International Journal of Mechanical, Industrial and Aerospace Sciences
ISSN: 2517-9950
Vol:11, No:10, 2017

Numerical Modeling of Wave Run-Up in Shallow Water Flows Using Moving Wet/Dry Interfaces

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Abstract**—**We present a new class of numerical techniques to solve shallow water flows over dry areas including run-up. Many recent investigations on wave run-up in coastal areas are based on the well-known shallow water equations. Numerical simulations have also performed to understand the effects of several factors on tsunami wave impact and run-up in the presence of coastal areas. In all these simulations the shallow water equations are solved in entire domain including dry areas and special treatments are used for numerical solution of singularities at these dry regions. In the present study we propose a new method to deal with these difficulties by reformulating the shallow water equations into a new system to be solved only in the wetted domain. The system is obtained by a change in the coordinates leading to a set of equations in a moving domain for which the wet/dry interface is the reconstructed using the wave speed. To solve the new system we present a finite volume method of Lax-Friedrich type along with a modified method of characteristics. The method is well-balanced and accurately resolves dam-break problems over dry areas.

Keywords**—**Run-up waves, Shallow water equations, finite volume method, wet/dry interface, dam-break problem.

I. Introduction

SHALLOW water equations have a very wide range of applications in coastal, environmental engineering and hydraulics. These equations have also been used to model complex flows including shocks and discontinuity. Development of numerical method to solve shallow water equations has been a very active field of research for the last decade. However, the most challenging problem remain the solution of the wave run-up and treatment of wet/dry interfaces in the shallow water flows. Many problems of interest involve wetting and drying zones occur for example, in the inter-tidal flats and/or coastal flood pains [1]. The difficulty is in numerically model the dry area where no water exists in these area, and the depth- averaged velocity components are normally determined by dividing the discharge per unit width, which can leads to predictions of unacceptable negative water depth and numerical instability [2].

In recent years, attention is widely given to the solution of the wet/dry interface problem in shallow water flows using approximate Riemann solver including HLL scheme [3], Roe scheme [4], Osher scheme [5] and a weighted average schemes [6]-[8]. Numerical simulations have also been performed to understand the effect of tsunami waves impact run-up in the presence of coastal slopes [9]. However, when these methods are employed to solve shallow water flows over wetting and drying areas, numerical inaccuracies usually occur at the wet/dry interfaces due to the loss of entropy property in the discretization. To overcome this difficulty, many techniques and methods have been developed but some of the resultant methods may not maintain the conservation property in the process of enforcing states at the moving wet/dry interface so as to suppress any undesired numerical oscillations. The objective of this study is to develop a stable, monotone and accurate numerical method able to approximate solutions to shallow water flows over wetting and drying areas.

In the current work, we propose a method for solving moving wet/dry interfaces in shallow water equations using a new system coordinates and the point-wise Riemann solver. Here, we reformulate the shallow water equations in a new set of moving coordinates and treat them as the model variables to be predicted at every time step. To solve the new shallow water system we consider a well-balanced finite volume method. The object of this study is to develop a numerical approach able to accurately approximate solutions to moving wet/dry fronts in shallow water flows. Our aim is to develop a class of numerical methods that are simple, easy to implement, and accurately solve the moving wet/dry fronts in shallow water flows without relying on complicated techniques. The proposed finite volume method can be interpreted as a fractional-stage scheme. In the first stage, the transport terms are solved by integrating the system along the characteristics defined by the interface velocity, while the numerical solutions are computed through a finite volume formulation of flux form in the second stage. Numerical results are presented for dam-break problems over dry areas using both flat and non-flat bed topography.

II. A New System of Shallow Water Equations

In this study we consider shallow water flows over dry areas as illustrated in Fig. 1. The conventional governing equations for this class of flows consist of

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} &= 0, && x \in [0,a], \\ \frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{h} + \frac{1}{2}gh^2\right) &= -gh\frac{\partial z}{\partial x}, && x \in [0,a], \end{aligned} \tag{1}$$

where $[0,a]$ is the flow domain, $z(x)$ is the bottom topography, $h(t,x)$ is the water height above the bottom, g is the acceleration due to gravity and $q(t,x)$ is the flow discharge, see Fig. 1. Equation (1) have been widely used to model water flows, flood waves and dam-break problems over wetted

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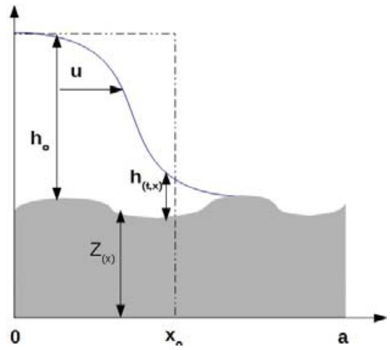

