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Box Counting Dimension of the Union L of Trinomial Curves When $\alpha \geq 1$

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***Abstract*—In the present work, we consider one category of curves denoted by $L(p,k,r,n)$. These curves are continuous arcs which are trajectories of roots of the trinomial equation $z^n = \alpha z^k + (1-\alpha)$, where z is a complex number, n and k are two integers such that $1 \leq k \leq n-1$ and α is a real parameter greater than 1. Denoting by L the union of all trinomial curves $L(p,k,r,n)$ and using the box counting dimension as fractal dimension, we will prove that the dimension of L is equal to $3/2$.**

***Keywords*—Feasible angles, fractal dimension, Minkowski sausage, trinomial curves, trinomial equation.**

I. Introduction

CONSIDER the subset L of the plane defined as the union of all trinomial curves $L(p,k,r,n)$, located outside the unit disk. These continuous arcs are trajectories of roots of the trinomial equation

$$z^n = \alpha z^k + (1-\alpha) \tag{1}$$

where z is a complex number, n and k are two integers such that $1 \leq k \leq n-1$ and α is a real parameter greater than 1. The main goal of this work is to calculate the box counting dimension [2], [5] of L.

Box counting dimension is one of the most widely used fractal dimensions. Its popularity is largely due to its relative ease of mathematical calculation. This dimension is also called Minkowski dimension. Let E be any non-empty bounded subset of IR^n and let $N_\varepsilon(E)$ be the smallest number of sets of diameter at most ε which can cover E. The *lower box counting dimension* of E denoted by $\underline{\dim}_B E$ or $\delta(E)$ and the *upper box counting dimension* of E denoted by $\overline{\dim}_B E$ or $\Delta(E)$ are respectively defined as follows [2], [5]:

$$\begin{aligned}\delta(E) &= \liminf_{\varepsilon \longrightarrow 0} \log N_\varepsilon(E) / -\log\varepsilon \\ \Delta(E) &= \limsup_{\varepsilon \longrightarrow 0} \log N_\varepsilon(E) / -\log\varepsilon\end{aligned}$$

If these are equal, the *box counting dimension of* E is this common value, simply denoted by $\dim_B E$

$$\dim_B E = \lim_{\varepsilon \longrightarrow 0} \log N_\varepsilon(E) / -\log\varepsilon.$$

This is sometimes referred to as fractal dimension. Let us note that there are several equivalent definitions of box counting dimension that are occasionally more convenient to use.

The main purpose of this work is to calculate the box counting dimension for the union of all trinomial curves $L(p,k,r,n)$ introduced in [4]. In order to prove that this dimension is equal to 1.5, we will demonstrate some auxiliary results.

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II. Box Counting Dimension in the Plane

In [2], it was proved that the box counting dimension of a subset can be expressed as follows.

Proposition. If F is a subset of IR^n, then

$$\begin{aligned}\underline{\dim}_B F &= n - \liminf_{\gamma \longrightarrow 0} \log vol^n(F_\gamma) / \log\gamma \text{ and} \\ \overline{\dim}_B F &= n - \limsup_{\gamma \longrightarrow 0} \log vol^n(F_\gamma) / \log\gamma\end{aligned}$$

where $F_\gamma = \{x \in IR^n : |x-y| \leq \gamma \text{ for some } y \in F\}$ is the γ-parallel body of F and $vol^n(F_\gamma)$ the n-dimensional volume of F_γ.

We can also find in [2, p. 44], the two following properties:

(i) $\underline{\dim}_B$ and $\overline{\dim}_B$ are monotonic.

(ii) $\overline{\dim}_B$ is finitely stable, i.e.

$$\overline{\dim}_B(E \cup F) = \max\{\overline{\dim}_B E, \overline{\dim}_B F\},$$

though $\underline{\dim}_B$ is not.

