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A Quadratic Approach for Generating Pythagorean Triples

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Abstract**—**The article explores one of the important relations between numbers-the Pythagorean triples (triplets) which finds its application in distance measurement, construction of roads, towers, buildings and wherever Pythagoras theorem finds its application. The Pythagorean triples are numbers, that satisfy the condition "In a given set of three natural numbers, the sum of squares of two natural numbers is equal to the square of the other natural number". There are numerous methods and equations to obtain the triplets, which have their own merits and demerits. Here, quadratic approach for generating triples uses the hypotenuse leg difference method. The advantage is that variables are few and finally only three independent variables are present.

Keywords**—**Arithmetic progression, hypotenuse leg difference method, natural numbers, Pythagorean triplets, quadratic equation.

I. Introduction

THE method of generating Pythagorean triples is known for about 2000 years [1]. Some of the famous approaches are Euclid's formula, [2] Berggren approach [3], Matrix method etc. [4], [5] and have their own merits and demerits. New theorems are built to define as many triples as possible and have been successful up to a certain level. The proposed technique approaches the triples in "difference method". Similar approaches have been done by various Mathematicians [6]-[8], but this kind of generalization is first of its kind. Difference method is implacable for both even, odd dependency and Arithmetic series with first and second order taken into the consideration, which relates each other with a representation of any natural number, by product of two natural numbers, where one is a non-square and other is a perfect square number. Many tables and at the end Graphical representation of these numbers have been obtained [9].

II. Difference Method

If *a*, *b* and *c* are Natural numbers satisfying the condition

$$a^2+b^2=c^2 \tag{1}$$

Then '*a*',' *b*' and '*c*' together are called as Pythagorean triples. Suppose:

$$c - b = d \tag{2}$$

TABLE I
Pythagorean Triples up to 100

3	4	5	27	36	45	24	70	74	28	96	100
6	8	10	30	40	50	45	60	75			
5	12	13	14	48	50	21	72	75			
9	12	15	24	45	51	30	72	78			
8	15	17	20	48	52	48	64	80			
12	16	20	28	45	53	18	80	82			
15	20	25	33	44	55	51	68	85			
7	24	25	40	42	58	40	75	85			
10	24	26	36	48	60	36	77	85			
20	21	29	11	60	61	13	84	85			
18	24	30	39	52	65	60	63	87			
16	30	34	33	56	65	39	80	89			
21	28	35	25	60	65	54	72	90			
12	35	37	16	63	65	35	84	91			
15	36	39	32	60	68	57	76	95			
24	32	40	42	56	70	65	72	97			
9	40	41	48	55	73	60	80	100			

a	b	c	Difference (c-b)	a	b	c	Difference (c-b)
3	4	5	1	12	9	15	6
5	12	13	1	18	24	30	6
7	24	25	1	24	45	51	6
9	40	41	1	30	72	78	6
11	60	61	1	21	28	35	7
13	84	85	1	35	84	91	7
4	3	5	2	12	5	13	8
6	8	10	2	16	12	20	8
8	15	17	2	20	21	29	8
10	24	26	2	24	32	40	8
12	35	37	2	28	45	53	8
14	48	50	2	32	60	68	8
16	63	65	2	36	77	85	8
18	80	82	2	15	8	17	9
9	12	15	3	21	20	29	9
15	36	39	3	27	36	45	9
21	72	75	3	33	56	65	9
8	6	10	4	39	80	89	9
12	16	20	4	20	15	25	10
16	30	34	4	30	40	50	10
20	48	52	4	40	75	85	10
24	70	74	4	33	44	55	11
28	96	100	4	24	18	30	12
15	20	25	5	36	48	60	12
25	60	65	5	39	52	65	13

Fig. 1 List of triples with Successive hypotenuse leg difference

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Since these numbers are related as second order equation (Quadratic equation) it can be represented as

$$a = a_2 n^2 + a_1 n + a_0 \quad (3)$$

$$b = b_2 n^2 + b_1 n + b_0 \quad (4)$$

$$c = c_2 n^2 + c_1 n + c_0 \quad (5)$$

where n is also a natural number which represents the n^{th} number from a series of numbers while keeping d as a constant. Since d is independent of n, it can written as

$$c_0 = b_0 = d \quad (6)$$

$$c_2 = b_2 \quad (7)$$

$$c_1 = b_1 \quad (8)$$

Substituting these values in (3)-(5) and (1), relations between variables can be obtained as follows

