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Adomian's Decomposition Method to Functionally Graded Thermoelastic Materials with Power Law

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Abstract**—**This paper presents an iteration method for the numerical solutions of a one-dimensional problem of generalized thermoelasticity with one relaxation time under given initial and boundary conditions. The thermoelastic material with variable properties as a power functional graded has been considered. Adomian's decomposition techniques have been applied to the governing equations. The numerical results have been calculated by using the iterations method with a certain algorithm. The numerical results have been represented in figures, and the figures affirm that Adomian's decomposition method is a successful method for modeling thermoelastic problems. Moreover, the empirical parameter of the functional graded, and the lattice design parameter have significant effects on the temperature increment, the strain, the stress, the displacement.

Keywords**—**Adomian, Decomposition Method, Generalized Thermoelasticity, algorithm, empirical parameter, lattice design.

I. Introduction

IN the last decade, much attention has been devoted to the numerical methods which do not require discretization of time-space variables or to the linearization of the nonlinear equations [1]. Adomian constructed the decomposition method to solve the linear and nonlinear partial and ordinary differential equations [2]-[4]. This method leads to computable, accurate, approximately convergent solutions to linear and nonlinear partial and ordinary differential equations. The solution can be verified to any degree of approximation. Recently, Adomian decomposition approach has been applied to obtain formal solutions to a wide class of partial and ordinary differential equations [5]-[16]. Adomian solved mathematical models of the dynamic interaction of immune response with a population of viruses, bacteria, antigens or tumor cells which had been modeled as systems of nonlinear differential equations or delay-differential equations by the ADM [4].

Adomian's decomposition method (ADM) is to divide the given equation into linear and nonlinear parts of the equation, invert the highest-order derivative in both sides, calculate Adomian's polynomials, and find the successive terms of the series solution by recurrent relation [1], [13]. Several modifications have been done to the Adomian decomposition method by many researchers to improve the accuracy and to expand its applications [10], [12], [16]. Recently, the decomposition method has been used in fractional differential equations [17]-[19].

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II. Formulation of the Problem by Using Laplace Transform Techniques

An isotropic and thermo-elastic body in one-dimensional has been considered to fill the region Ψ which is defined by $\Psi=\{x:\ 0\le x<\infty\}$ where the body is initially at rest and has been loaded by a harmonic thermal wave and the surface traction free [20].

The displacement components for one-dimensional medium have the form:

$$u_x(x,t)=u(x,t),\ u_u=u_z=0 \tag{1}$$

The equation of motion is:

$$\left[\lambda(x)+2\mu(x)\right]\frac{\partial^2 e(x,t)}{\partial x^2}-\gamma(x)\frac{\partial^2\theta(x,t)}{\partial x^2}=\rho(x)\frac{\partial^2 e(x,t)}{\partial t^2} \tag{2}$$

The generalized equation of heat conduction has the form:

$$\frac{\partial}{\partial x}\left[K(x)\frac{\partial\theta(x,t)}{\partial x}\right]=\left[\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right]\left[\begin{array}{l}\rho(x)c_E(x)\theta(x,t)\\+\gamma(x)T_0\,e(x,t)\end{array}\right] \tag{3}$$

which gives:

$$K(x)\frac{\partial^2\theta(x,t)}{\partial x^2}+\frac{\partial K(x)}{\partial x}\left(\frac{\partial\theta(x,t)}{\partial x}\right)=\rho(x)c_E(x)\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)\theta(x,t)+\gamma(x)T_0\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)e(x,t) \tag{4}$$

In (2)-(4), $\theta=(T-T_0)$ is the temperature increment where T_0 is the reference temperature such that $|\theta|/T_0\ll 1$ reference temperature, and e is the cubical dilation given by:

$$e(x,t)=\frac{\partial u(x,t)}{\partial x}, \tag{5}$$

ρ is the density, λ and μ are Lame's constants, K is the thermal conductivity, γ is a material constant given by $\gamma=(3\lambda+2\mu)\alpha_T$,

α_t being the coefficient of linear thermal expansion, and c_E is the specific heat at constant strain, τ_0 is the relaxation time.