Moreover, according to [5], Δ is monotonic: if E_1 is included in E_2, then

$$\Delta(E_1) \leq \Delta(E_2).$$

The properties of Δ are also checked by δ, with the exception of the stability. An example of the non-stability of the lower box counting dimension δ is given in [5, p. 32-33]; it's an example of perfect sets E_1 and E_2 on the line, such as $\delta(E_1 \cup E_2) \neq \max\{\delta(E_1), \delta(E_2)\}$.

On account of this last inconvenience of the lower box-counting dimension δ given by the property (ii), we will only use in this work the upper dimension Δ. Moreover, we are interested in the box counting dimension Δ in the plane IR^2. Assuming that F is a subset of IR^2, for $\gamma > 0$, the γ-parallel body $F(\gamma)$ of F will be called *Minkowski sausage* of F. The 2-dimensional volume $vol^2(F(\gamma))$ of $F(\gamma)$ is exactly the plane Lebesgue measure or the area of $F(\gamma)$, which will be denoted by $|F(\gamma)|_2$. Thus,

$$\Delta(F) = 2 - \lim_{\gamma \longrightarrow 0} \sup \log |F(\gamma)|_2 / \log\gamma.$$

According to section 2.6 of [5], to easily calculate $\Delta(E)$, it is interesting to know some equivalent definitions, in order to use one or the other, depending on how the problem arises. It may be more convenient to replace the continuous variable

ε with a discrete sequence ε_n that tends to 0. Here is what condition:

Lemma [5]. For any sequence (ε_n) of real numbers that converges to 0, such as the ratio

$$\log \varepsilon_n / \log \varepsilon_{n+1}$$

converges to 1, we have

$$\Delta(E) = \lim_{n \longrightarrow \infty} \log N(\varepsilon_n) \ / \ |\log \varepsilon_n| \,.$$

This condition imposed on the sequence ε_n indicates that it must not tend too fast to 0.

Remark. $(\gamma_n) = (1/n^2)$ is an example of sequences that can be used with the previous lemma.

Now, we will state some results of [1] which we shall make use to calculate the box counting dimension Δ for the union L of trinomial curves $L(p,k,r,n)$.

In Theorem 3 of [1], the area of the Minkowski sausage of a curve is increased in the following manner:

Theorem. Suppose that C is a curve with finite length $L(C)$. For any $\varepsilon > 0$, we have

$$|C(\varepsilon)|_2 \leq 2\,\varepsilon L\,(C) + \pi\varepsilon^2.$$

Furthermore, Theorem 2 of [1] state the result below, giving a bound for the length of a monotonic curve (see Fig. 1).

Theorem. Suppose that C is a curve defined by:

$$x(\theta) = \rho(\theta)\cos\theta, \qquad y(\theta) = \rho(\theta)\sin\theta,$$

where $\theta_1 \leq \theta \leq \theta_2$ and $\rho(\theta_1) = R_1$ and $\rho(\theta_2) = R_2$.

If $\rho(\theta)$ is a monotonic function, then the curve C has a finite length $L(C)$ such that

$$L(C) \leq |R_1 - R_2| + \max(R_1, R_2)\,|\theta_1 - \theta_2|\,.$$

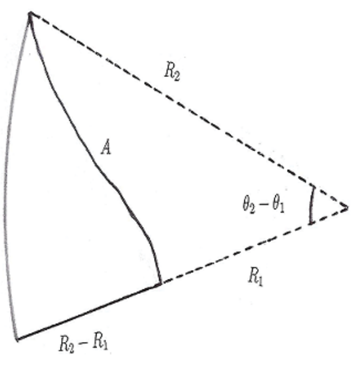

Fig. 1 A curve A with finite length

III. Box Counting Dimension of the Subset L

Putting $z = \rho e^{i\theta}$ in the trinomial equation (1), we have:

$$\rho^n e^{in\theta} \;=\; \alpha\rho^k e^{ik\theta} + (1-\alpha).$$

This means that

$$\rho^n[\cos n\theta + i\sin n\theta] \;=\; \alpha\rho^k[\cos k\theta + i\sin k\theta] + (1-\alpha).$$

