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# Contribution to the Analytical Study of Barrier Surface Waves: Decomposition of the Solution

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***Abstract*****—**When a partially or completely immersed solid moves in a liquid such as water, it undergoes a force called hydrodynamic drag. Reducing this force has always been the objective of hydrodynamic engineers to make water slide better on submerged bodies. This paper deals with the examination of the different terms composing the analytical solution of the flow over an obstacle embedded at the bottom of a hydraulic channel. We have chosen to use a linear method to study a two-dimensional flow over an obstacle, in order to understand the evolution of the drag. We set the following assumptions: incompressible inviscid fluid, irrotational flow, low obstacle height compared to the water height. Those assumptions allow overcoming the difficulties associated with modelling these waves. We will mathematically formulate the equations that allow the determination of the stream function, and then the free surface equation. A similar method is used to determine the exact analytical solution for an obstacle in the shape of a sinusoidal arch.

***Keywords*****—**Free-surface wave, inviscid fluid, analytical solution, hydraulic channel.

## I. Introduction

THE calculation of the free surface induced by the deformation of the bottom of a channel is not an easy task. Several authors have tried and ended up choosing a certain form of disturbance, such as Boutros et al. [1]. Indeed, methods calculating the free-surface elevation often involve integrals that are quite difficult to carry out analytically. To pass this milestone, each researcher has his own method:

- The integral is replaced by an approximation.
- We choose an obstacle for which the integral is calculated; this is, for example, the case of the triangular obstacle.

In this paper, we highlight all the analytical terms that make up the free surface equation and use a method that will allow us to address a particular bump lying in the bottom of a hydraulic channel.

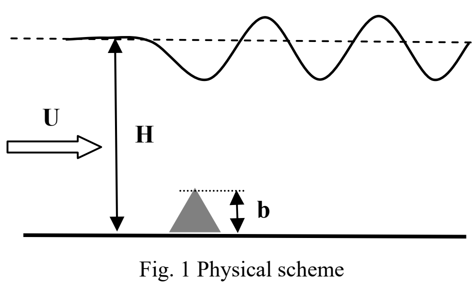


Fig. 1 Physical scheme

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## II. Mathematical Equations

Let's consider a free-surface flow in the presence of an obstacle of length l and maximum height b. The upstream water level is H.

The fluid is assumed to be incompressible and inviscid. Surface tension forces are neglected. Upstream, the flow is steady, uniform, and therefore irrotational.

The bottom equation can be written as:

$$y = - H + f(x) \qquad (1)$$

f(x) is the function describing the obstacle assumed to satisfy Dirichlet's conditions.

The velocity components are given by:

$$u = \frac{\partial \Psi}{\partial y} \quad \text{et} \quad v = - \frac{\partial \Psi}{\partial x} \qquad (2)$$

ψ(x,y) is the stream function which must satisfy adequate boundary conditions at the appropriate borders. In the case where b/H is very small, the small disturbance method can be used and we write:

$$\Psi(x,y) = -Uy + \phi(x,y) \text{ with } \Delta\Psi = 0 \quad \Rightarrow \quad \Delta\phi = 0 \qquad (3)$$

The free surface is a streamline; we can write Bernoulli's relationship on it, as:

$$\frac{g\,\varphi(x,y)}{U} - U\frac{\partial \varphi}{\partial y}(x,0) = \text{Constant} \qquad (4)$$

The bottom is also a streamline.

$$\varphi(x,y) = UH \qquad (5)$$

At the first order of the expansion, (4) and (5) give:

$$U\,f(x) = \varphi(x,-H) \qquad (6)$$

f(x) is written in the form:

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} dk \int_{-\infty}^{+\infty} f(s)\cos k(x-s)\,ds \qquad (7)$$

The stream function disturbance φ(x,y) is written [1]:

$$\varphi(x,y)=\frac{1}{\pi}\int_0^{\infty}\frac{\mathrm{ch}\,ky+\frac{g}{ku^2}\mathrm{sh}\,ky}{\mathrm{ch}\,kH-\frac{g}{ku^2}\mathrm{sh}\,kH}\int_{-\infty}^{+\infty}f(s)\cos k(x-s)\,ds \qquad (8)$$

*A. Triangular Bump*

Let's write the triangle equation:

$$f(x)=\begin{cases}0 & \text{for } x\le 0\\ x\tan\gamma & \text{for } 0<x\le l1\\ x\tan\theta+b(1+\tan\gamma\,\mathrm{cotg}\,\theta) & \text{for } l1\le x\le l1+l2\\ 0 & \text{for } x\le l1+l2\end{cases} \qquad (9)$$