The constitutive relation takes the form:

$$\sigma(x,t)=\left[\lambda(x)+2\mu(x)\right]\frac{\partial u(x,t)}{\partial x}-\gamma(x)\theta(x,t). \tag{6}$$

We consider that all the material parameters depend on the position with a power-function as:

$$\begin{gathered} K(x)=K_0\left(1+\alpha_0 x\right)^n, \quad \lambda(x)=\lambda_0\left(1+\alpha_0 x\right)^n, \\ \mu(x)=\mu_0\left(1+\alpha_0 x\right)^n C_E(x)=C_{0E}\left(1+\alpha_0 x\right)^n, \\ \rho(x)=\rho_0\left(1+\alpha_0 x\right)^n, \quad \alpha_T(x)=\alpha_{0T}\left(1+\alpha_0 x\right)^n \end{gathered} \tag{7}$$

where α_0 is a small constant which is called empirical parameter and "n" is a positive parameter depends on the lattice design of the materials.

Substitute from (7) into (1)-(4) and (6), we get:

$$\left(\lambda_0+2\mu_0\right)\frac{\partial^2 e(x,t)}{\partial x^2}-\gamma_0\left(1+\alpha_0 x\right)^n\frac{\partial^2\theta(x,t)}{\partial x^2}=\rho_0\frac{\partial^2 e(x,t)}{\partial t^2} \tag{8}$$

where $\gamma_0=(3\lambda_0+2\mu_0)\alpha_{0T}$

The heat equation takes the form:

$$\begin{gathered} K_0\frac{\partial^2\theta(x,t)}{\partial x^2}+n K_0\alpha_0\left(1+\alpha_0 x\right)^{-1}\frac{\partial\theta(x,t)}{\partial x}= \\ \left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)\left(1+\alpha_0 x\right)^n\left[\rho_0 C_{0E}\theta(x,t)+\gamma_0 T_0 e(x,t)\right] \end{gathered} \tag{9}$$

For a small value of $\alpha_0 x \ll 1$, we have

$$\left(1+\alpha_0 x\right)^{-1}\approx\left(1-\alpha_0 x\right) \tag{10}$$

Thus, we obtain:

$$\begin{gathered} \frac{\partial^2\theta(x,t)}{\partial x^2}+n\alpha_0\left(1-\alpha_0 x\right)\left(\frac{\partial\theta(x,t)}{\partial x}\right)= \\ \left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)\left(1+\alpha_0 x\right)^n\left[\frac{\rho_0 C_{0E}}{K_0}\theta(x,t)+\frac{\gamma_0 T_0}{K_0}e(x,t)\right] \end{gathered} \tag{11}$$

and

$$\sigma=\left(\lambda_0+2\mu_0\right)\left(1+\alpha_0 x\right)^n e-\gamma_0\left(1+\alpha_0 x\right)^{2n}\theta \tag{12}$$

For simplicity, the following non-dimensional variables will be used [20]:

$$(x',u')=c_0\eta\,(x,u),\ (t',\tau_0')=c_0^2\eta\,(t,\tau_0),\ \theta'=\frac{\theta}{T_0},\ \alpha_0'=\frac{\alpha_0}{c_0\eta},$$

$$\sigma=\frac{\sigma'}{\left(\lambda_0+2\mu_0\right)}, \tag{13}$$

where

$$\eta=\frac{\rho_0 C_{0E}}{K_0}\ \text{and}\ c_0=\sqrt{\frac{\lambda_0+2\mu_0}{\rho_0}}.$$

