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Vibration of a Beam on an Elastic Foundation Using the Variational Iteration Method

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***Abstract*—**Modelling of Timoshenko beams on elastic foundations has been widely used in the analysis of buildings, geotechnical problems, and, railway and aerospace structures. For the elastic foundation, the most widely used models are one-parameter mechanical models or two-parameter models to include continuity and cohesion of typical foundations, with the two-parameter usually considered the better of the two. Knowledge of free vibration characteristics of beams on an elastic foundation is considered necessary for optimal design solutions in many engineering applications, and in this work, the efficient and accurate variational iteration method is developed and used to calculate natural frequencies of a Timoshenko beam on a two-parameter foundation. The variational iteration method is a technique capable of dealing with some linear and non-linear problems in an easy and efficient way. The calculations are compared with those using a finite-element method and other analytical solutions, and it is shown that the results are accurate and are obtained efficiently. It is found that the effect of the presence of the two-parameter foundation is to increase the beam's natural frequencies and this is thought to be because of the shear-layer stiffness, which has an effect on the elastic stiffness. By setting the two-parameter model's stiffness parameter to zero, it is possible to obtain a one-parameter foundation model, and so, comparison between the two foundation models is also made.

***Keywords*—**Timoshenko beam, variational iteration method, two-parameter elastic foundation model.

I. Introduction

The use of beams resting and vibrating on elastic foundations is common in engineering applications in buildings, geotechnical applications, railway applications especially for soil-structures interaction, and aerospace structures [1]-[4]. For modelling of beams on elastic foundations, the Winkler mechanical model is well known, where the foundation is modelled using transverse, or, transverse and rotational springs, respectively. However, in the Winkler model, the springs are independent, so this model presents no cohesion, hence making the displacement localized directly under the applied load, which is a major drawback to the representation of the foundation.

To improve matters, two-parameter models were developed to include continuity and cohesion of typical foundations. One of the most popular ones is the Pasternak model which takes into account the foundation cohesion by a shear layer of incompressible vertical elements [5].

Research related to this work includes early studies of natural frequencies of a Timoshenko beam on a Pasternak foundation [6] and the use of a finite element method to analyse free vibrations and transient responses of a Timoshenko beam on both Winkler and Pasternak foundations [7]. Later studies include Chen et al. [8] who used a mixed method that combines the state space method and the differential quadrature method for free vibration of Euler-Bernoulli beams on a Pasternak foundation. Lee et al. [9] studied the flexural-torsional free vibrations of finite uniform beams resting on a finite Pasternak foundation. Recently, Ghannadiasi and Mofid [10] calculated an exact solution for natural frequencies of an elastically restrained Timoshenko beam on an arbitrary variable elastic foundation using the Green function.

The variational iteration method [11], [12] is used here as the method of solution and is capable of dealing with some linear and non-linear problems in an easy and efficient way [13]. The technique has already been used to solve engineering problems such as, the calculation of heat and wave-like equations [14], solutions for linear and non-linear waves [15], and, to calculate transverse natural frequencies of a Euler-Bernoulli beam [16].

In the present work, calculations are made to obtain natural frequencies of a Timoshenko beam on a Pasternak foundation with the results compared with those obtained in literature. The effect of the presence of the Pasternak foundation is examined, and comparison is also made with the results obtained using the simpler Winkler foundation.

II. Governing Equations

The following derivation is based on the Timoshenko beam theory, where shear and rotary effects are not negligible, and, also the Pasternak foundation model is illustrated in Fig. 1.

The potential energy of the beam/foundation system is [7]

$$U = \frac{1}{2}\int_0^l EI\left(\frac{\partial\theta(x,t)}{\partial x}\right)^2 \mathrm{d}x + \frac{1}{2}\int_0^l \kappa AG\left(\theta(x,t) - \frac{\partial w(x,y)}{\partial x}\right)^2 \mathrm{d}x + \frac{1}{2}\int_0^l k_w (w(x,t))^2 \mathrm{d}x + \frac{1}{2}\int_0^l k_p\left(\frac{\partial w(x,t)}{\partial x}\right)^2 \mathrm{d}x. \quad (1)$$

Here, l is the length of the beam, A is the cross-sectional area, I is the moment of inertia of the cross section, E is the modulus of elasticity, G is the modulus of rigidity, κ is the shear coefficient, k_w is the foundation stiffness coefficient, k_p is the foundation shear coefficient, $w(x,t)$ is the transverse deflection, and $\theta(x,t)$ is the beam slope due to bending at the axial location and time t.

