International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:13, No:9, 2019

Relation between Roots and Tangent Lines of Function in Fractional Dimensions: A Method for Optimization Problems

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***Abstract*—**In this paper, a basic schematic of fractional dimensional optimization problem is presented. As will be shown, a method is performed based on a relation between roots and tangent lines of function in fractional dimensions for an arbitrary initial point. It is shown that for each polynomial function with order N at least N tangent lines must be existed in fractional dimensions of $0 < \alpha < N+1$ which pass exactly through the all roots of the proposed function. Geometrical analysis of tangent lines in fractional dimensions is also presented to clarify more intuitively the proposed method. Results show that with an appropriate selection of fractional dimensions, we can directly find the roots. Method is presented for giving a different direction of optimization problems by the use of fractional dimensions.

***Keywords*—**Tangent line, fractional dimension, root, optimization problem.

I. Introduction

RECENT developments in all fields have led to a renewed interest in optimization methods and to find efficient numerical algorithms for solving the optimization problems. One of the most significant discussions on the optimization problems is Gradient-Based Method (GBM). Newton's, quasi Newton's, Broyden–Fletcher–Goldfarb–Shanno (BFGS) and limited-memory version of BFGS (L-BFGS) methods are important methods based on Gradient [1]. In recent years, there has been an increasing amount of literature on Newton iteration method and generalized newton methods [1]-[7]. The considerable note is when the above mentioned methods are applied to solve the equation f(x) = 0, it is needed to calculate the derivative of the function. Uses and example applications of GBM can be found in [1].

Recently, fractional (non-integer) derivatives and integrals play an important role in theory and applications. The idea of the fractional calculus was planted over 300 years ago in the letters between Leibniz and L'Hospital [8]-[10]. In 1823, Abel investigated the generalized tautochrone problem, and he was the pioneer to apply fractional calculus techniques in a physical problem [8]. Later, Liouville has applied fractional calculus to solve problems in potential theory [8]. Since then, the fractional calculus has triggered the attention of many researchers in all areas of sciences such as fluid mechanics, biology, physics and engineering [11]-[14]. Several attempts have been made to improve the fractional calculus in many different forms of fractional operators [15]-[20] and the solutions of fractional diffrentioal and integral equations such as homotopy perturbation method [21]. However, a method by the use of fractional dimensions can be required for optimization problems. In this regard, the present study provides a first demonstration that the fractional dimensions can be related to roots of function and with an orientation to be used in the optimization problems. To this end, it is created to give a schematic route for a direct solution of the optimization problem.

II. Geometrical Analysis

Idea is started with a question of what are the red-dash and blue-dash lines in Fig. 1. It can be observed that there is a line between an arbitrary initial point and root of the function. Generally, the purpose of this study is a solution to find the red-dash (blue-dash) line equation. As we will see, these lines are tangent lines of function in fractional dimensions. Then, it is possible to achieve a method for direct optimization method or finding the root directly instead of an iterative algorithm.

III. Fractional Derivative

Among several definitions of the fractional derivative [8]-[9], the following definitions are used:

Definition 1. Fractional arbitrary order derivative of the f function of order $0 < \alpha \le 1$ is defined by

$$D_a^{\alpha} f(x) = \frac{d}{dx}\int_a^x \frac{(x-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau)d\tau = \frac{d}{dt} I_a^{1-\alpha} f(x) \tag{1}$$

The Leibniz rule in the fractional calculus is:

$$D_a^{\alpha}[f(x)g(x)] = \sum_{k=0}^{\infty}\binom{\alpha}{k} D_a^{\alpha-k} f(x) D_a^k g(x) = \sum_{k=0}^{\infty}\binom{\alpha}{k} D_a^{\alpha-k} g(x) \tag{2}$$

In this research, (4) is used for fractional derivative of polynomial functions as a result of definition 1 when a=0.

$$D_x^{\alpha}(x^n) = \frac{d^{\alpha}}{dx^{\alpha}}(x^n) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha} \tag{3}$$

Fractional derivative for constant value is as following and for $\alpha = 1$ is equal to zero.