The primes have been canceled for simplicity. Thus, we get the governing equations in the forms:

$$\frac{\partial^2 e}{\partial x^2}-\beta\left(1+\alpha_0 x\right)^n\frac{\partial^2\theta}{\partial x^2}=\frac{\partial^2 e(x,t)}{\partial t^2} \tag{14}$$

$$\begin{gathered} \frac{\partial^2\theta}{\partial x^2}+n\alpha_0\left(1-\alpha_0 x\right)\frac{\partial\theta}{\partial x}= \\ \left[\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right]\left(1+\alpha_0 x\right)^n\left[\theta+\varepsilon e\right] \end{gathered} \tag{15}$$

$$\sigma=\left(1+\alpha_0 x\right)^n e-\beta\left(1+\alpha_0 x\right)^{2n}\theta \tag{16}$$

where

$$\beta=\frac{\gamma_0 T_0}{\lambda_0+2\mu_0}\ \text{and}\ \varepsilon=\frac{\gamma_0}{\rho_0 C_{0E}}.$$

To solve the governing equations (14)-(16), we will apply the following boundary conditions:

The thermal boundary conditions are:

$$\theta(x,t)\big|_{x=0}=\theta^0\sin(\omega t),\quad \left.\frac{\partial\theta(x,t)}{\partial x}\right|_{x=0}=0 \tag{17}$$

and the mechanical boundary conditions are:

$$\sigma(x,t)\big|_{x=0}=0,\quad \left.\frac{\partial e(x,t)}{\partial x}\right|_{x=0}=0 \tag{18}$$

III. Adomian's Decomposition Method

Adomian's decomposition method usually defines the equation in an operator form by considering the highest-ordered derivative in the problem. We define the differential operator L in terms of the two derivatives contained in the problem [4], [5].

We consider (14) and (15) in the operator form as:

$$L_{xx}e(x,t)=L_{tt}e(x,t)+\beta\left(1+\alpha_0 x\right)^n L_{xx}\left[\theta(x,t)\right] \tag{19}$$

$$\begin{gathered} L_{xx}\theta=\left(1+\alpha_0 x\right)^n\left(L_t\theta+\tau_0 L_{tt}\theta\right)+ \\ \varepsilon\left(1+\alpha_0 x\right)^n\left(L_t e+\tau_0 L_{tt}e\right)-n\alpha_0\left(1-\alpha_0 x\right)L_x\theta \end{gathered} \tag{20}$$

where

$$L_t=\frac{\partial}{\partial t},\ L_{tt}=\frac{\partial^2}{\partial t^2},\ L_x=\frac{\partial}{\partial x},\ L_{xx}=\frac{\partial^2}{\partial x^2} \quad (21)$$

Assuming that the inverse of the operators L_x^{-1} and L_{xx}^{-1} exists and are taken as definite integrals as [1], [4], [5], [8]:

$$L_x^{-1}f(x)=\int_0^x f(x_1)\,dx_1,\ L_{xx}^{-1}f(x)=\int_0^x\int_0^{x_2} f(x_1)\,dx_1\,dx_2 \quad (22)$$

Thus, applying the inverse operator on both the sides of (19), (20) and using the boundary and the initial conditions in, we obtain:

$$e(x,t)=e(0,t)+\left.\frac{\partial e(x,t)}{\partial x}\right|_{x=0}+L_{xx}^{-1}\begin{bmatrix}L_{tt}e(x,t)+\\ \beta(1+\alpha_0 x)^n L_{xx}\left[\theta(x,t)\right]\end{bmatrix} \quad (23)$$

$$\theta(x,t)=\theta(0,t)+\left.\frac{\partial \theta(x,t)}{\partial x}\right|_{x=0}+L_{xx}^{-1}\begin{bmatrix}(1+\alpha_0 x)^n(L_t\theta+\tau_0 L_{tt}\theta)+\\ \varepsilon(1+\alpha_0 x)^n(L_t e+\tau_0 L_{tt}e)-\\ n\alpha_0(1-\alpha_0 x)L_x\theta\end{bmatrix} \quad (24)$$

