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A Unification and Relativistic Correction for Boltzmann's Law

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Abstract**—**The distribution of velocities of particles in plasma is a well understood discipline of plasma physics. Boltzmann's law and the Maxwell-Boltzmann distribution describe the distribution of velocity of a particle in plasma as a function of mass and temperature. Particles with the same mass tend to have the same velocity. By expressing the same law in terms of energy alone, the author obtains a distribution independent of mass. In summary, for particles in plasma, the energies tend to equalize, independent of the masses of the individual particles. For high-energy plasma, the original law predicts velocities greater than the speed of light. If one uses Einstein's formula for energy ($E=mc^2$), then a relativistic correction is not required.

Keywords**—**Cosmology, EMP, Euclidean, plasma physics, relativity.

I. Introduction

THE Maxwell-Boltzmann distribution function describes the distribution of velocity of a particle in plasma as a function of mass and temperature (see [1]-[3]). This paper expresses the same law in terms of energy alone, independent of mass.

For particles with smaller mass, the Maxwell-Boltzmann distribution predicts that the velocities will be much higher than for particles with lower mass. The velocity increase is inversely proportional to the inverse of the square root of the mass decrease. For electrons, a factor of 42.85 times faster than a proton (see Fig. 1). This effect causes the electromagnetic pulse (EMP) associated with nuclear blasts.

When an atomic bomb is detonated, the electrons separate from the protons, producing enormous electromagnetic fields, sufficient to knock out electric power and damage electronic equipment.

When the Maxwell-Boltzmann distribution is expressed in terms of energy, mass disappears from the resulting probability density functions and cumulative distribution functions.

II. Results

Expressed in the energy domain, the probability density function of energy, E, for particles in plasma becomes

$$f_E(E)=\sqrt{\frac{4E}{\pi(kT)^3}}e^{\frac{-E}{kT}} \tag{1}$$

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where E is kinetic energy, v is velocity, k is Boltzmann's constant, m is mass, and T is temperature.

The corresponding cumulative distribution function is

$$F_E(E)=erf(\sqrt{\frac{E}{kT}})-\sqrt{\frac{4E}{\pi kT}}e^{\frac{-E}{kT}} \tag{2}$$

where *erf* is the normal (Gaussian) error function,

$$erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du \tag{3}$$

This same formulation applies to all of the species of the plasma, independent of a particle's mass. It therefore provides a unification for the Maxwell-Boltzmann distribution.

III. Analysis

According to definition, the cumulative distribution function of a random variable, X is defined in terms of probability (see [4], [5])

$$F_X(x)=P(X<x) \tag{4}$$

where $P($event$)$ is the probability of an event.

Expressed verbally, for a real number x, the cumulative distribution function $F_X(x)$ is the probability that the random variable, X, is less than x.

The probability density function is the derivative of the cumulative distribution function (see [4], [5])

$$f_X(x)=\frac{d\,F_X(x)}{dx} \tag{5}$$

Turning our attention to the Boltzmann distribution [1], the probability density function of any of the species of the gas is given by

$$f_B(v)=\sqrt{\left(\frac{m}{2\pi kT}\right)^3}4\pi v^2 e^{\frac{-mv^2}{2kT}} \tag{6}$$

where v is velocity, k is Boltzmann's constant, m is mass, and T is temperature.

This classical formulation of the Maxwell-Boltzmann distribution yields a different probability density function for each of the species of the plasma, according to mass.

From the same reference, cumulative distribution function is computed

$$\begin{aligned} F_B(v) &= P(V < v) \\ &= erf\left(\frac{v}{\sqrt{\frac{2kT}{m}}}\right) - \sqrt{\frac{2}{\pi}}\frac{v e^{\frac{-mv^2}{2kT}}}{\sqrt{\frac{kT}{m}}} \end{aligned} \tag{7}$$

where *P(*event*)* is the probability of an event, and *erf* () is the normal error function.