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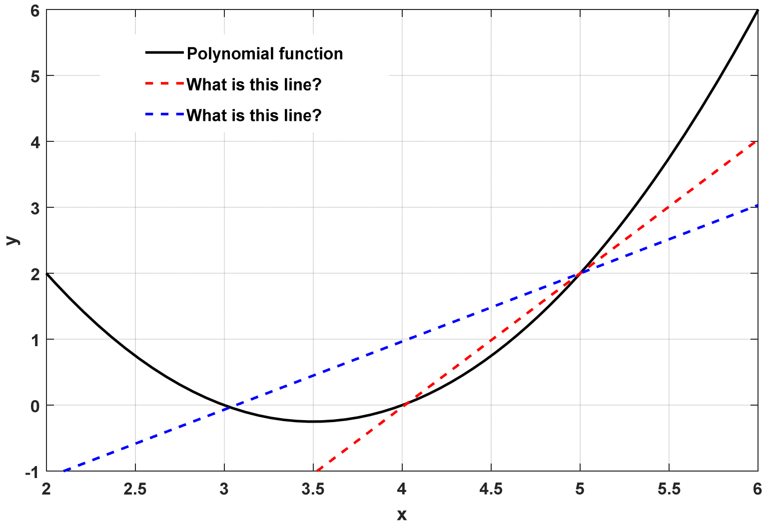

Fig. 1 Geometric schematic for lines crossing the roots

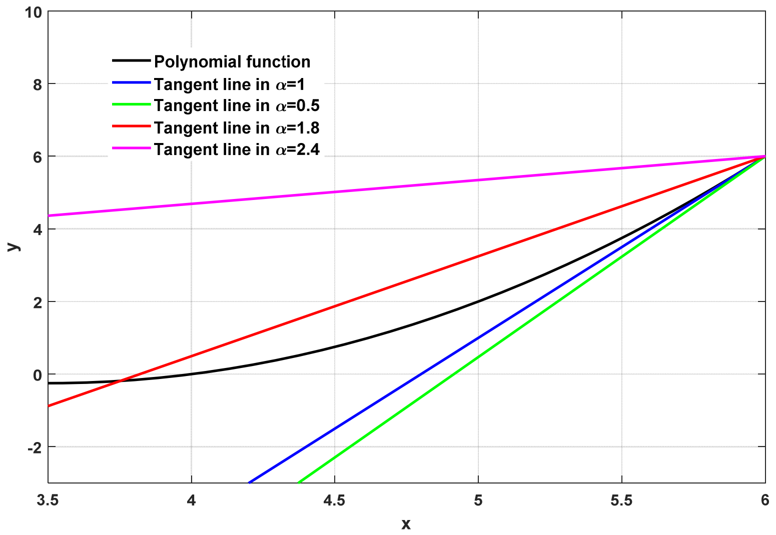

Fig. 2 Tangent lines in some fractional dimensions of function

$$D_x^{\alpha}C = \frac{d^{\alpha}}{dx^{\alpha}}(C) = C\frac{\Gamma(1)}{\Gamma(1-\alpha)}x^{-\alpha} \quad (4)$$

A definition for local fractional derivative is as follows:

Definition 2. A local fractional derivative of $f(x)$ of order α at $x=x_0$ is defined by

$$D^{\alpha}f(x_0) = \frac{d^{\alpha}f(x)}{dx^{\alpha}}\Big|_{x=x_0} = \lim_{x\to x_0}\frac{\Delta^{\alpha}(f(x)-f(x_0))}{(x-x_0)^{\alpha}} \quad (5)$$

and

$$\Delta^{\alpha}(f(x)-f(x_0)) \cong \Gamma(1+\alpha)\Delta(f(x)-f(x_0)) \quad (6)$$

First derivative gives the slope of tangent line of function for the proposed point. However, when the fractional derivative is applied for a certain point, it gives the slope of tangent line of function in fractional dimension at that point. Order of fractional derivative corresponds to fractional dimension. Fig. 2 shows the tangent lines in some fractional dimensions of function for a fixed initial point. It can be observed that the tangent lines in fractional dimensions can be varied around tangent line from first derivative of function (blue line). In other words, slopes of tangent lines in fractional dimensions can be decreased or increased in comparison with slope of tangent line from first derivative. In this configuration as shown in Fig. 2, tangent lines in dimensions of 0.5 and 2.4 have higher and lower slope values respectively while tangent line in dimension of 1, first derivative, is between them.