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The kinetic energy of the system is

$$T = \frac{1}{2}\int_0^l \rho A \left(\frac{\partial w(x,t)}{\partial t}\right)^2 \mathrm{d}x + \frac{1}{2}\int_0^l \rho I \left(\frac{\partial \theta(x,t)}{\partial t}\right)^2 \mathrm{d}x, \quad (2)$$

where ρ is the mass per unit volume. The equation of motion can be obtained using the Hamilton's principle

$$\int_{t_1}^{t_2} \delta(T - U)\mathrm{d}t + \int_{t_1}^{t_2} \delta W \mathrm{d}t = 0. \quad (3)$$

Here, δW is the virtual work done, t_1 and t_2 are the times at which the configuration is known, and δ denotes the virtual change. On substituting (1) and (2) into (3), two coupled equations can be found for the free vibration response

$$\rho A \frac{\partial^2 w(x,t)}{\partial t^2} + \kappa AG \left(\frac{\partial \theta(x,t)}{\partial x} - \frac{\partial^2 w(x,t)}{\partial x^2}\right) + k_w w(x,t) - k_p \frac{\partial^2 w(x,t)}{\partial x^2} = 0, \quad (4)$$

$$EI \frac{\partial^2 \theta(x,t)}{\partial x^2} - \rho I \frac{\partial^2 \theta(x,t)}{\partial t^2} - \kappa AG \left(\theta(x,t) - \frac{\partial w(x,t)}{\partial x}\right) = 0. \quad (5)$$

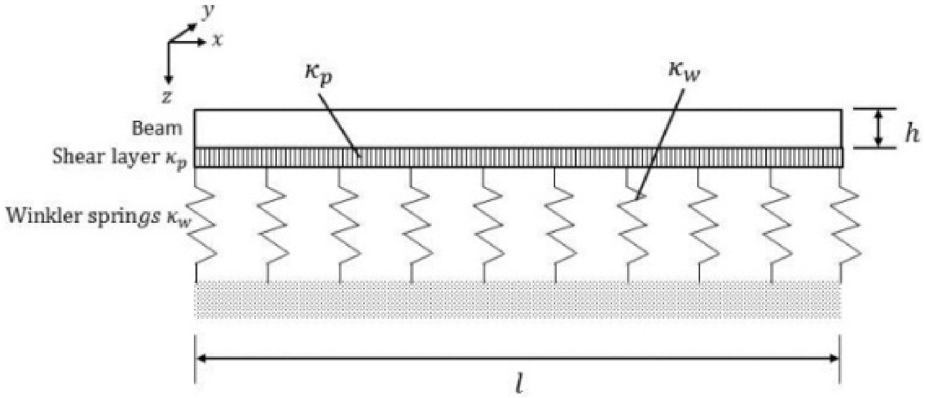

Fig. 1 A beam on a Pasternak foundation

TABLE I
BOUNDARY CONDITIONS FOR TIMOSHENKO BEAM

B.C.	Right-hand side		Left-hand side	
C-F	$\varphi\|_{\eta=0} = 0$	$\Theta\|_{\eta=0} = 0$	$d\varphi/d\eta\|_{\eta=1} - \Theta\|_{\eta=1} = 0$	$d\Theta/d\eta\|_{\eta=1} = 0$
C-C	$\varphi\|_{\eta=0} = 0$	$\Theta\|_{\eta=0} = 0$	$\varphi\|_{\eta=1} = 0$	$\Theta\|_{\eta=1} = 0$
P-P	$\varphi\|_{\eta=0} = 0$	$d\Theta/d\eta\|_{\eta=0} = 0$	$\varphi\|_{\eta=1} = 0$	$d\Theta/d\eta\|_{\eta=1} = 0$

The beam is harmonically excited with an angular frequency ω and

$$w(x,t) = W(x)e^{i\omega t}, \theta(x,t) = \Theta(x)e^{i\omega t} \quad (6)$$

where $i = \sqrt{-1}$. On substituting (6) into (4) and (5) gives

$$k_p \frac{\mathrm{d}^2 W(x)}{\mathrm{d}x^2} + \kappa GA \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}W(x)}{\mathrm{d}x} - \Theta(x)\right) - k_w W(x) + \rho A \omega^2 W(x) = 0, \quad (7)$$

$$EI \frac{\mathrm{d}^2 \Theta(x)}{\mathrm{d}x^2} + \kappa GA \left(\frac{\mathrm{d}W(x)}{\mathrm{d}x} - \Theta(x)\right) + \rho I \omega^2 \Theta(x) = 0. \quad (8)$$