Let's call: $kH = t$, $l1/H = L1$, $l2/H = L2$, $b/H = B$, $y_0 = \varphi(x,o)/H$, $Q(t)=1/(t^2(\mathrm{ch}t - \mathrm{sh}t/Fr^2\, t))$, $B/L1 = \tan\gamma$, $B/L2 = \tan\theta$. Fr is the Froude number defined by: $Fr^2 = U^2/gH$

By using (9) in (8), the free surface is written as:

$$y_0=\left[\begin{array}{l}\frac{\tan\gamma}{\pi}\int_0^{+\infty}\frac{\cos t(x-l1)-\cos tx}{t^2\left(\mathrm{ch}t-\frac{\mathrm{sh}\,t}{Fr^2\,t}\right)}dt\\ +\frac{\tan\theta}{\pi}\int_0^{+\infty}\frac{\cos t(x-L2)-\cos(tx-L1)}{t^2\left(\mathrm{ch}t-\frac{\mathrm{sh}\,t}{Fr^2\,t}\right)}dt\end{array}\right] \qquad (10)$$

The poles of the function to be integrated are the roots of Q(t) which are: $t = 0$, $t = \pm i\beta n$ $n= 1,2,.....$ with $\tan\beta n = Fr^2\beta n$, $t = \pm\alpha$ with $\mathrm{th}\ \alpha = Fr^2\alpha$.

The last pole exists only if $Fr<1$. This condition allows the subcritical regime to be distinguished from the supercritical regime. Q(z) is then written as:

$$Q(z)=\left[\begin{array}{l}\frac{Fr^2}{Fr^2-1}\frac{1}{z^2}+2Fr^2\sum_{n=1}^{\infty}\frac{\cos\beta n}{Fr^2\cos\beta n-1}\frac{1}{z^2-\beta n^2}\\ +\frac{2\,Fr^2\,\mathrm{ch}\,\alpha}{Fr^2\,\mathrm{ch}^2\alpha-1}\frac{1}{z^2-\alpha^2}\end{array}\right] \qquad (11)$$

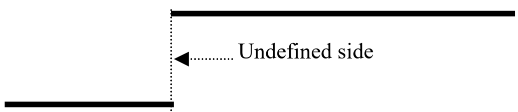


Fig. 2 Semi-infinite step

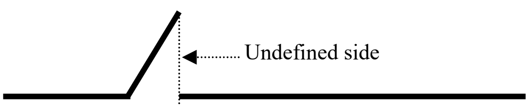


Fig. 3 Semi-infinite step

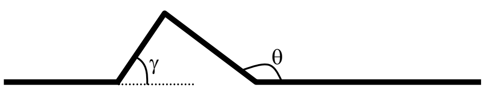


Fig. 4 Angles of the triangle

$$y_0 = \frac{Fr^2\tan\gamma}{2(Fr^2-1)}\left(|x|-|x-L1|\right)+$$

$$\breve{g}Fr^2\,\mathrm{tg}\,\gamma\left(\sum_{n=1}^{\infty}\frac{\cos\beta n\left(e^{-\beta n|x-L1|}-e^{-\beta n|x|}\right)}{\beta n\left(Fr^2\cos^2\beta n-1\right)}\right)+$$

$$\frac{Fr^2\tan\gamma}{\alpha\left(Fr^2\,\mathrm{ch}^2\alpha-1\right)}\left(\sin\alpha\,|x|+\sin\alpha\,x\right.-$$

$$\frac{Fr^2\tan\gamma}{\alpha\left(Fr^2\,\mathrm{ch}^2\alpha-1\right)}\left(\sin\alpha\,|x-L1|+\sin\alpha(x-L1)\right)+$$

$$\frac{Fr^2\tan\theta}{2(Fr^2-1)}\left(|x-L1|-|x-L1-L2|\right)+$$

$$Fr^2\tan\theta\left(\sum_{n=1}^{\infty}\frac{\cos\beta n\left(e^{-\beta n|x-L1-L2|}-e^{-\beta n|x-L1|}\right)}{\beta n\left(Fr^2\cos^2\beta n-1\right)}\right)+$$

$$\frac{Fr^2\tan\theta}{\alpha\left(Fr^2\,\mathrm{ch}^2\alpha-1\right)}\left(\sin\alpha\,|x-L1|+\sin\alpha(x-L1)\right)-$$

$$\frac{Fr^2\tan\theta}{\alpha\left(Fr^2\,\mathrm{ch}^2\alpha-1\right)}\left(\sin\alpha\,|x-L1-L2|+\sin\alpha(x-L1-L2)\right)$$