Fig. 1 Illustration of a shallow water system over dry area

domains. In the present work we are interested in solving shallow water flows over dry areas and therefore (1) are solved subject to the following initial condition

$$h(0,x) = \begin{cases} h_0, & \text{if} \quad x \leq x_0, \\ 0, & \text{if} \quad x > x_0, \end{cases} \tag{2}$$

where $x_0 \in [0,a]$ is the initial location of the wet/dry interface and h_0 is a given water height. It should be stressed that most of numerical methods for solving (1), (2) perturb the dry state using a wetted threshold above which the solution state is considered to be dry. This is mainly used to avoid division by zero for updating the water velocity u during the simulation process. However, perturbing the water height may result in inaccuracy in the computed solutions and may lead to false location of the wet/dry fronts on the coastal zones. In the current study we reformulate (1), (2) in a moving domain and solve the obtained model only for the wetted area and the advection of the wet/dry interface is obtained using the speed of the water. Thus, we solve the following system

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} &= 0, \qquad x \in \Big[0, \chi(t)\Big], \\ \frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{h} + \frac{1}{2}gh^2\right) &= -gh\frac{\partial z}{\partial x}, \quad x \in \Big[0, \chi(t)\Big], \end{aligned} \tag{3}$$

where the interface $\chi(t)$ is defined by the ordinary differential equation

$$\dot{\chi}(t) = |u| + \sqrt{g\,h}, \qquad \chi(0) = x_0. \tag{4}$$

Next we introduce the new coordinate

$$y = \frac{x}{\chi(t)},$$

and the shallow water equations (3) are rewritten in the new coordinates as

$$\begin{aligned} \chi\frac{DH}{Dt} + \frac{\partial Q}{\partial y} &= 0, \qquad y \in [0,1], \\ \chi\frac{DQ}{Dt} + \frac{\partial}{\partial y}\left(\frac{Q^2}{H} + \frac{1}{2}gH^2\right) &= -gH\frac{\partial Z}{\partial y}, \quad y \in [0,1], \end{aligned} \tag{5}$$

where $H(t,y) = h\left(t, y\left(\chi(t)\right)\right)$, $U(t,y) = u\left(t, y\left(\chi(t)\right)\right)$, $Z(t,y) = z\left(t, y\left(\chi(t)\right)\right)$ and the advective derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} - \frac{\dot{\chi}\,y}{\chi}\frac{\partial}{\partial y}. \tag{6}$$

For simplicity in the presentation we rewrite (5) in a conservative form as

$$\chi\frac{D\mathbf{W}}{Dt} + \frac{\partial \mathbf{F}(\mathbf{W})}{\partial y} = \mathbf{S}(\mathbf{W}), \tag{7}$$

where $\mathbf{W} = (H,Q)^T$,

$$\mathbf{F}(\mathbf{W}) = \begin{pmatrix} HU \\ \dfrac{Q^2}{H} + \dfrac{1}{2}gH^2 \end{pmatrix}, \quad \mathbf{S}(\mathbf{W}) = \begin{pmatrix} 0 \\ -gH\dfrac{\partial Z}{\partial x} \end{pmatrix}.$$

It is clear that the shallow water equations (5) are now solved for (H, HU) in the fixed domain $y \in [0,1]$.

III. Numerical Procedures

We solve (7) we consider a fractional step procedure for which the advective part is decoupled from the conservative part in the temporal discretization. Thus, at each time step the new water height and discharge are updated by solving first the advective equation

$$\chi\frac{D\mathbf{W}}{Dt} = 0, \qquad y \in [0,1], \tag{8}$$

followed by solving the conservation system

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{W})}{\partial y} = \mathbf{S}(\mathbf{W}), \qquad y \in [0,1]. \tag{9}$$

Hence, we discretize the space domain in cells $[y_{i-1/2}, y_{i+1/2}]$ with same length Δy and we divide the time interval into subintervals $[t_n, t_{n+1}]$ with uniform size Δt. Here, $t_n = n\Delta t$, $y_{i-1/2} = i\Delta y$ and $y_i = (i+1/2)\Delta y$ is the center of the control volume. We use the notation $w_i(t)$ to denote the space average of a function $\mathbf{W}(t,x)$ in the cell $[y_{i-1/2}, y_{i+1/2}]$ at time t, by $\mathbf{W}_i^n = w_i(t^n)$, and by $\mathbf{W}_{i+1/2}$ to denote the numerical flux at $y = y_{i+1/2}$ and time t,

$$w_i(t) = \frac{1}{\Delta y}\int_{y_{i-1/2}}^{y_{i+1/2}} \mathbf{W}(y,t)dy, \qquad \mathbf{W}_{i+1/2} = \mathbf{W}(y_{i+1/2}, t).$$