$$a_0^2 = \frac{d b_1^2}{2 b_2} \quad (9)$$

$$a_1^2 = 2 b_2 d \quad (10)$$

$$a_2 = 0 \quad (11)$$

$$b_0 = \frac{1}{2}\left(\frac{b_1^2}{2 b_2} - d \right) \quad (12)$$

This implies,

$$a, b, c \in f(d, n) \quad (13)$$

$$a_0, b_0, a_1 \in f(d, b_1, b_2) \quad (14)$$

From the above equations, b_1, b_2 are proved to be functions of d. To get the values of a_x, b_x, c_x (x = [0, 1, 2]) examples can be taken from calculated table and can be verified as

$$b_0 = 0 \quad (15)$$

$$a_0 = c_0 = d \quad (16)$$

$$a_1 = b_1 = c_1 \quad (17)$$

To satisfy a, b and $c \in \mathbf{N}$, it is impossible to obtain generalized equation from the above relation. The Pythagorean triples are computed with the help of Microsoft Excel, then analyzed and verified with the MATLAB code. The first and foremost, the expressions for d, b_1 and b_2 have to be known. In order to find the expression for d, b_1 and b_2, the concept of representation of any Natural number with product of two Natural numbers where one is not a perfect square number and other is a perfect square is taken. Using this d is represented as

$$d = k m^2 \quad (18)$$

$k, m \in \mathbf{N}$ and k should be kept as small as possible and m should be kept as large as possible. Analyzing k and m, the values of b_2, b_1 are obtained.

TABLE II
EQUATIONS FOR A, B, C UP TO A DIFFERENCE OF 25

d	a	b	c
1	2n+1	2n(n+1)	2n(n+1)+1
2	2n+2	n(n+2)	n(n+2)+2
3	3(2n+1)	6n(n+1)	6n(n+1)+3
4	2(2n+2)	2n(n+2)	2n(n+2)+4
5	5(2n+1)	10n(n+1)	10n(n+1)+5
6	3(2n+2)	3n(n+2)	3n(n+2)+6
7	7(2n+1)	14n(n+1)	14n(n+1)+7
8	4(n+2)	n(n+4)	n(n+4)+8
9	3(2n+3)	n(2n+6)	n(2n+6)+9
10	5(2n+2)	5n(n+2)	5n(n+2)+10
11	11(2n+1)	22n(n+1)	22n(n+1)+11
12	6(2n+2)	6n(n+2)	6n(n+2)+12
13	13(2n+1)	26n(n+1)	26n(n+1)+13
14	7(2n+2)	7n(n+2)	7n(n+2)+14
15	15(2n+1)	30n(n+1)	30n(n+1)+15
16	8(n+2)	2n(n+4)	2n(n+4)+16
17	17(2n+1)	34n(n+1)	34n(n+1)+17
18	6(n+3)	n(n+6)	n(n+6)+18
19	19(2n+1)	38n(n+1)	38(n+1)+19
20	10(2n+2)	10n(n+2)	10n(n+2)+20
21	21(2n+1)	42n(n+1)	42n(n+1)+21
22	11(2n+2)	11n(n+2)	11(n+2)+22
23	23(2n+1)	46n(n+1)	46n(n+1)+23
24	12(n+2)	3n(n+4)	3n(n+4)+24
25	5(2n+5)	2n(n+5)	2n(n+5)+25

$$b_2 = \begin{cases} k/2, & k \text{ is even} \\ 2k, & k \text{ is odd} \end{cases} \quad (19)$$

$$b_1 = \begin{cases} km, & k \text{ is even} \\ 2km, & k \text{ is odd} \end{cases} \quad (20)$$

Thus a, b, c can be written in terms of k and m and it satisfies all the criteria and holds good for all values of $d \in \mathbf{N}$.

$$a = a_1 n + a_0$$
$$b = b_2 n^2 + b_1 n$$
$$c = c_2 n^2 + c_1 n + c_0$$
If k is even
$$a = kmn + km^2$$
$$b = (k/2)n^2 + kmn$$
$$c = (k/2)n^2 + kmn + km^2$$
If k is odd
$$a = 2kmn + km^2$$
$$b = 2kn^2 + 2kmn$$
$$c = 2kn^2 + 2kmn$$

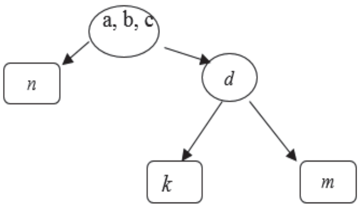

Fig. 2 Variable dependency flow. ▭ represents independent variables, ○ represents dependent variable

III. Analysis

The above equations are obtained by keeping d as constant, which gives series of triples (many a, b and c). Among these are the series of a, b and c, forming arithmetic series, both first as well as second order. For arithmetic series with first order formed by set of a, difference is given by $2km$ for odd values of k and km for even values of k. Also first term is given by $(m+1)km$ for even $(2+m)km$. Second order arithmetic series formed by the b, second order difference is given by k, if k is even and $4k$, if k is odd and first term is given as $(k/2)*(1+2m)$ for even values of k and $2k(1+m)$ for odd values of k. Second order arithmetic series formed by $\boldsymbol{c}$ that also follows the same values of b for second order difference, but first term gets added with km^2. Hence, geometrically for a constant d, series of a will form a line with slope equals to km with y intercept as difference km^2 and b forms an upper parabola with vertex $(-m,0)$ for even values of k and $((-m/2),0)$ for odd values of k.

