International Journal of Business, Human and Social Sciences
ISSN: 2517-9411
Vol:13, No:5, 2019

Developing Proof Demonstration Skills in Teaching Mathematics in the Secondary School

M. Rodionov, Z. Dedovets

Abstract**—**The article describes the theoretical concept of teaching secondary school students proof demonstration skills in mathematics. It describes in detail different levels of mastery of the concept of proof-which correspond to Piaget's idea of there being three distinct and progressively more complex stages in the development of human reflection. Lessons for each level contain a specific combination of the visual-figurative components and deductive reasoning. It is vital at the transition point between levels to carefully and rigorously recalibrate teaching to reflect the development of more complex reflective understanding. This can apply even within the same age range, since students will develop at different speeds and to different potential. The authors argue that this requires an aware and adaptive approach to lessons to reflect this complexity and variation. The authors also contend that effective teaching which enables students to properly understand the implementation of proof arguments must develop specific competences. These are: understanding of the importance of completeness and generality in making a valid argument; being task focused; having an internalised locus of control and being flexible in approach and evaluation. These criteria must be correlated with the systematic application of corresponding methodologies which are best likely to achieve success. The particular pedagogical decisions which are made to deliver this objective are illustrated by concrete examples from the existing secondary school mathematics courses. The proposed theoretical concept formed the basis of the development of methodological materials which have been tested in 47 secondary schools.

Keywords**—**Education, teaching of mathematics, proof, deductive reasoning, secondary school.

I. Introduction

THE role of proofs in school mathematical education is, according to most methodologists, the most significant contribution of mathematics to human culture. Accordingly, it is crucial that students realise and are motivated by this importance. Students' intellectual and motivational potential is optimally achieved by the consistent application of the correct level of logical rigour and reasoning. This is first expressed in the character of the definitions of concepts and proofs of theorems, which obviously should be logically rigorous and, if possible, be based on the available intuitive visual representations by the students themselves. The optimal correlation of rigour and visual evidence in teaching school mathematics has not yet been achieved. At the same time, the dominant "regulators" of rigour are not always consistent with each other. Methodologists and teachers must consider the role of the concept or fact being studied when considering course design. Factors to be borne in mind include the importance of the concept or fact (the more important, the greater the rigour required); student age (the older the student the higher the rigour); the specificity of the material (geometry requires more rigour than algebra), and the level of students' knowledge and skills, which will determine acceptable understanding of concepts such as proof. In the present conditions of school teaching, the question "to prove or not to prove?" is passively considered to be the prerogative of the author of a particular textbook and or the teacher who uses it. At the same time, when considering motivation, the automatic transfer of the rigour of proofs from mathematical science to general school practice is significantly connected with the clarity, authenticity and conveyed importance of what is being studied and on student awareness and preoccupation. Accordingly, it is more expedient to talk not only about the degree of logical, mathematical rigour, but also about the overall educational standards of rigour and the generalised and internalised expectation for justified explanation and an eagerness to continuously learn [3], [9]. This approach allows us to define several levels of rigour which reflect the relationship between the proof itself and the one who seeks it-assimilating this not only into any particular learning situation, but also creating the conditions and means for successfully moving the students along this "ladder of levels".

M. Rodionov is with the Department of Mathematics and Computer Science Teaching, Penza State University, Penza, Russia (e-mail: do7tor@gmail.com).

Z. Dedovets is with the Department of School of Education, The University of West Indies, Trinidad and Tobago (e-mail: Zhanna.Dedovets@sta.uwi.edu).

II. Mathematical Rigour in Proofs

The setting of the levels of rigour for proof and the corresponding teaching stages should reflect student awareness [5]. In this respect, it is possible to distinguish three basic levels of rigour of the proof, corresponding to the three stages of development of reflection in students [1], [4], [6], [8].