The following non-dimensional parameters are now defined

$$\eta = \frac{x}{l}, \varphi = \frac{W}{l}, \mu = \Omega_n^2 = \frac{\rho A l^4}{EI}\omega^2, \gamma = \frac{I}{Al^2}, \nu = \frac{\kappa GAl^2}{EI}, \bar{k}_w = \frac{k_w l^4}{EI}, \bar{k}_p = \frac{k_p l^2}{EI}.$$

The non-dimensional form of the governing equations can be written as

$$\frac{\mathrm{d}^2 \varphi(\eta)}{\mathrm{d}\eta^2} - \frac{\nu}{\bar{k}_p}\frac{\mathrm{d}\Theta(\eta)}{\mathrm{d}\eta} - \left(\frac{\bar{k}_w - \mu}{\bar{k}_p}\right)\varphi(\eta) = 0, \quad (9)$$

$$\frac{\mathrm{d}^2 \Theta(\eta)}{\mathrm{d}\eta^2} + \nu\left(\frac{\mathrm{d}\varphi(\eta)}{\mathrm{d}\eta} - \Theta(\eta)\right) + \gamma\mu\Theta(\eta) = 0, \quad (10)$$

where μ is the non-dimensional natural frequency, ν is the non-dimensional shear deformation parameter, and γ is the non-dimensional rotary inertia parameter.

The governing equations for the natural frequencies of a Timoshenko beam, resting on a Pasternak foundation, are a system of differential equations. These can be solved using four boundary conditions, two of which may be prescribed at the right-hand side, and, two at the left-hand side.

Typical boundary conditions for clamped-free (C-F), clamped-clamped (C-C) and pinned-pinned (P-P) conditions are listed in Table I.

III. VARIATIONAL ITERATIONAL METHOD

A. Basic Idea

To illustrate the basic concept of the variational iteration method, consider

$$\mathrm{L}u + \mathrm{N}u = g(x), \quad (11)$$

where L is a linear operator, N is a non-linear operator, and

$g(x)$ is a forcing term. According to the variational iteration method [5], [6], the following correction functional can be constructed

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\mathrm{L}u_n(s) + \mathrm{N}\tilde{u}_n(s) - g(s))\,\mathrm{d}s, \quad (12)$$

where λ is a Lagrange multiplier which can be identified optimally via the variational iteration method. The subscripts n denote the n[th] approximation, and $\tilde{u}$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$.

In general, the application of the variational iteration method follows three steps: the establishment of the correction functional, the identification of the Lagrange multipliers, and the determination of the initial iteration.

B. Application

To find the closed-form solution for the system (9) and (10) is difficult, but the system is in fact amenable to some numerical techniques. In this work, the natural frequencies and associated mode shapes for a Timoshenko beam resting on a Pasternak foundation are found by developing the variational iteration method as an eigenvalue problem. To solve (9) and (10), the correction functionals are written as

$$\varphi_{n+1}(\eta) = \varphi_n(\eta) + \int_0^\eta \lambda_1(s)\left[\frac{\mathrm{d}^2\varphi_n(s)}{\mathrm{d}s^2} - \frac{\nu}{\bar{k}_p}\frac{\mathrm{d}\Theta_n(s)}{\mathrm{d}s} - \left(\frac{\bar{k}_w-\mu}{\bar{k}_p}\right)\varphi_n(s)\right]\mathrm{d}s, \quad (13)$$

$$\Theta_{n+1}(\eta) = \Theta_n(\eta) + \int_0^\eta \lambda_2(s)\left[\frac{\mathrm{d}^2\Theta_n(s)}{\mathrm{d}s^2} + \nu\left(\frac{\mathrm{d}\varphi_n(s)}{\mathrm{d}s} - \Theta_n(s)\right) + \gamma\mu\Theta_n(s)\right]\mathrm{d}s. \quad (14)$$