The following identities were used:

$$\int_0^{\infty}\frac{\cos tx1-\cos tx2}{t^2-p^2}dt=-\frac{\pi}{2p}\left[\begin{array}{l}\sin|px1|+\sin px1\\ -\sin|px2|-\sin px2\end{array}\right]$$

$$\int_0^{\infty}\frac{\cos tx1-\cos tx2}{t^2+p^2}dt=\frac{\pi}{2p}\left(e^{-\beta n|x1|}-e^{-\beta n\,x2}\right) \qquad (12)$$

$$\int_0^{\infty}\frac{\cos tx1-\cos tx2}{t^2}dt=-\frac{\pi}{2}\left(|x1|-|x2|\right)$$

It is very clear that the free-surface is the sum of three contributions, each due to a pole.

The pole $t = \pm$ a exists only if the Froude number is less than one; it gives rise to two trains of the same wavelengths that start at the beginning and end of the obstacle.

The following remarks can then be highlighted:

- the length of the obstacle modulates the amplitude of the final waves.
- waveless for a supercritical regime.
- the influence of this pole on the free surface goes from zero to $+\infty$ where it is the only one to contribute to the final solution.

The $i\beta n$ poles contribute by two exponential decreases, symmetrical with respect to the beginning and end of the obstacle; it is at these levels that the free-surface is located. As for the previous pole, the variation in length L changes the shape of the free surface, but this influence is located in the vicinity of the obstacle.

The zero pole is very particular. Its contribution is written as: $1- Fr^2 f(x) / 2 (Fr^2-1)$. From this result, it is noted that if the beginning and the end levels of the obstacle are not the same, the average downstream free-surface does not extend the undisturbed upstream free-surface.

*B. Semi-Infinite Step*

For the semi-infinite step and the rectangular triangle, the vertical sides of the obstacles create a mathematical discontinuity of the free-surface due to the fact that the function describing the obstacle is itself discontinuous. The semi-infinite step is represented by Fig. 2. The rectangular triangle is represented by Fig. 3. In the equation of non-particular triangle, the angles at the base are shown in Fig. 4.

To obtain the equation of the free-surface for a rectangular triangle, the angle value of $\gamma$ is increased up to $\pi/2$. If $\theta = 0$ and $\gamma = \pi/2$, we obtain the free-surface expression of the semi-infinite step. The obtained equation for the semi-infinite step, for $x < 0$ is:

$$y_0 = -\frac{Fr^2 B}{2(Fr^2-1)} - Fr^2 B \sum_{n=1}^{\infty} \frac{\cos\beta n \, e^{x\beta n}}{Fr^2\cos^2\beta n - 1} + C1$$

C1 is a constant to be determined. Let's write the undisturbed free-surface condition far upstream:

$$\lim_{\to -\infty} y_0 = -\frac{Fr^2 B}{2(Fr^2-1)} + C1 = 0$$

This allows us to determine the constant C1 and the expression of the free-surface y0 for $x < 0$, namely:

$$y_0 = -Fr^2 B \sum_{n=1}^{\infty} \frac{\cos\beta n \, e^{x\beta n}}{Fr^2\cos^2\beta n - 1}$$

For $x < 0$,

$$y_0 = \begin{cases} \dfrac{Fr^2 B}{2(Fr^2-1)} + Fr^2 B \displaystyle\sum_{n=1}^{\infty} \frac{\cos\beta n \, e^{-x\beta n}}{Fr^2\cos^2\beta n - 1} + \\ 2\,Fr^2 B \dfrac{ch\,\alpha \cos\alpha x}{Fr^2 ch^2\alpha - 1} + C2 \end{cases}$$

The function is continuous for the $x = 0$; therefore:

$$\lim_{\to 0-} y_0 = \lim_{\to 0+} y_0$$

This allows us to determine C2:

$$y_0 = \{Fr^2 B \sum_{n=1}^{\infty} \frac{\cos\beta n\,( e^{-x\beta n} - 2 )}{Fr^2\cos^2\beta n - 1} - 2\,Fr^2 B \frac{ch\,\alpha\,(1 - \cos\alpha x)}{Fr^2 ch^2\alpha - 1}$$

By combining the two writings, the free-surface for a semi-infinite step can be written, for any x:

$$y_0 = \{Fr^2 B \sum_{n=1}^{\infty} \frac{\cos\beta n\,( e^{-|x|\beta n} - 1 - \frac{|x|}{x} )}{Fr^2\cos^2\beta n - 1} - Fr^2 B \frac{ch\,\alpha\,(1 - \cos\alpha x)}{Fr^2 ch^2\alpha - 1}\left(1 + \frac{|x|}{x}\right)$$

This solution is identical to the one given by Bloor et al. [2] by another means. The contribution of each pole is summarized in Fig. 5.