1. The level of "common sense".
2. The level of "mental experience".
3. The level of "thought of thoughts".

At the first level, the very term "prove" for the student means that something needs to be done to verify and confirm the existence of an object or phenomenon the student cannot otherwise explain. The student will not be able to explain how this happens – to reproduce the course of thoughts involved in obtaining this result. The explanation of one's own actions in this case is replaced by ascertainment of the arithmetic operations performed, of measuring procedures or elementary geometric constructions, or an artificial construction of sequence. For example, when answering the question: How can this problem be solved? A fairly typical response for a student is either a simple reproduction of each stage of the solution, or in the case of an unfamiliar situation, a blockbuster utilisation of all possible known actions to obtain an intuitively predictable result. These features of mathematical activity are also manifested in the implementation of one-step deductive reasoning, which directly results from the need to use inductively derived previously common rules and definitions. This is manifested, in particular, in "reinforcement", and sometimes in the replacement of the rule by experimental practice or indicating that other possible alternatives are incompatible according to common sense.

The underlying reason for this is the domination of visual-figurative thinking, the logic of which is subject to its own specific laws, different from the laws of functioning of verbal logic. In particular, the "grasp" of the situation under consideration occurs here without detailed analysis, often on the basis of random (insufficient or superfluous) relationships. Conclusions cannot be verbally formulated, and the thought process itself is a flurry of images (plane images, space images, symbolic images, graphic images), as a result of which the solution arises suddenly, in the form of a kind of mental "picture". Thus, for example, the proof of the theorem sum of the angles of a triangle at (Fig. 1) can be represented as the displacement of a straight line *AC* parallel to itself until the moment when this line passes through point *B*. Moreover, all the angles of triangle *ABC* seem to be "will gather together" at this point, making up a 180 degree angle.

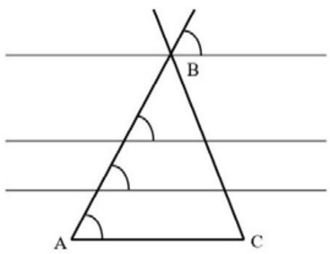

Fig. 1 The isosceles triangle *ABC* with base *AC* [8]

At the entry level, students use visual, intuitive and experience-based approaches to establish the correspondence between the intermediate and final results of this proof. The evidence is of the nature of the sequence of acts that pass from one kind of image to another. The need for logical reasoning at this stage is not realized by the students. At the next level, the student can understand the relationships established in the process of mathematical activity. In other words, the student is interested in proof as "from outside". Personal interest in the proof is not expressed by attempting to reveal the meaning of the logical steps being implemented, but in a desire to convince the teacher and peers that the personally derived method of solution is correct and useful.

The considered level is transitive in the sense that schoolchildren here are not limited to reproducing the assimilated patterns of reasoning, but are making attempts to combine them, adapting to new goals and circumstances. Here, students can already operate with sentences that have the character of axioms, and suggestions whose evidence is explicitly based on other reasoning. However, they are not yet fully aware of the logical foundations of these proposals. Such awareness can only be realized by establishing a closer connection with the visual-figurative components of mathematical evidence. The images themselves in this case serve both as direct arguments in justifying the decision, and as a means to stimulate deductive reasoning and interpret logical conclusions. Accordingly, at this stage of mastery of logical reasoning, it is permissible to use "not completely mathematical" arguments of varying degrees of depth and completeness, as well as omissions of some steps that students would find intuitively obvious. The statements, which initially belong to the number of obvious ones, as well as the claims, whose justifications require too delicate arguments, can in general be considered without proof.

The triangle ABC on Fig. 2 helps the students to find the sum of the angles of the triangle.

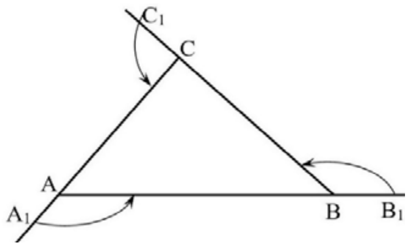

Fig. 2 The triangle *ABC* [8]

They must perform a complete rotation around this triangle to demonstrate that the sum of these angles equal to 360°. Such an approach is more likely achieve motivation than traditional teaching methods [8].

The highest level of rigour at school level presupposes the possibility of students understanding not only the existence of significant links and relationships that are singled out in the process of thinking activity in support of certain statements, but also the nature of this activity itself. At this level, students can not only comment on the sequence of steps taken in the justifications, but also go outside the scope of this "justification field", answering questions such as from what general knowledge can a particular proposal be deduced, or what suggestions can follow from this premise? This reflects a substantive reflection of the proof, its logical structure and the rules of inference used in it-demonstrating the students' mastery of the simplest meaningful schemes of reasoning, and adequate self-control over the course of the implementation of deductive procedures [8].