IV. Fractional Dimensional Method

Local and global solutions for finding the roots are presented in Theorems 1 & 2, respectively. It gives a relation between roots and tangent lines of the function in the fractional dimensions.

Theorem 1. Let f (R→R), a polynomial function of order N, be continuous and C^{N+1} differentiable function in an interval [a,b]. (x_1, y_1) is an initial point and $(x^*,0)$ is a root of function. Suppose there are fractional dimensions $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ which follow:

$$D^{\alpha_{\min}} f(x)\big|_{x=x_1} = \frac{(f(a)-f(x_1))}{(a-x_1)} < D^{\alpha} f(x)\big|_{x=x_1} < \frac{df(x)}{dx}\big|_{x=x_1} = D^{\alpha_{\max}} f(x)\big|_{x=x_1}$$

or

$$D^{\alpha_{\min}} f(x)\big|_{x=x_1} = \frac{df(x)}{dx}\big|_{x=x_1} < D^{\alpha} f(x)\big|_{x=x_1} < \frac{(f(b)-f(x_1))}{(b-x_1)} = D^{\alpha_{\max}} f(x)\big|_{x=x_1},$$

thus, there is a tangent line of the function in fractional dimension of $0 < \alpha \le N+1$, which exactly passes through the root of the proposed function.

$$\forall x_1, f(x_1), x^*, f(x^*) = 0 \;\exists \alpha, D^{\alpha} f(x)\big|_{x=x_1} \Rightarrow x^* = x_1 - \frac{f(x_1)}{D^{\alpha} f(x)\big|_{x=x_1}} \quad (7)$$

Proof. Line equation of function in α dimension with an initial point $(x_1, f(x_1))$ is equal to $f(x) - f(x_1) = D^{\alpha} f(x)\big|_{x=x_1}(x - x_1)$. Suppose that this line for fractional dimension of α collides with root:

$$f(x^*) - f(x_1) = D^{\alpha} f(x)\big|_{x=x_1}(x^* - x_1) \Rightarrow 0 - f(x_1) = D^{\alpha} f(x)\big|_{x=x_1}(x^* - x_1)$$
$$\Rightarrow x^* = x_1 - \frac{f(x_1)}{D^{\alpha} f(x)\big|_{x=x_1}}$$

For Generalization of the method have:

Theorem 2. Let f (R→R) be a continuous and C^{N+1} differentiable function of order N in an interval of (-∞,+∞) with maximum N roots where x_k^* is a set of roots of the function. Suppose $D^{\alpha} f(x)\big|_{x=x_1}$ can be as follows:

$$0 \le \tan^{-1}(D^{\alpha} f(x)\big|_{x=x_1}) \le \pi$$

Thus, there are minimum N tangent lines of the function in fractional dimensions of $0 < \alpha \le N+1$, which exactly cross whole roots of the objective function.

$$\forall x_1, f(x_1), x_k^*, f(x_k^*) = 0 \;\exists \alpha_k, D^{\alpha_k} f(x)\big|_{x=x_1} \Rightarrow x_k^* = x_1 - \frac{f(x_1)}{D^{\alpha_k} f(x)\big|_{x=x_1}} \quad (8)$$

Proof. Line equations for a set of fractional dimensions, α_k, with an initial point $(x_1, f(x_1))$ is equal to $f(x) - f(x_1) = D^{\alpha_k} f(x)\big|_{x=x_1}(x - x_1)$. Suppose these lines for fractional dimensions of α_k cross the roots:

$$f(x_k^*) - f(x_1) = D^{\alpha} f(x)\big|_{x=x_1}(x_k^* - x_1) \Rightarrow 0 - f(x_1) = D^{\alpha_k} f(x)\big|_{x=x_1}(x_k^* - x_1)$$
$$\Rightarrow x_k^* = x_1 - \frac{f(x_1)}{D^{\alpha_k} f(x)\big|_{x=x_1}}$$

A. Calculation of Fractional Dimensions for Roots

Two initial points (x_1, y_1) and (x_2, y_2) are considered for calculation of fractional dimensions. In this regard, it is considered two different tangent lines in fractional dimensions that pass through the roots, as depicted in Fig. 4. Two lines are crossing the root of function. Thus, root based on theorem 1 can be written as:

$$x^* = x_1 - \frac{f(x_1)}{D^{\alpha_1} f(x)\big|_{x=x_1}} = x_2 - \frac{f(x_2)}{D^{\alpha_2} f(x)\big|_{x=x_2}} \quad (9)$$