On making the correction functional stationary, the following conditions are found

$$1 - \left.\frac{\mathrm{d}\lambda_1(s)}{\mathrm{d}s}\right|_{s=\eta} = 1 - \left.\frac{\mathrm{d}\lambda_2(s)}{\mathrm{d}s}\right|_{s=\eta} = 0,$$
$$\lambda_1(s)|_{s=\eta} = \lambda_2(s)|_{s=\eta} = 0, \quad \frac{\mathrm{d}^2\lambda_1(s)}{\mathrm{d}s^2} = \frac{\mathrm{d}^2\lambda_2(s)}{\mathrm{d}s^2} = 0. \quad (15)$$

The Lagrange multipliers $\lambda_1(s)$ and $\lambda_2(s)$ can be identified from (15) as

$$\lambda_1(s) = \lambda_1(s) = s - \eta. \quad (16)$$

To start the iterative process, the first term of each series, $\varphi_0(\eta)$ and $\Theta_0(\eta)$ are written as

$$\varphi_0(\eta) = a_1\eta + a_2, \quad (17)$$
$$\Theta_0(\eta) = b_1\eta + b_2$$

where a_1, a_2, b_1, b_2 are unknown constants with some of them being determined from boundary conditions. Using the iteration formula, the following can be obtained

$$\varphi_1(\eta) = \varphi_0(\eta) + \int_0^\eta (s-\eta)\left[\frac{\mathrm{d}^2\varphi_0(s)}{\mathrm{d}s^2} - \frac{\nu}{\bar{k}_p}\frac{\mathrm{d}\Theta_0(s)}{\mathrm{d}s} - \left(\frac{\bar{k}_w-\mu}{\bar{k}_p}\right)\varphi_0(s)\right]\mathrm{d}s, \quad (18)$$

$$\Theta_1(\eta) = \Theta_0(\eta) + \int_0^\eta (s-\eta)\left[\frac{\mathrm{d}^2\Theta_0(s)}{\mathrm{d}s^2} + \nu\left(\frac{\mathrm{d}\varphi_0(s)}{\mathrm{d}s} - \Theta_0(s)\right) + \gamma\mu\Theta_0(s)\right]\mathrm{d}s, \quad (19)$$

$$\varphi_2(\eta) = \varphi_1(\eta) + \int_0^\eta (s-\eta)\left[\frac{\mathrm{d}^2\varphi_1(s)}{\mathrm{d}s^2} - \frac{\nu}{\bar{k}_p}\frac{\mathrm{d}\Theta_1(s)}{\mathrm{d}s} - \left(\frac{\bar{k}_w-\mu}{\bar{k}_p}\right)\varphi_1(s)\right]\mathrm{d}s, \quad (20)$$

$$\Theta_2(\eta) = \Theta_1(\eta) + \int_0^\eta (s-\eta)\left[\frac{\mathrm{d}^2\Theta_1(s)}{\mathrm{d}s^2} + \nu\left(\frac{\mathrm{d}\varphi_1(s)}{\mathrm{d}s} - \Theta_1(s)\right) + \gamma\mu\Theta_1(s)\right]\mathrm{d}s, \quad (21)$$

$$\varphi_k(\eta) = \varphi_{k-1}(\eta) + \int_0^\eta (s-\eta)\left[\frac{\mathrm{d}^2\varphi_{k-1}(s)}{\mathrm{d}s^2} - \frac{\nu}{\bar{k}_p}\frac{\mathrm{d}\Theta_{k-1}(s)}{\mathrm{d}s} - \left(\frac{\bar{k}_w-\mu}{\bar{k}_p}\right)\varphi_{k-1}(s)\right]\mathrm{d}s, \quad (22)$$

$$\Theta_k(\eta) = \Theta_{k-1}(\eta) + \int_0^\eta (s-\eta)\left[\frac{\mathrm{d}^2\Theta_{k-1}(s)}{\mathrm{d}s^2} + \nu\left(\frac{\mathrm{d}\varphi_{k-1}(s)}{\mathrm{d}s} - \Theta_{k-1}(s)\right) + \gamma\mu\Theta_{k-1}(s)\right]\mathrm{d}s. \quad (23)$$

After $\varphi_k(\eta)$ and $\Theta_k(\eta)$ are obtained, the solution for (9) and (10) can be stated as

$$\varphi(\eta) = \lim_{k\to\infty}\varphi_k(\eta), \quad \Theta(\eta) = \lim_{k\to\infty}\Theta_k(\eta) \quad (24)$$

As ∞ is not possible, a large number n is used according to the accuracy required.

