to be as $\sqrt{3\times I_2}-\sigma_Y=0$ which is same as that of Von Mises.

A comparison of the original and the modified descriptions, obtained by incorporating the values of the principal stresses derived from the solution of the cubic equation of stress invariants for a definite condition, of the different failure theories, is given in Table I. The simple and convenient forms of the modified descriptions of the failure theories can be easily observed and all take the form given as - $F(I_2,\theta)=0$.

It may be observed that for a definite value of $\theta$ these failure theories can be expressed in a general form as $C\sqrt{I_2}-\sigma_Y=0$, where C is a constant having different values for different theories.

## V. Conclusion

The aim of this study was to put forward an explicit solution for the cubic equation of stress invariants for a definite condition and apply them to different failure theories. The solution had been obtained by considering the similarity between the cubic equation of invariants when the linear invariant is zero and the trigonometric equation of $\sin\theta$. The approach excludes the need for any trial and error method or any iterative method. The solution obtained was then applied to different failure theories to determine their modified descriptions and it has been observed that the failure criterion can be expressed in a simple and convenient form as $F(I_2,\theta)=0$. It has also been observed that the failure criterion can be expressed in a general form.

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