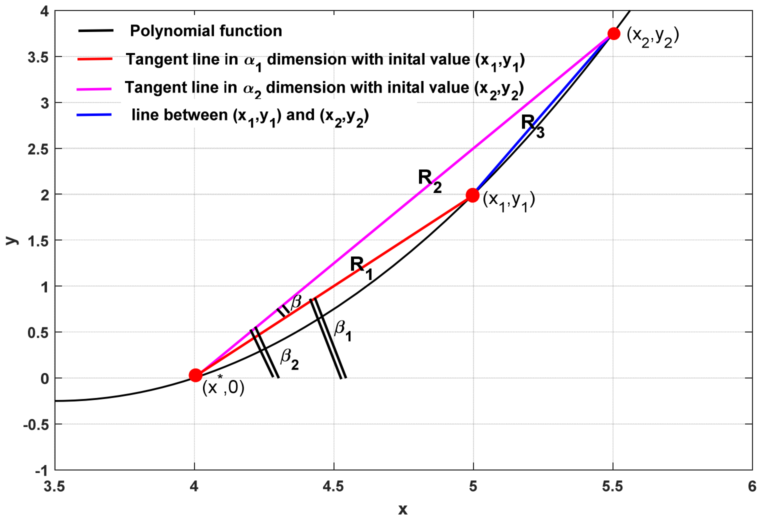

Fig. 3 Geometric configuration for calculation of appropriate fractional dimensions for root

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:13, No:9, 2019

Then, if see the problem in vector analysis, the relation between all lines can be expressed by (10):

$$R_3^2 = R_1^2 + R_2^2 - 2R_1R_2\cos\beta \tag{10}$$

where R_1 and R_2 are a distance from root to (x_1, y_1) & root to (x_2, y_2) and are equal to $\sqrt{(x_1-x^*)^2+(y_1-0)^2}$ & $\sqrt{(x_2-x^*)^2+(y_2-0)^2}$ respectively, while R_3 is equal to $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ and is a distance between (x_1, y_1) and (x_2, y_2), see Fig. 3. β is a difference angle between angles of R2 (β_2), and R1 lines $(\beta_1), \beta=\beta_2-\beta_1$. where $\beta_1=\tan^{-1}(D^{\alpha_1}f(x)\big|_{x=x_1})$ and $\beta_2=\tan^{-1}(D^{\alpha_2}f(x)\big|_{x=x_2})$ are as presented in Fig. 3.

With a substitution of all relations in (10), we have:

$$\begin{aligned}(x_1-x_2)^2+(y_1-y_2)^2 &= (x_1-x^*)^2+(y_1)^2+(x_2-x^*)^2+(y_2)^2 \\ &-2\sqrt{(x_1-x^*)^2+(y_1)^2}\sqrt{(x_2-x^*)^2+(y_2-0)^2}\cos\beta\end{aligned} \tag{11}$$

Based on (9), two fractional derivatives of function are related to each other and can be presented as:

$$D^{\alpha_2}f(x)\big|_{x=x_2} = \frac{f(x_2)}{x_2-x_1+\dfrac{f(x_1)}{D^{\alpha_1}f(x)\big|_{x=x_1}}} \tag{12}$$

With substituting of (9) and (12) to (11), the general relation for fractional dimension can be presented as (13). It can be described by an equation like $f(\alpha)=0$ where the solutions of $f(\alpha)=0$ give the whole dimensions that satisfy (8) and consequently give the all roots.