The substitution of (24) into the boundary condition given in Table I produces four simultaneous equations, which can be given in matrix form as

$$[A][B] = [0] \quad (25)$$

where $[A]$ is a four by four matrix and $[B] = \langle a_1, a_2, b_1, b_2\rangle^T$.

For non-trivial solutions, the determinant of the matrix $[A]$ must be equal to zero, giving a polynomial for the eigenvalue μ and hence ω. The modal shapes associated with the natural frequencies can be readily calculated, and in this work, the mode shape is normalised using [17]

$$\hat{\varphi}(\eta) = \frac{\varphi(\eta)}{\sqrt{\int_0^1|\varphi(\eta)|^2\mathrm{d}\eta}}, \quad \hat{\Theta}(\eta) = \frac{\Theta(\eta)}{\sqrt{\int_0^1|\Theta(\eta)|^2\mathrm{d}\eta}} \quad (26)$$

IV. Numerical Examples

A. Convergence

The efficiency of obtaining converged solutions was important for this study. An example of the rate of convergence is given on Fig. 2 for the case of clamped-clamped boundary conditions, with $\bar{k}_p = 5, \bar{k}_w = 100$ and $\gamma = 2.5\times10^{-3}$. As can be seen, the variational iteration method is proven to be fast with convergence for the first mode obtained after very few iterations. Convergence for higher modes took longer time.

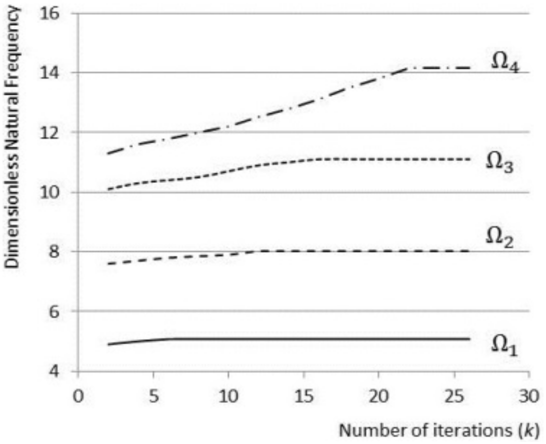

Fig. 2 Convergence for the first four natural frequencies when $\bar{k}_p = 5, \bar{k}_w = 100, \gamma = 2.5 \times 10^{-3}$

B. Clamped-Clamped Boundary Conditions

The following is for a beam with clamped-clamped (C-C) boundary condition of rectangular cross-section and unit breath resting on an elastic foundation. The results are given for both slender thin beams and short thick beams and it can be seen that the natural frequency results for the Euler-Bernoulli beam agree well with those of [18].

It can be seen from Table II that the natural frequencies increase with the increasing Pasternak foundation parameter for a given Winkler foundation parameter. Also, as the beams become slenderer, the natural frequency increases for a given Pasternak and Winkler parameter.

TABLE II
FIRST THREE NATURAL FREQUENCIES FOR C-C BEAMS

γ	$\bar{k}_w$	$\bar{k}_p$ 0	5	25
$1/416$	0	22.3733 (22.3729)	23.6877 (23.7072)	28.2971 (28.3024)
		61.6728 (61.6853)	63.4890 (63.4890)	70.2445 (70.2244)
		120.903 (120.9120)	122.800 (122.8993)	130.485 (130.6449)
	10,000	102.454 (102.475)	102.759 (102.759)	103.918 (103.917)
		117.462 (117.484)	118.418 (118.440)	122.189 (122.213)
		156.775 (156.901)	158.332 (158.458)	164.352 (164.481)
$1/17$	0	18.190	19.536	24.110
		41.800	43.904	51.380
		54.775	54.878	55.205
	10,000	54.834	54.760	54.967
		73.103	74.650	76.913
		102.212	109.202	106.214

Results in brackets are those of [18].

C. Pinned-Pinned Boundary Conditions

Results using the variational iteration method (VIM) are now compared with those obtained using finite element analysis [19] for a simply supported (P-P) uniform beam with finite length, and $l = 0.5$m, $E = 210$ GPa, $G = 80.8$ GPa, $\kappa = 5/6$, $\rho = 7850$ kg/m^3.

Table III shows the natural frequencies for the beam without a foundation and comparison of the VIM calculation method with results found by the finite element method [19] when 70 elements were used. It can be seen again that the natural frequencies increase when there is a Pasternak foundation, i.e. the presence of the elastic stiffness and shear layer increment the beam stiffness, and hence, the natural frequencies increase. As mode number increases, however, this increase in natural frequency reduces.