Now, we will decompose the unknown functions $\theta(x,t)$ and $e(x,t)$ by a sum of components defined by the following series:

$$\theta(x,t)=\sum_{k=0}^{\infty}\theta_k(x,t)=\theta_0+\sum_{k=1}^{\infty}\theta_k(x,t) \quad (25)$$

and,

$$e(x,t)=\sum_{k=0}^{\infty}e_k(x,t)=e_0+\sum_{k=1}^{\infty}e_k(x,t) \quad (26)$$

The zero-components are defined by the terms that arise from the boundary conditions, which give:

$$e_0=e(0,t)+\left.\frac{\partial e(x,t)}{\partial x}\right|_{x=0} \quad (27)$$

and

$$\theta_0=\theta(0,t)+\left.\frac{\partial \theta(x,t)}{\partial x}\right|_{x=0} \quad (28)$$

Substituting from (27) and (28) in (25) and (26), we get

$$\sum_{k=0}^{\infty}e_k(x,t)=e(0,t)+\frac{\partial e(0,t)}{\partial x}+L_{xx}^{-1}\begin{bmatrix}L_{tt}\sum_{k=0}^{\infty}e_k(x,t)-\\ \beta(1+\alpha_0 x)^n L_{xx}\sum_{k=0}^{\infty}\theta_k(x,t)\end{bmatrix} \quad (29)$$

and,

$$\begin{aligned}\sum_{k=0}^{\infty}\theta_k(x,t)&=\theta(0,t)+\left.\frac{\partial\theta(x,t)}{\partial x}\right|_{x=0}+\\ &L_{xx}^{-1}(1+\alpha_0 x)\left[L_t\sum_{k=0}^{\infty}\theta_k(x,t)+\tau_o L_{tt}\sum_{k=0}^{\infty}\theta_k(x,t)\right]+\\ &L_{xx}^{-1}\left[(1+\alpha_0 x)^n\left(\varepsilon L_t\sum_{k=0}^{\infty}e_k(x,t)+\varepsilon\tau_o\sum_{k=0}^{\infty}e_k(x,t)\right)\right]+\\ &L_{xx}^{-1}\left[n\alpha_0(1-\alpha_0 x)L_x\sum_{k=0}^{\infty}\theta_k(x,t)\right]\end{aligned} \quad (30)$$

We obtain these components of $e_k(x,t)$ and $\theta_k(x,t)$ as recursive formulas:

$$e_{k+1}(x,t)=L_{xx}^{-1}\left[L_{tt}e_k(x,t)+\beta(1+\alpha_0 x)^n L_{xx}\theta_k(x,t)\right],\ k\ge 1 \quad (31)$$

and,

$$\begin{aligned}\theta_{k+1}(x,t)=L_{xx}^{-1}(1+\alpha_0 x)^n&\begin{bmatrix}L_t\theta_k(x,t)+\tau_o L_{tt}\theta_k(x,t)+\\ \varepsilon L_t e_k(x,t)+\varepsilon\tau_o L_{tt}e_k(x,t)\end{bmatrix}-\\ &L_{xx}^{-1}\left[n\alpha_0(1-\alpha_0 x)L_x\theta_k(x,t)\right],\ k\ge 1\end{aligned} \quad (32)$$

We calculate the zero components by using the boundary conditions in (27) and (28), hence, we obtain:

$$\theta_0=\theta^0\sin(\omega t) \quad (33)$$

and,

$$e_0=\beta\theta^0\sin(\omega t) \quad (34)$$

also, the first components take the forms:

$$\theta_1(x,t)=\frac{\omega(1+\varepsilon\beta)}{6}(3+\alpha_0 x)x^2\left[\cos(\omega t)-\tau_0\omega\sin(\omega t)\right] \quad (35)$$

and,

$$e_1(x,t)=-\frac{\beta\omega^2}{2}x^2\sin(\omega t) \quad (36)$$

The rest of the components of the iteration formulas, (31) and (32), have been calculated similarly by using MAPLE 17 program. Moreover, the decomposition series solutions, (29) and (30), are convergent very rapidly in real physical problems [9]-[11]. The convergence of the decomposition series has been investigated by several authors [10], [12], [16].