At the same time c also forms a parabola with vertex $(-m, km^2/2)$ for even values of k and

$(-m/2, km^2/2)$ for odd values of k.

c-b=1=d		
a	b	c
3	4	5
5	12	13
7	24	25
9	40	41
11	60	61
13	84	85

b difference analysis		
b	1 st difference	2 nd difference
4		
12	8	
24	12	4
40	16	4
60	20	4
84	24	4

c difference analysis		
c	1 st difference	2 nd difference
5		
13	8	
25	12	4
41	16	4
61	20	4
85	24	4

c-b=2=d		
a	b	c
4	3	5
6	8	10
8	15	17
10	24	26
12	35	37
14	48	50
16	63	65
18	80	82

b difference analysis		
b	1 st difference	2 nd difference
3		
8	5	
15	7	2
24	9	2
35	11	2
48	13	2
63	15	2
80	17	2

c difference analysis		
c	1 st difference	2 nd difference
5		
10	5	
17	7	2
26	9	2
37	11	2
50	13	2
65	15	2
82	17	2

Fig. 3 Difference analysis

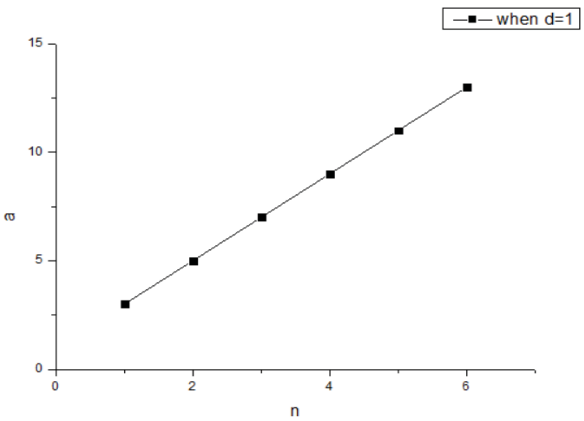

Fig. 4 a plotted against n, when d=1

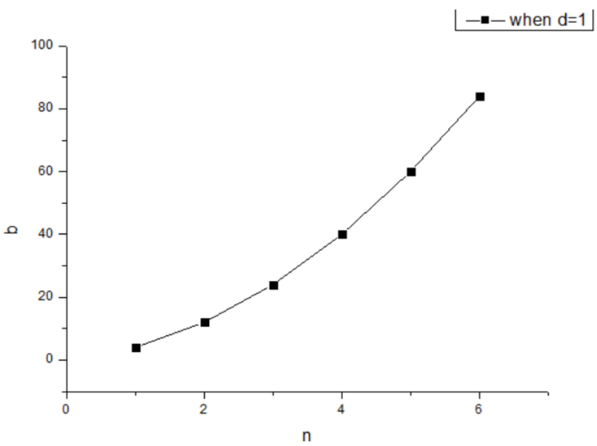

Fig. 5 b plotted against n, when d=1

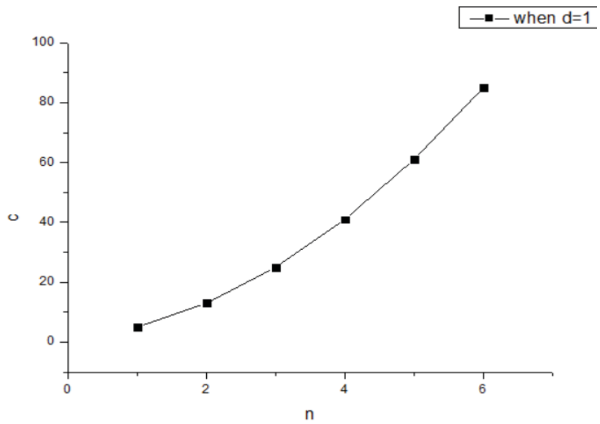

Fig. 6 c plotted against n, when d=1

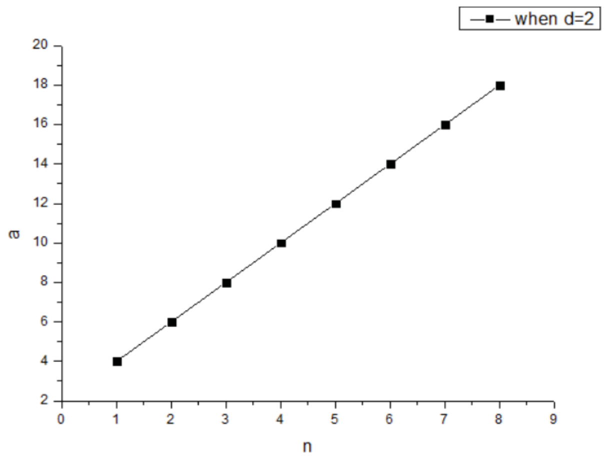

Fig. 7 a plotted against n, when d=2

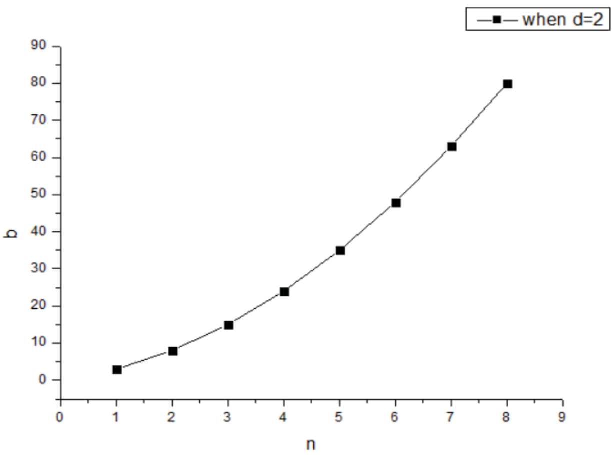

Fig. 8 b plotted against n, when d=2

IV. Conclusion

From the above method all the natural numbers are obtained, variables are few and finally only 3 independent variables present, are some of the advantages. Whenever applications like distance measurement, construction of roads, towers, buildings come to the picture, where Pythagoras theorem is applied one can use these equations.