The proofs themselves begin to be presented not as means of persuasion of the validity of singular mathematical facts and their properties, but in the form of a method of justification, a logical ordering of the corresponding system of these facts. In other words, students begin to realize the need to detect a certain "similarity", the generality of techniques, expressed in the formulation of generalized methods of proof, which can be transferred to a relatively large class of mathematical regularities.

The potential deployability of the justifications for this level does not mean a complete replacement of the visually intuitive forms of such justifications with their logical equivalents, but implies their mutual enrichment. Such a restructuring, in the first place, means changing the role of the visual-intuitive tools used, which in this case are considered solely as incentives to search for logical conclusions. Leaving aside the theory of teaching of deductive reasoning as a whole, we point out two basic conditions for the purposeful formation and actualization of the motivational mechanisms inherent in mathematical activity.

1) Determination of the possibilities of choosing the level of rigour appropriate in the specific mathematical activity and determining a methodical support adequate to this level;
2) Identification and updating of motivational means, providing a natural transition through the "ladder of levels".

The fulfilment of these conditions presupposes a diagnosis of the nature of the reflection of the students' deductive reasoning, and fully accounting for this in determining the content of the work at respective levels of teaching of proof. This must take into account the following two challenges.

1) Different types of reflection between students of the same age range.
2) Unequal deductive potential of material used in various mathematical courses. In particular, the material of some sections of geometry exceeds the deductive possibilities of algebra and calculus in the upper grades of the secondary school due to the forced application of the visually-intuitive approach in this course when studying concepts such as the limit, continuity, and derivative of a function.

The second of these difficulties can be addressed by how a school constructs the content of courses on mathematics. The first of these difficulties can be partially overcome with the help of an appropriate combination of reasoning at different age stages corresponding to different levels of complexity. So, if the assimilation of "ready proof" in the first stage (7-11 years) is possible only by a fixation of reasoning based on "common sense", then in the second stage (12-14 years), these arguments play the role of an "introductory passage" directed towards encouraging the perception of the idea of the proof and its structure as a whole. The main passage at the same time corresponds to the level of "mental experience", supported by all possible types of mathematical activities at this level. Finally, at the final stage (15-17 years) the rationale for theorems is expedient at all three levels. This sequence optimises the possibilities of students developing an understanding of the logical steps of proof beyond the intuitively obvious or already familiar.

III. The Formation of Student Needs in Relation to Mathematical Proofs

It is clear that the provision of a natural transition of students through the "ladder of levels" of rigor presupposes the selection of appropriate methodical techniques that would stimulate the need of schoolchildren in improving their skills to conduct deductive reasoning. This can be distorted by external requirements (to prove, to isolate, to deduce, to refute) over regulating the process, without taking into account the presence or absence of the student's inner desire.

In particular, the demonstration of the need for mathematical proofs in school practice traditionally uses the so-called visual illusions, reinforced by "convincing explanations" of the limited role of observation and experience in the knowledge of facts and patterns, and also involves the task of comparing the linear or angular quantities of geometric objects that are in particular configurations. Such a comparison is generally supposed to be conducted on the basis of an implicit use of laws which are intuitively obvious to students: the value of the part is less than the value of the whole, if we deduct equalities from equal parts, we will get equal parts, etc. The main drawback of such methods from a motivational point of view (despite their external attractiveness) is the possibility of an experimental verification of a fact that is more preferable for a large part of schoolchildren than an attempt to construct deductive inference. Accordingly, these students are not so much convinced of the limited use of observation and experience but are instead confirmed in the thought that deduction, although an important element of mathematical knowledge (as imparted by the teacher), can be completely replaced by direct measurements and calculations. The latter, in turn, negatively affects the development of the internal need for a deductive justification.

Elimination of the indicated difficulties obviously presupposes the selection of such tasks which cannot be achieved, or only with great difficulty, by the means of direct experimental justification [2].

Let us give an example. A square is inscribed in the circle (Fig. 3) [8]. Students should investigate and compare the areas of: 1) the circle and the square, 2) the doubled square and the circle, 3) the square and the half circle, 4) the rectangle and the circle in Fig. 3.