In an algorithmic form, the ADM can be expressed and implemented in linear coupling in thermoelasticity models as:

Algorithm

Set a suitable value for the tolerance $Tol=10^{-6}$ and let k be the iteration index,

Step1. Compute the initial approximations $\theta_0=\theta(0,t)$ and $e_0=e(0,t)$ by using (33) and (34).

Step2. Use the calculated values of θ_k and e_k to compute θ_{k+1} and

e_{k+1} from (31) and (32).

Step3. If $\max|\theta_{k+1}-\theta_k|<Tol$ and $\max|e_{k+1}-e_k|<Tol$ stop, otherwise continue and go back to step 2.

Step4. Calculate θ and e which complete the solution.

IV. The Numerical Results

The copper material has been chosen for the numerical evaluations, and the values' constants were taken as [20]:

$$K=386\,W/(mK)\ ,\ \alpha_T=1.78\times10^{-5}K^{-1}\ ,\ c_E=383.1\,J/(kg\,K)\ ,$$
$$\eta=8886.73\ s/m^2,\ T_0=293K,\ \mu=3.86\times10^{10}\ N/m^2,$$
$$\lambda=7.76\times10^{10}\ N/m^2,\ \rho=8954\ \text{kg/m}^3\ ,\ \tau_0=6.5\times10^{-14}\,\text{s}\,.$$

Thus, we get the following dimensionless parameters:

$$\tau_0=0.01\,,\alpha\text{=}0.0104443\,,\varepsilon=1.60862\,,\ \omega=\pi\ ,\ \theta^0=1.0\,.$$

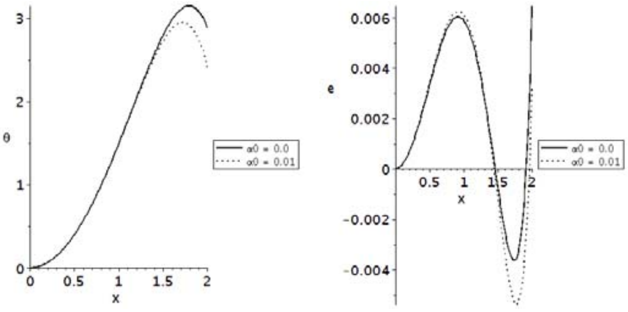
We calculate the numerical solutions when the non-dimensional value of the time is $t=2.0$, and the non-dimensional value of the distance is $0\le x\le 2.0$. According to the above algorithm, we stopped the calculation on the 7th component $e_7(x,t)$ and $\theta_7(x,t)$.

Fig. 1 represents the temperature increment, the strain, the stress, the displacement distributions when n = 1, and for various values of the parameter $\alpha_0=(0,0.01)$ to stand on the effect of the empirical parameter on all the studied functions. It is observed that the empirical parameter has significant effects on all the studied functions. In the presence of the constant α_0, the values of the temperature increment distribution curve decrease as the distance increases and this appears at the end of the curve, but the situation is reversed in the case of strain, stress, and displacement, where the absolute value of those functions increases as the coefficient is valued. We also find the effect of the coefficient on the values of the maximum points of the curves, as the coefficient α_0 works to increase the value of the maximum points while in the distribution of the temperature it is in inverse mode.