The above function provides a finite probability that the velocity will exceed the speed of light, *c*. This cannot be realized in the physical universe. For extremely energetic plasmas, this probability becomes significant, and this classical formulation for the Maxwell-Boltzmann distribution is useless. To resolve this issue, the Maxwell-Boltzmann distribution can be expressed in terms of energy instead of velocity.

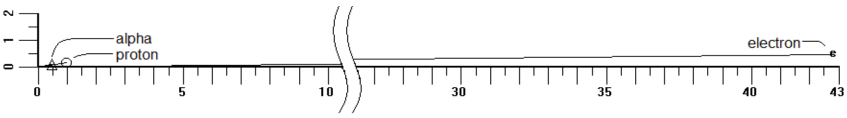

Fig. 1 Relative Velocities of Particles in Plasma

Before Einstein (in Boltzmann's era), kinetic energy was always computed

$$E = \frac{mv^2}{2} \tag{8}$$

Solving for velocity,

$$v = \sqrt{\frac{2E}{m}} \tag{9}$$

Substituting into (7),

$$F_E(E) = erf\left(\frac{\sqrt{\frac{2E}{m}}}{\sqrt{\frac{2kT}{m}}}\right) - \sqrt{\frac{2}{\pi}}\frac{\sqrt{\frac{2E}{m}}e^{\frac{-E}{kT}}}{\sqrt{\frac{kT}{m}}} \tag{10}$$

Simplifying,

$$F_E(E) = erf\left(\sqrt{\frac{E}{kT}}\right) - \sqrt{\frac{4E}{\pi kT}}e^{\frac{-E}{kT}} \tag{11}$$

where *k* is Boltzmann's constant, *E* is kinetic energy, and *T* is temperature.

Astonishingly, the values containing mass all cancel. This is clearly a more unified and robust approach for characterizing gas or plasma. It says that the kinetic energy, *E*, is essentially mass independent.

Reiterating, this equation applies to all particles of the gas or plasma, independent of a particle's mass. The distribution of energy is therefore strictly a function of temperature.

Moving on, the probability density function is the derivative of the cumulative distribution function [4].

$$\begin{aligned} f_E(E) &= \frac{d\,F_E(E)}{dE} \\ &= \frac{d\left(erf\left(\sqrt{\frac{E}{kT}}\right) - \frac{2}{\sqrt{\pi}}\sqrt{\frac{E}{kT}}e^{\frac{-E}{kT}}\right)}{dE} \end{aligned} \tag{12}$$

where *erf(x)* is the normal distribution error function.

Performing the differentiation,

$$\begin{aligned} f_E(E) = &\frac{2}{\sqrt{\pi}}e^{\frac{-E}{kT}}\frac{1}{\sqrt{kT}}\frac{1}{2\sqrt{E}} \quad - \\ &\frac{2}{\sqrt{\pi}}\frac{e^{\frac{-E}{kT}}}{\sqrt{kT}}\frac{1}{2\sqrt{E}} \quad + \\ &+\frac{2}{\sqrt{\pi}}\sqrt{\frac{E}{kT}}e^{\frac{-E}{kT}}\frac{1}{kT} \end{aligned} \tag{13}$$

The first two terms cancel, leaving

$$f_E(E) = \sqrt{\frac{4E}{\pi(kT)^3}}e^{\frac{-E}{kT}} \tag{14}$$

This concludes the derivation of the density and distribution functions as a function of energy and temperature.

A further examination of (11) will reveal that energy increases in direct proportion to temperature. Let

$$\Psi = \frac{E}{kT} \tag{15}$$

then the distribution of Ψ is dimensionless.

$$F_\Psi(\psi) = erf(\sqrt{\psi}) - \sqrt{\psi}e^{-\psi} \tag{16}$$

For any particular temperature, the value of Ψ increases monotonously with values of *E*. Therefore

$$P(E < e) = P(\Psi < \psi) \tag{17}$$

From the definition of cumulative distribution functions, it follows that

$$\begin{aligned} F_E(E) &= F_\Psi(\Psi) \\ &= F_\Psi(\Psi kT) \end{aligned} \tag{18}$$

Using this representation for simulations, energy can be generated using a single distribution function.