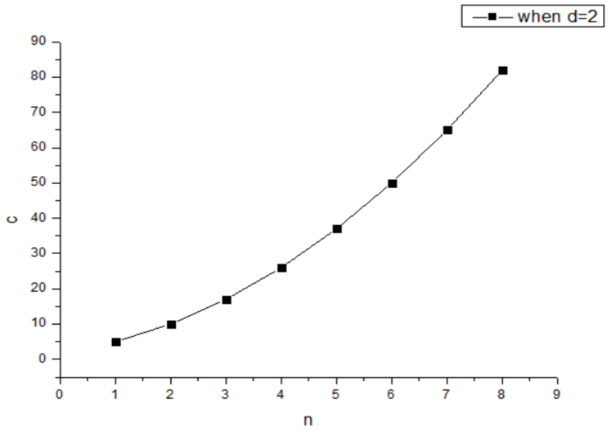

Fig. 9 c plotted against n, when d=2

References

[1] William J. Spezeski "Rethinking Pythagorean Triples" Applications and Applied Mathematics: An International Journal (AAM) ISSN: 1932-9466 Vol. 3, Issue 1 (June 2008), pp.100–112 (Previously, Vol. 3, No. 1).

[2] Coxeter, H. S. M. (1989). Introduction to geometry (2nd ed.). New York: John Wiley & Sons. (First Edition 1969).

[3] Mack, J., & Czernezkyj, V. (2010). The tree in Pythagoras' garden. Australian Senior Mathematics Journal, 24(2), 58–63.

[4] Martin William Bredenkamp: Matrices that define series of Pythagorean triples that have a triangle with one irrational side as limit. European Scientific Journal September 2014 /SPECIAL/ edition Vol.3 ISSN: 1857 – 7881 (Print) e – ISSN 1857- 7431 388.

[5] W.-K. Chen, Linear Networks and Systems (Book style). Belmont, CA: Wadsworth, 1993, pp. 123–135.

[6] Manuel Benitoa;1, Juan L.Varonab; ∗; Pythagorean triangles with legs less than n Journal of Computational and Applied Mathematics 143 (2002) 117–12.

[7] Gerstein, L. J. (2005). Pythagorean triples and inner products. Mathematics Magazine, 78(3), 205–213.

[8] John Roe, Elementary Geometry, Oxford University Press Inc., New York 1993.

[9] Pythagorean Triples and Cryptographic Coding by Subhas Kak Submitted on 21 Apr 2010 (v1), last revised 30 Sep 2010 (this version, v2) https://arxiv.org/ftp/arxiv/papers/1004/1004.3770.pdf.