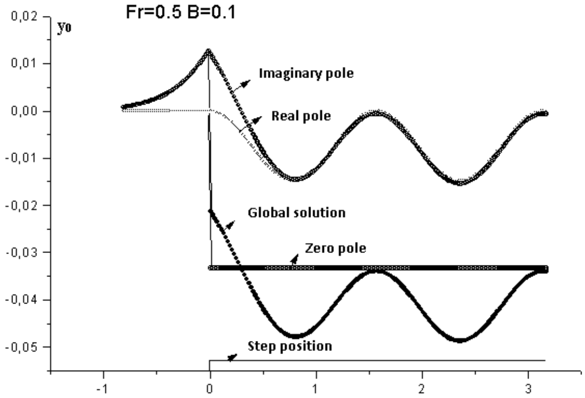


Fig. 5 Contribution of each pole

By using the same way, we obtain the solution of the rectangular triangle:

$$y_0 = \begin{cases} \dfrac{Fr^2\tan\gamma}{2(Fr^2-1)}\left(|x| - |x - L1|\right) + \dfrac{Fr^2\,B}{2(Fr^2-1)} - \\ \displaystyle\sum_{n=1}^{\infty} \frac{Fr^2\cos\beta n\,\tan\gamma}{Fr^2\cos^2\beta n - 1}\left(e^{-\beta n|x|} - e^{-\beta n|x - L1|}\right) + \\ \dfrac{Fr^2 ch\,\alpha\,\tan\gamma}{Fr^2 ch^2\alpha - 1}\left[\begin{array}{c} \sin\alpha x + \sin\alpha|x| - \\ \sin\alpha(x - L1) + \sin\alpha|x - L1| \end{array}\right] - \\ \dfrac{Fr^2 ch\,\alpha\,(\cos\alpha(x - L1) - 1)}{Fr^2 ch^2\alpha - 1}\left(1 + \dfrac{|x - L1|}{x - L1}\right) + \\ \displaystyle\sum_{n=1}^{\infty} \frac{Fr^2\cos\beta n}{Fr^2\cos^2\beta n - 1}\left(1 + \frac{|x - L1|}{x - L} - \frac{|x - L1|}{x - L1} e^{-\beta n|x - L1|}\right) \end{cases}$$

In view of the number of papers devoted to the subject, we will mention only a few of them in chronological order. Lamb [1] considered a very small obstacle. Gazdar [2] showed that some obstacles do not produce waves. Bouhadef et al. [3] conducted, in addition to the analytical study, an experimental investigation. Forbes and Schwartz [4] treated the case of a semi-circular obstacle using a numerical method. Boutros et al. [5] considered a linear obstacle (the triangle). King and Bloor [6] were interested in the case of a semi-infinite step. Bouhadef and Peube [7], [8] took into account, in their study within the framework of linear theory; the fact that the velocity profile in the hydraulic channel is sheared and not constant. Zitoun and Bouhadef [9], Guendouzen-Dabouz et al. [10] highlighted the influence of the distance, separating two drowned obstacles, on the phenomenon of gravity waves. Bouinoun and Bouhadef [11] used the conformal mapping theory to treat the nonlinear free-surface problem for an uneven bottom.

*C. Sinusoidal Arch*

Let's consider the flow over a sinusoidal obstacle defined by:

$$f(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{b}{2}\left(1 - \cos\dfrac{2\pi x}{L}\right) & 0 \le x \le L \\ 0 & L \le x \end{cases} \qquad (13)$$

Instead of the streamfunction, we consider the velocity potential $\Phi$ decomposed into the sum of a velocity potential corresponding to the horizontal bottom and a velocity potential of the disturbance.

$$\Phi = \varphi - Cx \text{ with } \Delta\Phi = 0 \rightarrow \Delta\varphi = 0 \tag{14}$$