By identifying the imaginary part of the two members of the equality and when $\theta \neq l\pi/n$, where l is an integer, it follows that

$$\rho^{n-k} \;=\; \alpha\sin k\theta / \sin n\theta.$$

On the other side, we can divide (1) by z^n and consider the imaginary part. If $\alpha \neq 0$ and $\theta \neq l\pi/(n-k)$, where l is an integer, we can deduce that

$$\rho^k \;=\; \frac{(\alpha-1)}{\alpha}\,\frac{\sin n\theta}{\sin(n-k)\theta}.$$

Moreover, we can find the α-free equation of trajectories of roots of (1), linking only ρ and θ, which is as follows

$$\rho^{n-k}\sin n\theta - \rho^n\sin(n-k)\theta = \sin k\theta.$$

In this paper, we will restrict our study to the case $1 \leq \alpha < +\infty$. According to [3], un angle θ is called *feasible* in the case $1 \leq \alpha < +\infty$, if θ verify the equalities of signes:

$$signe(\sin n\theta) = signe(\sin k\theta) = signe(\sin(n-k)\theta).$$

At the present time, let us define the curves $L\,(p,k,r,n)$ illustrated in Fig. 2.

According to [4], the cases $n = 2$ and $n = 3$ are particular cases, because the trajectories solutions of (1), where $\alpha > 1$ for these cases are linear. Hence, the definition of a trinomial curve $L\,(p,k,r,n)$ is as follows:

Assume that n is an integer larger than or equal to 4. The trinomial curve $L(p,k,r,n)$ is the set of roots of (1) with $\alpha > 1$ and the feasible angles belong to the interval $[2\pi p/(n-k),\ (2r+1)\pi/n]$, where p and r are nonzero integers verifying $r \geq p$ and k is an integer such that $(r-p)n/r < k < [2(r-p)+1]n/(2r+1)$.

Proposition 2.1 of [4] prove the existence of the curves $L(p,k,r,n)$. In addition, the proof of this proposition gives us that:

$$\sin n\theta > 0, \qquad \sin k\theta > 0, \qquad \sin(n-k)\theta > 0,$$

for any θ such that $2\pi p/(n-k) < \theta < (2r+1)\pi/n$.

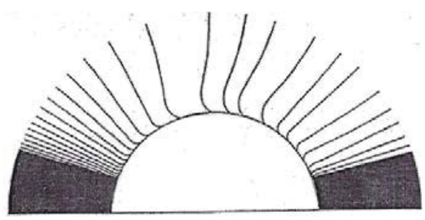

Fig. 2 Trinomial curves $L\,(p,k,r,n)$

Now, let us recall that a very important property for the trinomial arcs $L(p,k,r,n)$ was proved in [4]. This property gives that the function $\rho\,(\theta)$ for these curves is monotonic. In fact, each curve $L(p,k,r,n)$ can be expressed in polar coordinates (ρ,θ) by a function $\rho\,(\theta)$. Therefore, we have the following result.

Theorem. Let $L(p,k,r,n)$ be a trinomial curve. For any integer k such that $(r-p)n/r < k < [2(r-p)+1]n/(2r+1)$, $\rho(\theta)$ is an increasing function on the interval $[2\pi p/(n-k),\ (2r+1)\pi/n]$.

This main result will be used in the estimation of the fractal dimension Δ of the union L of the curves $L(p,k,r,n)$. At first, there are some basic remarks.

Remark. According to [4], the subset L is symmetric with respect to the real axis. Then, because the upper box counting dimension Δ is finitely stable, we will estimate the fractal dimension $\Delta(L)$ only for the trinomial curves $L(p,k,r,n)$ located on the upper half plane.