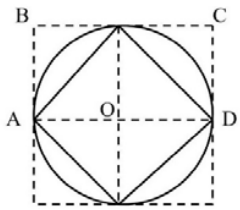

Fig. 3 The square ABCD inscribed in circle (O, OD) [8]

When performing this task, students need to use deduction. To answer the first question, it is sufficient to indicate that the square occupies only a part of the circle, and therefore its area is smaller than the area of the circle. The second question already implies going beyond the basic configuration and considering the square described around the circle. Since the described square is twice as large as inscribed, and at the same time includes a circle, the doubled area of the original square will be larger than the area of the circle. Finally, the answers to the third and fourth questions are considered as immediate consequences of the results of the previous reasoning using the obvious facts: half of equal parts are equal; equal to one and the same are equal to each other [8].

The necessity to develop students' needs to prove is not limited to the framework of the first stage. Teaching deductive reasoning should be included in all stages. There may be stipulations in connection with the need to verify the existence of objects appearing in the condition of certain theorems before applying the corresponding algorithms. Among these theorems we can include, for example, Viet's theorem (before using it, it is necessary to verify the existence of real roots of the quadratic equation), Weierstrass's theorem (before finding the greatest and least value of the function students need to be sure that these values exist at a given interval) and some others. The solution of this problem can be based on consideration of paradoxes of the type.

Problem 1.

Prove by contradiction that 1 is the largest integer.

Let x be the largest integer. Assume that $x > 1$. Then, multiplying the left and right sides of the inequality by x, we obtain $x > x$. The last conclusion contradicts the assumption made that x is the largest integer. Hence $x = 1$.

The absurdity of the resulting conclusion stems from the incompleteness of the thesis. All the reasoning is meaningful only in the case when the existence of the largest integer is initially known.

Since in the second stage of training, the centre of attention of students shifts from the justified fact to the process of its proof from the motivational point of view it is important to ensure the "braking" of potential impulsiveness and the deployment of an indicative search activity which underlies their readiness to analyze this process. This can be achieved through the organization of any activity requiring systematic examination, by the observance of a strict sequence of actions. When teaching mathematics, the so-called "tasks for planning actions" have special significance in the considered key. They are specially oriented on the need for a consistent transition from one operation to another by highlighting the specific content of these actions and observing step-by-step control over their implementation. At the same time, the process of solving such problems can be connected with the choice of the optimal variant of the solution of the problem in question from several alternatives, which in turn necessitates a constant monitoring and evaluation of the measure of the conformity of the actions to their originally generalized grounds. We will illustrate the activity of students in solving problems with the planning of actions using the next example.

Problem 2.

The travellers decided to go around a lake (Fig. 4) by car to explore the lake's surroundings. It is known that the lake is almost round in shape and they should drive for five days. The tank of the car holds fuel only for one day's journey. In addition, the car has a fuel reserve for another two days. To provide themselves with fuel for the whole journey, the travellers decided to pre-arrange places in various locations of the coast for its storage. How can the trip be organized to minimise preparation and storage?

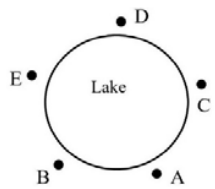

Fig. 4 The lake

In solving this problem after a series of unsuccessful attempts of an intuitive nature, some of which proceeded from patently erroneous premises (bring a canister on foot or refuel at a station), the schoolchildren attempted to determine the locations for possible stops on the coast of the lake.

As a result of a joint discussion with the teacher, a hypothesis was put forward that since the machine can only carry a 3-day supply, these bases can be from the starting point *A* at a distance of no more than one day's travel. The travellers must get to the stop, go back and leave fuel. Points that are at a distance of a one-day move from point *A*, as seen from Fig. 4, will be two: *B* and *C*. Further arguments were reduced to verifying the possibility of moving from point *C* to point *B* around the lake with a full supply of fuel. As a result, it was shown that the preparation for the journey took four days, and the journey itself five days. This application of reasoning on the justification of the proposed ways of solving caused genuine interest among some students in this case, as evidenced, in particular, by their subsequent attempts to find other possible locations for fuel bases. In the next lesson, the students returned to the discussion of the problem without a specific instruction from the teacher and proposed an exhaustive (according to the age stage) set of options, noting that either these databases cannot be built (the arc *CDEB*), or it will need additional time (the arc *BAC*). The students tried to justify their answers before classmates, answering the questions they asked. Considering and justifying or discounting options and processes in this way significantly contributes to the process of justifying the solution in the form of a series of successively arising situations, each of which is formed after passing the previous one, and thus, contributing to the formation of the activity of adolescents in carrying out logical reasoning.