The sliding condition on the bottom of equation:

$$y_f = -H + f(x) \tag{15}$$

gives:

$$\left(\frac{v}{u}\right)_{y=y_f} = f'(x) \tag{16}$$

where u and v are respectively the horizontal and vertical velocity components and f'(x) the first derivative of f(x). Thus:

$$f'(x) = -\frac{1}{c}\left(\frac{\partial}{\partial y}\right)_{y=-H} \tag{17}$$

On the free-surface $y_0(x)$, the cinematic boundary condition is written as:

$$\left(\frac{\partial\varphi}{\partial y}\right)_{y=0} = -C\frac{dy_0}{dx} \tag{18}$$

At the same level, the dynamic Bernoulli relation boundary condition gives:

$$y_0 = \frac{C}{g}\left(\frac{\partial\varphi}{\partial x}\right)_{y=0} \tag{19}$$

By combining the derivative of (19) with (18) we obtain:

$$\left(\frac{\partial^2\varphi}{\partial x^2}\right)_{y=0} + \frac{g}{C^2}\left(\frac{\partial\varphi}{\partial y}\right)_{y=0} = 0 \tag{20}$$

Using the Fourier transform, one can get the free-surface equation, for a subcritical regime (Fr<1), in the form:

$$\begin{aligned} y_0 = {} & \frac{\left(\frac{|x-L|}{x-L} - \frac{|x|}{x}\right)\left(\cos\frac{2\pi x}{L} - 1\right)}{\frac{2\pi H}{L}\left(\frac{2\pi H}{L}Fr^2 \operatorname{ch}\frac{2\pi H}{L} - \operatorname{sh}\frac{2\pi H}{L}\right)} + \\ & 2\operatorname{ch}\alpha\,\frac{\left(\frac{|x-L|}{x-L}+1\right)\left(\cos\alpha\frac{x-L}{H} - 1\right) - \left(\frac{|x|}{x}+1\right)\left(\cos\alpha\frac{x}{H} - 1\right)}{\left(\alpha^2 - \frac{4\pi^2H^2}{L^2}\right)\left(Fr^2\operatorname{ch}^2\alpha - 1\right)} \\ & + 2\sum\cos\beta_n\left(\frac{\left(\frac{|x-L|}{x-L}\right)\left(1 - e^{-\beta n\frac{|x-L|}{H}}\right) - \left(\frac{|x|}{x}\right)\left(1 - e^{-\beta n\frac{|x|}{H}}\right)}{\left(\beta n^2 + \frac{4\pi^2H^2}{L^2}\right)\left(Fr^2\cos^2\beta n - 1\right)}\right) \end{aligned} \tag{21}$$

For a supercritical regime (Fr>1), simply replace $\alpha$ by zero.

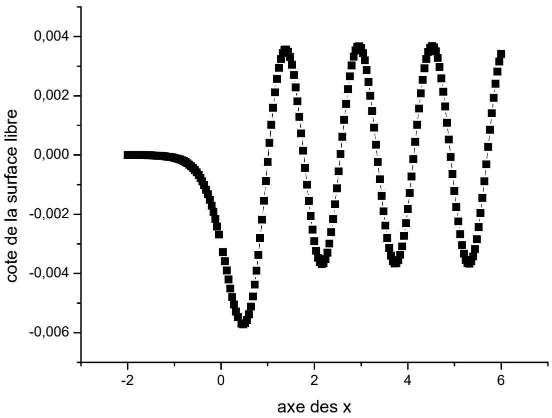


Fig. 6 Free-surface profile for a sinusoidal arch

It should be noted that the above solution was discontinuous. The expression of the free surface involves the derivative of the function describing the obstacle. Since this is continuous for each zone (upstream, on and downstream of the arch), it results that the solutions obtained form a family generated by inverts, this same continuous derivative, i.e. with one constant. To identify the solution to our problem, among all the other possible solutions, we use the boundary condition expressing the fact that the free-surface, far enough upstream, is horizontal at the same level as the undisturbed surface. It is this approach, from zone to zone, that we have adopted to obtain the right continuous solution.

## III. Conclusion

In this work, we have specified the contribution of each of the three poles of the complex function involved in the calculation of the free-surface. We were thus able to explain certain "anomalies" in particular cases of obstacles and identify their origin. Through this analytical solution, we have found results that are well known in the literature. On the other hand, the theoretical study we conducted for a very particular obstacle, namely a sinusoidal arch, made it possible, using the Fourier transform, to determine the analytical expression of the free surface. Since the obstacle has a specific characteristic, namely its period, it is shown that the amplitude of the generated waves is modulated by the parameters period and Froude number without however this amplitude becoming unbounded.

## Acknowledgment

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