Let us consider a trinomial curve $L(p,k,r,n)$. Let R be a real number greater than 1 and θ_0 the feasible angle which corresponds to R, so $2\pi p/(n-k) < \theta_0 < (2r+1)\pi/n$. In the rest of this section, we will consider the restrictions of the curves $L(p,k,r,n)$ located inside the disk with radius R. These restrictions will also be denoted by $L(p,k,r,n)$ and their union by L. From [4], we obtain that $\rho\left[2\pi p/(n-k)\right] = 1$. This implies that such a curve $L(p,k,r,n)$ joins the two points of polar coordinates $(1, 2\pi p/(n-k))$ and (R, θ_0).

To state the main result of this work, we need the following lemma which we shall make use.

Lemma. For any integer n greater than or equal to 4, the length of the trinomial curve $L(p,k,r,n)$ is smaller than $2R-1$.

Proof. Consider an arc $L(p,k,r,n)$ with finite length $L(L(p,k,r,n))$. By the previous results, we have

$$\begin{aligned} L(L(p,k,r,n)) &\leq R[\theta_0 - 2\pi p/(n-k)] + R - 1 \\ &< R[(2r+1)\pi/n - 2\pi p/(n-k)] + R - 1. \end{aligned}$$

Because $k > (r-p)n/r$, we obtain that

$$L(L(p,k,r,n)) < 2R-1.$$

Theorem. The fractal dimension $\Delta(L)$ of the union L of all trinomial curves $L(p,k,r,n)$ is equal to $3/2$.

Proof. Looking at the trinomial curves $L(p,k,r,n)$ outside the unit disk (see Fig. 2), we remark that we can divide the union L into the two families of trinomial curves:

a. The curves $L(p,k,r,n)$ located in the first quadrant of the plane. We denote by L_1 the union of these curves.

b. The curves $L(p,k,r,n)$ located in the second quadrant of the plane. We denote by L_2 the union of these curves.

Because the upper box-counting dimension Δ is finitely stable, we can have that

$$\Delta(L) = \Delta(L_1 \cup L_2) = \max[\Delta(L_1), \Delta(L_2)].$$

So, we proceed at the start by calculating $\Delta(L_1)$. Then, the estimation of $\Delta(L_2)$ can be established in the same way.

First, we have to show that $\Delta(L_1) \leq 3/2$. Next, we have to prove that $\Delta(L_1) \geq 3/2$. Thus, let us begin by proving that $\Delta(L_1) \leq 3/2$. Assume that n is an integer greater than 4 and $(n-3)$ the number of the n first curves $L(p,k,r,j)$ where $j \geq 4$. Let us set:

$$\varepsilon_n = 1/(n+1)^2.$$

One can remark that the area of the sausage of L_1 is smaller than or equal to the sum of the areas of the sausages of the $(n-3)$ first curves $L(p,k,r,j)$ which join the points of polar coordinates $(1, 2\pi p/(j-k))$ and (R, θ_0) and the area of the sausage of the sector: $S = \{(\rho,\theta) : 1 < \rho < R,\ 0 < \theta < \theta_c\ /\ 2\pi/(n+1) < \theta_c < 3\pi/(n+1)\}$. This implies that

$$|L_1(\varepsilon_n)|_2 \leq |S(\varepsilon_n)|_2 + \sum_{j=4}^{n} |L(p,k,r,j)(\varepsilon_n)|_2 .$$

Concerning the sector S, we can obtain that

$$\begin{aligned} |S(\varepsilon_n)|_2 &\leq \frac{1}{2}\left[R^2 - (1-\varepsilon_n)^2\right]\theta_c + 2\varepsilon_n(R-1+\varepsilon_n) \\ &+ R\varepsilon_n\theta_c + \frac{1}{2}\pi\varepsilon_n^2 \\ &= \frac{1}{2}(R^2-1)\theta_c + 2(R-1)\varepsilon_n + (R+1)\varepsilon_n\theta_c \\ &+ (2+\pi/2)\varepsilon_n^2 - \frac{1}{2}\varepsilon_n^2\theta_c \\ &\leq \frac{3\pi}{2}\left(R^2-1\right)\sqrt{\varepsilon_n} + 2(R-1)\varepsilon_n \\ &+ 3\pi(R+1)\varepsilon_n\sqrt{\varepsilon_n} + (2+\pi/2)\varepsilon_n^2 - \pi\varepsilon_n^2\sqrt{\varepsilon_n} \\ &= O(\sqrt{\varepsilon_n}) \end{aligned}$$