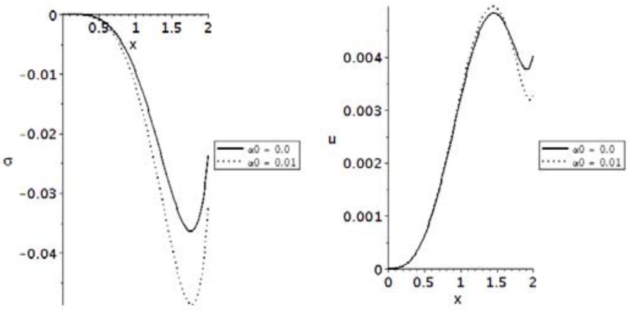
Fig. 2 has the same discerption of Fig. 1 but when n = 3 to stand on the effect of this parameter on all the studied functions. By comparing Fig. 1 with Fig. 3, it is noticed that when the value of the parameter n increases, the parameter α_0 got more impact on all the distributions and the difference between the two cases of $\alpha_0=0.0$ and $\alpha_0=0.01$ will be more visible.

Fig. 3 shows the significant effects of the parameter n on all the studied functions and that effects are more obvious on the strain, the stress, and the displacement.

Fig. 4 shows all the studied function on 3D-figures with a wide range of x $(0\le x\le 2.0)$ and wide range of the parameter n $(0\le n\le 3.0)$ when the time $t=2.0$. Those figures represent that the parameter n has significant effects on all the studied functions.

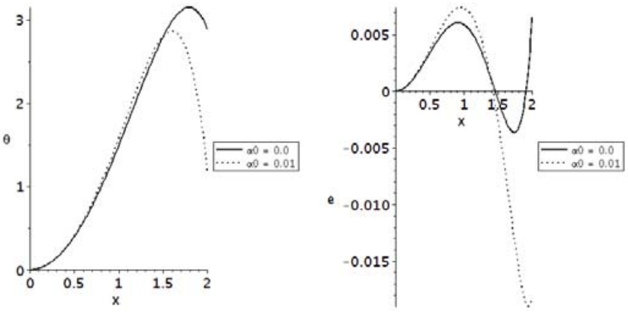


(a) The temperature increment distribution (b) The strain distribution

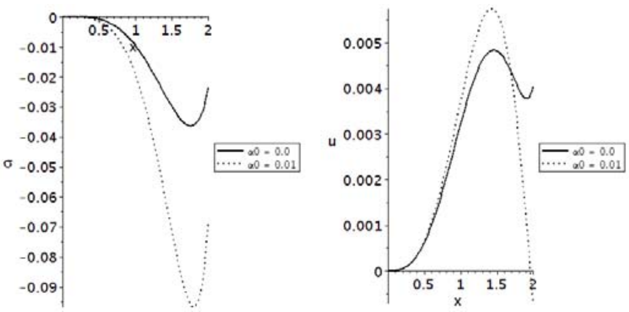


(c) The stress distribution (d) The displacement distribution

Fig. 1 The temperature increment, the strain, the stress, the displacement distributions when n = 1