In the transition to the highest possible level of rigour of deductive reasoning in a school setting, the student is interested not so much in clarifying the validity of the considered mathematical fact or the correctness of the proof, as in assessing the significance of this fact in the general structure of the theory. Accordingly, the described transition is characterized by replacing the question of "is the sum of the angles of the triangle actually equal *180*°?" to the elucidation of the existence of "a logical necessity" for this sum to be equal to *180°*.

Such an assessment, obviously assumes a certain level of systemic knowledge, enabling the expansion of the "angle of view" for this content, as well as the mastering of appropriate deductive skills. In addition, there is a significant change in student perspective. This change is manifested in the fact that a student in proving a statement (even if the proof is initially known) does not just convince him or herself and others of this justification, but also tries to try to understand the positions of any opponents (real or virtual) so that under the pressure of disputes and contradictions, the student can develop the habit of understanding the motives of one's own arguments.

Let us illustrate the latter situation with the help of the following problem, which can be considered as an example of a particularly stimulating character.

Problem 3:

"How many children do you have and how old are they?" the guest asked the mathematics teacher. "I have three boys," said the teacher. "The product of their years is 72, and the sum of their years is equal to the number of my house." The guest, having learned the number of the house, said: "The problem is not defined." "Yes, it is," the teacher replied, "But I hope that the eldest boy will still become a good mathematician".

Name the house number and the age of the boys, justifying your considerations.

The first step of the solution is obvious: we should represent number 72 as three factors. The second step is the definition of the sum of the ages of children as being equal to the number of the house. Such a situation is possible only when more than one sum of the factors of the number 72 is equal to the same number, which means that the house number can be equal only to the number 14, since 14 = 2 + 6 + 6 = 3 + 3 + 8. The third step of the solution is to choose from the two alternatives the only possible option. Here we stand in the position of a teacher who wants his older boy to become a mathematician. Since it is only one older boy (and not two or three), the correct answer is: 3, 3, 8.

Note that similar considerations can be used either when a student analyzes ready-made evidence or builds his or her own. In both situations, the student must alternately shift from the position of the initiator of action to the position of its critic and vice versa, while reflecting on the internal, essential features of the thought process.

Highlighting these or other methods of mathematical activity that stimulate the students to implement a meaningful reflection of their proof arguments, we proceed from the following basic criteria: the completeness and generality of the argumentation; internal locus of control; multivariance of choice; the retention of the supertask; orientation on the way of activity; and, range of criteria assessments. In Table I below, these criteria are presented together with the corresponding methods that contribute most to their achievement.

TABLE I
METHODOLOGICAL APPROACHES AND CRITERIA

Criterion \ Method	Filling in the blanks in the proof and discarding unnecessary arguments	The placement of the proof steps in the correct sequence	Denial of arguments	The search for rational methods of proof	Full study of the situation	Organization of logical experiments	Task development
completeness of argumentation	+	+	+	+	+	+	
internal locus of control,	+		+	+	+		
multivariance of choice				+		+	+
retention of super tasks		+		+	+		+
orientation on the way of activity	+	+	+	+	+	+	+
range of criteria assessments				+	+	+	+

Let us consider, for example, how to stimulate students to retain the supertask needed for a full study of the problem situation, or, in other words, how to embark on a kind of "discovering" of the variety of relationships implicit in this situation.

When considering the conditions for congruence of two triangles, the students can be asked to formulate a number of other conditions, which, in their view, are not represented in the textbook. A trivial search of possible alternatives leads to the formulation of three more conditions (two angles and a side opposite one of the angles, two sides and the angle opposite one of the sides, and also at three angles). If the last of the listed conditions is immediately easily refuted by a counterexample, then the question of the first two remains open. The failure of the first attempts to prove them by analogy with one of the previous signs enables students to realise the lack of knowledge available to them on this issue and begins to play the role of a long-term motivational factor in the study of a number of subsequent topics.