TABLE III
FIRST THREE NATURAL FREQUENCIES FOR P-P BEAMS

Mode	$\gamma = 0.04$, VIM $\bar{k}_w = \bar{k}_p = 0.0$	VIM $\bar{k}_w = \bar{k}_p = 2.5$	FEM [19] $\bar{k}_w = \bar{k}_p = 2.5$
1	3959.011	5209.460	5209.242
2	14610.321	15965.466	15965.908
3	29574.219	31051.876	31057.635

Fig. 3 presents the ratio of the natural frequency found for the Pasternak foundation ($\bar{k}_p = 10$) at various values of $\bar{k}_w$ to that found for a Pasternak foundation ($\bar{k}_p = 10, \bar{k}_w = 0$) for the first three modes. It can be seen that there is an increase in the natural frequencies of the Timoshenko beam as the Winkler parameter increases, with the first mode being significantly affected. For higher modes increasing the Winkler parameter has not such a great effect. Also shown on Fig. 3 is the effect of increasing the Pasternak parameter of the natural frequencies while keeping the Winkler parameter constant. Increase in the Pasternak parameter has more effect on the natural frequencies than experienced by increasing the Winkler parameter with an increase in the first mode clearly affected. The higher modes are also considerably affected by the $\bar{k}_P$ value.

D. Clamped-Free Boundary Conditions

From the variational iteration method, a polynomial can be obtained to describe the first mode shape function. The same procedure can be employed for other natural frequencies. The variation of the first and third mode shapes for various Winkler parameters is illustrated on Fig. 4, where the Pasternak parameter is set to zero. It can be seen from the third mode that, by increasing $\bar{k}_w$, both the amplitude and the phase of the shape function are both affected.

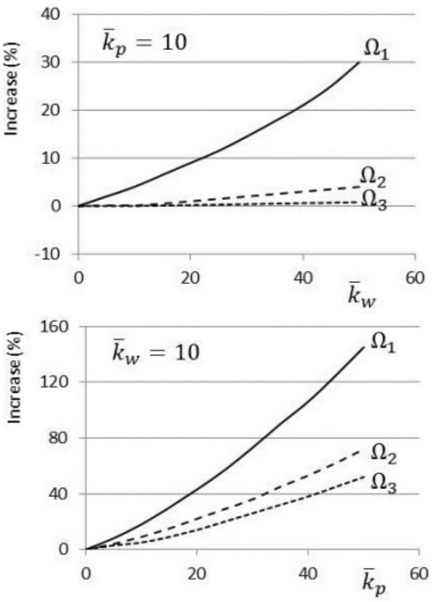

Fig. 3 Increases in natural frequencies due to the Winkler and Pasternak parameters

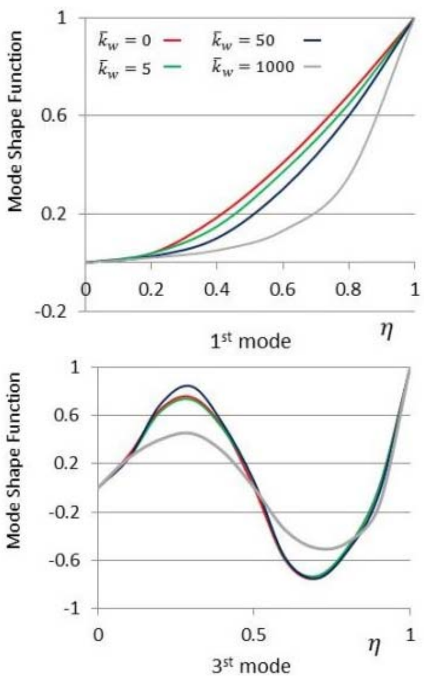

Fig. 4 First and third mode shapes for various Winkler parameter values

V. Conclusion

Beams resting on a Pasternak foundation are an important component of many mechanical, civil and geotechnical engineering applications. An accurate knowledge of natural frequencies and mode shapes of such components is important in engineering practice. Although this problem has been already extensively studied, most of the techniques in use are based on perturbation or discretization of the governing equations, so leading to tedious and sometimes complex calculations. An alternative procedure based on the variational iteration method is proposed in this paper, and the numerical results show that the convergence of the method is efficient with the ensuing results agreeing satisfactorily with established results found in the literature.

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