IV. Relativistic Considerations

If Boltzmann had expressed his law in terms of energy instead of mass and velocity, relativistic correction would not be required. Keep in mind that Boltzmann died just after Einstein published his first paper on relativity. In Boltzmann's era, kinetic energy was computed using the classical Newtonian formula,

$$E = \frac{mv^2}{2} \tag{19}$$

Einstein concluded that mass increases with energy.

$$E_T = mc^2 \tag{20}$$

where E_T is total energy.

$$E_T = E + m_0 c^2 \tag{21}$$

where E_T is total energy, E is kinetic energy, m is mass, and m_0 is rest mass.

Solving for kinetic energy,

$$E = (m - m_0)\, c^2 \tag{22}$$

Mass increases with velocity,

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{23}$$

where m_0 is rest mass, and v is velocity.

Solving for energy as a function of velocity,

$$E = \left(\frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0\right) c^2 \tag{24}$$

The term m_0 can be factored out from the right side.

$$E = \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1\right) m_0\, c^2 \tag{25}$$

This formula agrees with the classical Newtonian formula to within 0.1% for velocities less than 10^6 meters per second.

Unfortunately, the above equation is not computationally useful. For smaller values of velocity, it involves computing the difference between two numbers which are very close to 1. The precision of the computation is compromised.

To obtain good computational results for kinetic energy requires expressing the above equation as a Taylor-series expansion [6].

$$\begin{aligned} E = m_0 v^2 \Big(&\frac{1}{2} + \frac{3}{2^2}\frac{v^2}{2!c^2} + \ldots \\ &+ \frac{1 \cdot 3 \ldots (2n-1) v^{2n-2}}{2^n n! c^{2n-2}}\Big) + \ldots \end{aligned} \tag{26}$$

This series converges very rapidly. The first term is recognizable as the classical formula for kinetic energy, $m_0 v^2/2$. For each term, the next term decreases by a factor of

$$\frac{v^2}{c^2}\left(n - \frac{1}{2n}\right)\frac{1}{n!} \tag{27}$$

For velocities less than 10% of the speed light, the first term is less than 1%. For velocities less than 1% of the speed of light, it is less than 0.01%.

To solve for velocity as a function of energy, first divide both sides of (25) by $m_0 c^2$.

$$\frac{E}{m_0\, c^2} = \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1\right) \tag{28}$$

Adding 1 to both sides,

$$\frac{E}{m_0\, c^2} + 1 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{29}$$

Squaring and inverting both sides,

$$1 - \frac{v^2}{c^2} = \frac{1}{\left(\frac{E}{m_0\, c^2} + 1\right)^2} \tag{30}$$

Solving for v/c squared,

$$\frac{v^2}{c^2} = 1 - \frac{1}{\left(\frac{E}{m_0\, c^2} + 1\right)^2} \tag{31}$$

The formula on the right is always less than 1; therefore, the ratio on the left must also be less than 1. In conclusion, the resulting velocity will always be less than the speed of light. The conundrum with Maxwell-Boltzmann distribution is therefore repaired.

V. Conclusion

These results have implications for cosmology, particularly near the time of creation. Interacting particles will tend to have the same energy.

Plasma is a state of matter where electrons have too much energy to stay confined to atoms. Unless the geometry early universe is confined to a small finite volume, electrons in the early universe would simply dissipate into empty space. The remaining positively charged protons and helium nuclei repel each other, and would therefore follow soon after. An unconfined universe would have no atoms. No stars. No galaxies. This will occur unless the plasma is confined inside a finite volume. This concludes that the size of the universe must be finite, and therefore non-Euclidean. Solving this problem is, of course, a subject for a different paper.

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