In particular, the first of the condition compiled by the pupils can serve as a starting point for studying the theorem on the sum of the angles of a triangle, while the second (in the literature commonly called the 4^{th} condition of the congruence of triangles) is actualized when considering the question of the possibility of constructing a triangle when given its three elements. The construction of a triangle when given two sides and an angle opposing one of them, in the case when this angle is acute, it leads, as is known, to two possible solutions (Fig. 5).

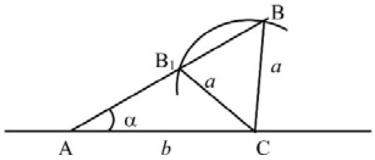

Fig. 5 Constructing the triangle *ACB*

The ambiguity of the decision indicates the invalidity of the previously formulated hypothesis. Nevertheless, its subjective "tangibility", reinforced by practice, stimulates to a certain extent the process of finding additional conditions under which the fourth condition will start to work. An elementary logical experiment, confirmed by an experimental test, shows that, as such a condition, the uniformity of the angles at the vertex *B* of one triangle and the corresponding vertex B_1 of another triangle can be chosen [8]. The proof of the complicated version of the condition is either carried out by the imposition method, or can be realized in the study of sine and cos theorems. In this example, we tried to show how, when teaching a sufficiently large amount of diverse mathematical materials that it is possible to ensure the development of the students' deductive reasoning by retaining in their experience a single motivational line, securing certain problem situations in the structure of the overall supertask.

IV. Conclusion

The deductive reasoning skills of students is one of the key conditions to develop logical thinking. Teachers should pay attention to methods of teaching simulation tools that influence the development to reasoning in mathematical activity [6], [7], [10]. There are many methodological approaches promoting the solution of the pointed task described in this article. These didactic techniques described here have three levels. Approaches used at lower levels can be applied at higher levels (but not vice versa), supporting students with different sources and the various degree of complexity for each activity. It is necessary to diagnose students` reasoning skills and reflexive abilities and consider them in the teaching mathematics. The teachers can overcome this difficulty by appropriately combining deductive reasoning with different degrees of complexity. At the first stage, students use reasoning with common sense, while at the second stage, students use reasoning related to their experience. At the third stage, theorem proving should be taught at all levels. Hence, effective teaching of mathematics in the secondary school motivates and guides students to their own discovery of proofs of mathematical statements. [6], [7], [10].

References

[1] I. S. Yakimanskaya, Characteristics of creative thinking of students, Russia, Moscow, Education, 1989.
[2] V. A. Dalinger, One method of prove. Mathematics at school, Russia, V.5, pp. 12-14, 1993.
[3] G. V. Dorofeev, The thesis about correctness of reasoning and details of presentation during task solving, Mathematics at school, Russia, V.1, pp. 44-47, 1982.
[4] J. Piaget, Judgment and Reasoning in the Child, Russia, St. Petersburg, Union, 1997.
[5] G. Polya, Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving, Russia, Moscow, Science, 1976.
[6] M. Rodionov, Motivational Strategies for Teaching Math: theoretical aspects and practical implementation. Germany, Saarbrücken, LAP LAMBERT Academic Publishing Gmbh& Co. KG, 2012.
[7] M. Rodionov, E. Marina, The formation of pupils thinking skills in construction problems-solving, Russia, Penza, PSPU, 2006.
[8] M. Rodionov, Z. Dedovets, Dynamic Peculiarities of the Establishment of Pupils' Valuable Attitude towards Mathematical Demonstrative Reasoning and their Application in School Practice. London International Conference on Education (LICE-2012) In Collaboration with World Congress on Sustainable Technologies (WCST-2012). Proceedings (November 19-22, 2012, London, GBR)- pp.620-624.
[9] E. L. Thorndike, The psychology of algebra. The Macmillan Company, New York, 1926
[10] Z. I. Slepkan, Teaching Proofs in Geometry, Russia, Moscow, High School, pp. 34–43, 1989.