On the other hand, consider a trinomial curve $L(p,k,r,j)$, where $j \geq 4$. Using the result that $\rho(\theta)$ is increasing for each $L(p,k,r,j)$ and by the majoration above of the length of a curve, we can find that

$$|L(p,k,r,j)(\varepsilon_n)|_2 \leq 2(2R-1)\varepsilon_n[1 + \frac{2}{(2R-1)}\varepsilon_n].$$

So, we can obtain that:

$$\begin{aligned} \sum_{j=4}^{n} |L(p,k,r,j)(\varepsilon_n)|_2 &\leq 2(2R-1)(n-3)\varepsilon_n \times \\ &\quad [1 + \frac{2}{(2R-1)}\varepsilon_n] \\ &\leq 2(2R-1)\sqrt{\varepsilon_n}[1 + \frac{2}{(2R-1)}\varepsilon_n] \\ &= O(\sqrt{\varepsilon_n}) \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} \Delta(L_1) &= \limsup_{n \longrightarrow +\infty} (2 - \log |L_1(\varepsilon_n)|_2 / \log \varepsilon_n) \\ &\leq \limsup_{n \longrightarrow +\infty} (2 - \log O(\sqrt{\varepsilon_n}) / \log \varepsilon_n) \\ &= 3/2. \end{aligned}$$

With an analogue argument, one gets

$$\Delta(L_2) = \limsup_{n \longrightarrow +\infty} (2 - \log |L_2(\varepsilon_n)|_2 / \log \varepsilon_n) \leq 3/2.$$

Consequently, it yields that

$$\Delta(L) = \max\{\Delta(L_1), \Delta(L_2)\} \leq 3/2.$$

At this stage, to complete the proof, we have to show that $\Delta(L) \geq 3/2$. Like before, we will demonstrate that $\Delta(L_1) \geq 3/2$. Thus, we consider a sequence (ε_j) which is distinct from that of the first part. Let $\varepsilon_j = \pi/j^2$, where $j > 4$. Remarking that the sausage $L_1(\varepsilon_j)$ contains the sector

$$S' = \{(\rho, \theta) : 1 < \rho < (R+2)/3,\ 0 < \theta < 2\pi/j\},$$

we conclude that

$$|L_1(\varepsilon_j)|_2 \geq [\frac{1}{9}(R+2)^2 - 1]\pi/j = O\sqrt{\varepsilon_j}$$

So, we obtain

$$\Delta(L_1) = \limsup_{j \longrightarrow +\infty}(2 - \log |L_1(\varepsilon_j)|_2 / \log \varepsilon_j) \geq 3/2.$$

As before, with a same manner, we can deduce that

$$\Delta(L_2) = \limsup_{j \longrightarrow +\infty}(2 - \log |L_2(\varepsilon_j)|_2 / \log \varepsilon_j) \geq 3/2.$$

Then, it follows that

$$\Delta(L) = \max\{\Delta(L_1), \Delta(L_2)\} \geq 3/2$$

and the result is so proved.

IV. Conclusion

In the present work, it was proved that the fractal dimension $\Delta(L)$ of the set L is equal to 1.5. The fact that this dimension is a non integer value prove the fractal structure of the set L. To calculate it, we considered the restrictions of these trinomial curves $L(p,k,r,n)$ inside a disk with radius R where R is a real number greater than 1. A further estimation of this fractal dimension when R tends to infinity would be interesting.

Currently, we are working on the implementation with a JAVA program of these trinomial arcs $L(p,k,r,n)$. On the other side, the programmation of this calculation of the upper box-counting dimension $\Delta(L)$ would be of great importance. Another perspective is to estimate the Hausdorff dimension of this subset L.

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