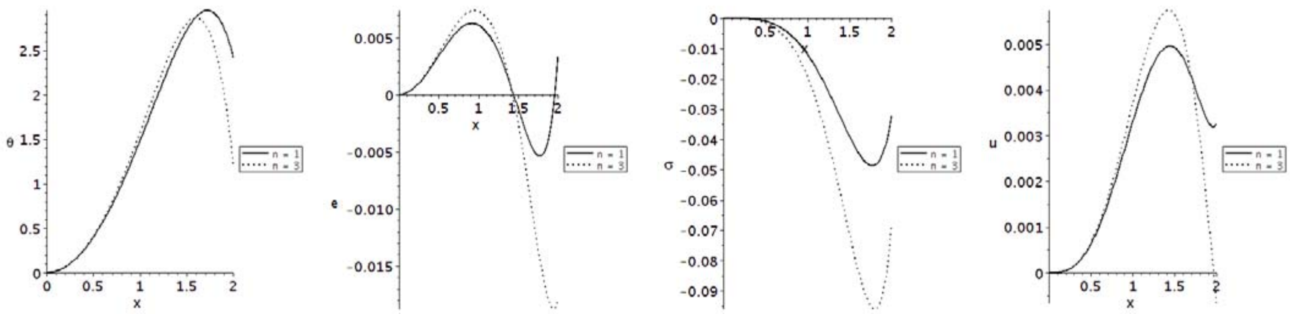


(a) The temperature increment distribution (b) The strain distribution



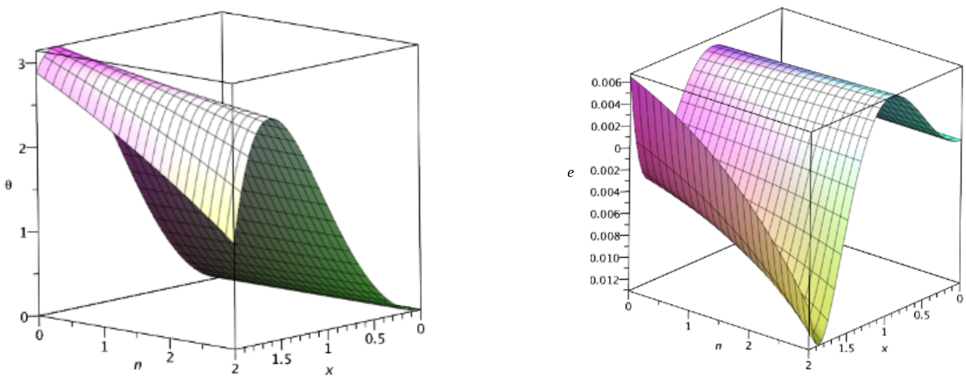
(c) The stress distribution (d) The displacement distribution

Fig. 2 The temperature increment, the strain, the stress, the displacement distributions when n = 3

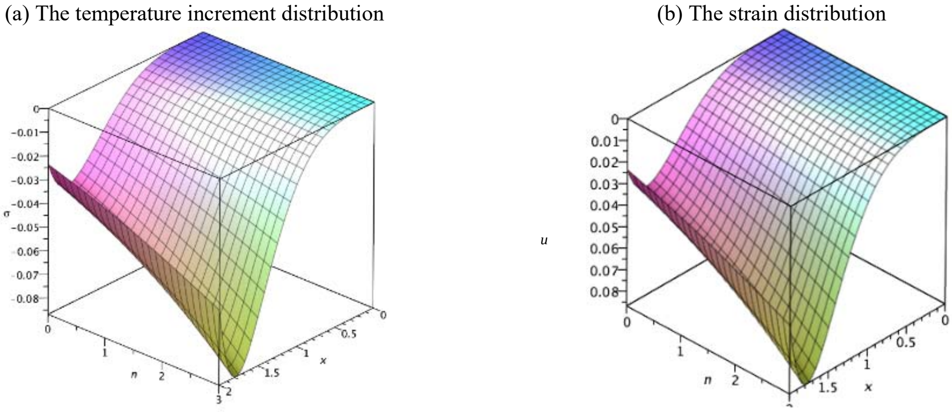


(a) The temperature increment distribution (b) The strain distribution (c) The stress distribution (d) The displacement distribution

Fig. 3 The temperature increment, the strain, the stress, the displacement distributions when $\alpha_0 = 0.01$



(a) The temperature increment distribution (b) The strain distribution



(c) The stress distribution (d) The displacement distribution

Fig. 4 The temperature increment, the strain, the stress, the displacement distributions when $t = 2.0$

V. Conclusions

This work introduced the numerical solutions of a one-dimensional problem of generalized thermoelasticity with power-functionally graded. The material properties have been dependent on the lattice shape. Adomian's decomposition method has been used. It is noted that, this method is a successful method with successful iteration to solve thermoelasticity models. The empirical parameter and the lattice design parameter have significant effects on the temperature increment, the strain, the stress, the displacement.

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