It can be observed from (13) that fractional dimensions around the one point have a nonlinear oscillating behavior. It is due to dependency to cosine function.

$$\begin{aligned}&(x_1-x_2)^2+(y_1-y_2)^2 = \\ &\left(\begin{aligned}&(x_1-x_1-\frac{f(x_1)}{D^{\alpha_1}f(x)\big|_{x=x_1}})^2+(y_1)^2+(x_2-x_1-\frac{f(x_1)}{D^{\alpha_1}f(x)\big|_{x=x_1}})^2+(y_2)^2 \\ &-2\left(\begin{aligned}&\sqrt{(x_1-x_1-\frac{f(x_1)}{D^{\alpha_1}f(x)\big|_{x=x_1}})^2+(y_1)^2}\sqrt{(x_2-x_1-\frac{f(x_1)}{D^{\alpha_1}f(x)\big|_{x=x_1}})^2+(y_2)^2} \\ &\times\cos\left(\tan^{-1}(\frac{f(x_2)}{x_2-x_1+\frac{f(x_1)}{D^{\alpha_1}f(x)\big|_{x=x_1}}})-\tan^{-1}(D^{\alpha_1}f(x)\big|_{x=x_1})\right)\end{aligned}\right)\end{aligned}\right)\end{aligned} \tag{13}$$

For instance, tangent lines in fractional dimensions for polynomial function x^2-7x+12=0 with roots of x=3 and x=4 are shown in Fig. 4. Tangent lines in fractional dimensions $\alpha=0.375$ and $\alpha=2$ pass through the root x=4. Also tangent line for $\alpha=2.228$ crosses the other root at x=3. It can be seen that the slope variations of tangent lines of function in fractional dimension is nonlinear with oscillating variations.

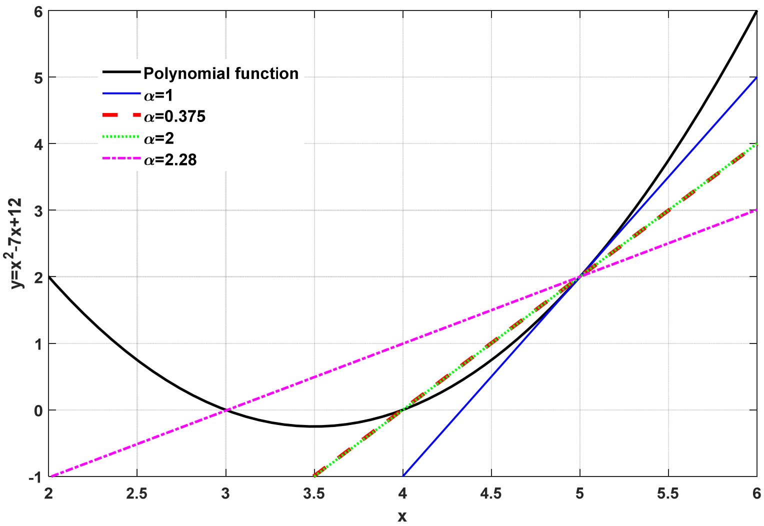

Fig. 4 Tangent lines of fractional dimensions pass through the roots

B. Optimization Problem

General form of optimization problem without constraint can be described as following for performing the best guess of optimal point:

$$\underset{x}{Min}\, g(x) \quad \text{or} \quad f(x)=g'(x)=0 \tag{14}$$

Suppose g(x) is locally minimized at x*. Now the equivalent problem presented above with initial point $(x_1, f(x_1))$ is as follows:

$$\underset{x}{Min}\, g(x) \equiv f(x^*) \quad where \quad x^* = x_1 - \frac{f(x_1)}{D^{\alpha} f(x)\big|_{x=x_1}} \tag{15}$$

If the objective function has several extremum points at x_k^*, then a global solution of problem can be replaced by:

$$\underset{x}{Min}\, g(x) \equiv f(x_k^*) \quad where \quad x_k^* = x_1 - \frac{f(x_1)}{D^{\alpha_k} f(x)\big|_{x=x_1}} \tag{16}$$

Now the main challenging for schematic of optimization problem is to find the appropriate fractional dimensions and what is $D^{\alpha_k} f(x)\big|_{x=x_1}$ which satisfy (13). However, further analysis of (13) is out of this paper and can be considered as a future work.

V. Conclusion

In this paper, the basic theoretical framework required to generate a fractional dimensional optimization method is introduced. It is shown that there are some tangent lines of function in fractional dimensions which pass through the roots. A general relation for calculation of fractional dimensions is also presented. It shows that the fractional dimensions variation around one point has an oscillating behavior as a cosine function. Results show that with appropriate fractional dimensions can directly find the roots.

Acknowledgment

The author likes to thank Isfahan Mathematics House in Iran for their supports.

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