To solve the advection equation (8) we used the well-established method of characteristics. Thus, for each mesh point $y_{i+1/2}$ the characteristic curves $Y_{i+1/2}$ associated with (6) are the solutions of the initial-value problem

$$\begin{aligned} \frac{dY_{i+1/2}(\tau)}{d\tau} &= v_{i+1/2}\Big(\tau, Y_{i+1/2}(\tau)\Big), \quad \tau \in [t_n, t_{n+1}], \\ Y_{i+1/2}(t_{n+1}) &= y_{i+1/2}, \end{aligned} \tag{10}$$

where $v(\tau,y) = -\dfrac{\dot{\chi}(\tau)y}{\chi(\tau)}$. To solve the ordinary differential equations (10) we use the standard second-order Runge-Kutta scheme. Once the characteristics curves $Y_{i+1/2}(t_n)$ are known, the method of characteristics advects the solution of (8) at instant t_{n+1} as

$$\widetilde{\mathbf{W}}\left(t_{n+1}, y_{i+1/2}\right) = \mathbf{W}\left(t_n, Y_{i+1/2}(t_n)\right). \tag{11}$$

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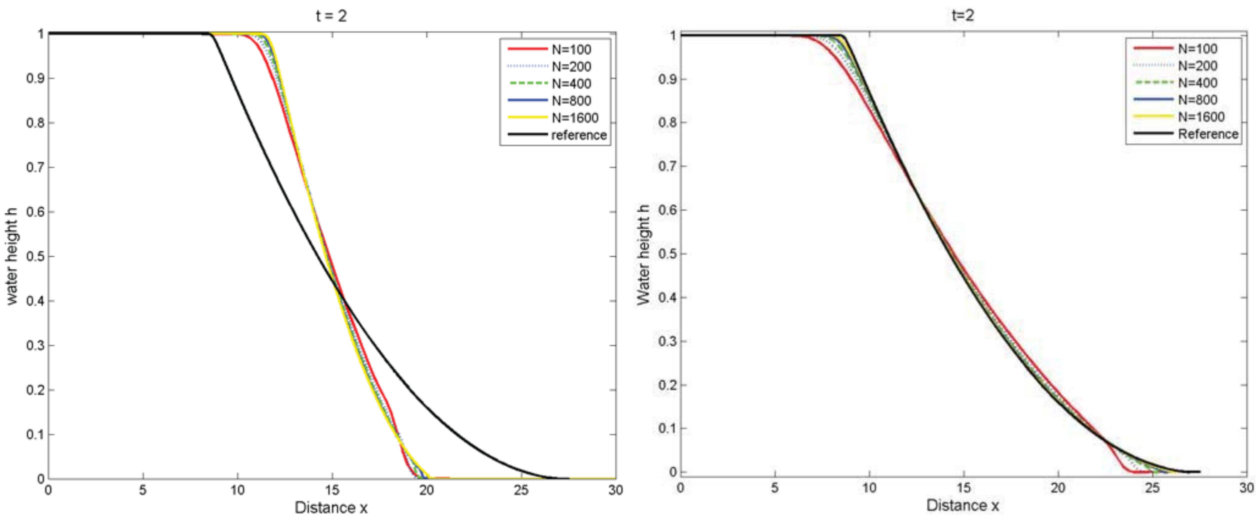
TABLE I
RELATIVE L^1-ERROR, MINIMUM TIME STEP Δt AND CPU TIMES (IN SECONDS) OBTAINED FOR THE DAM-BREAK PROBLEM OVER FALT BED AT TIME $t = 1$ USING THE CONVENTIONAL AND THE PROPOSED APPROACHES

	Conventional approach			Proposed approach		
Gridpoints	L^1-error	min Δt	CPU	L^1-error	min Δt	CPU
100	0.0019	0.0443	0.053851	0.0017	0.0543	0.307248
200	8.5455E-4	0.02078	0.098860	7.612E-4	0.0257	0.438689
400	4.2419E-4	0.0098	0.179201	3.2288E-4	0.0123	0.836681
800	1.7346E-4	0.0047	0.245428	1.2956E-4	0.0060	1.915621
1600	7.0205E-5	0.0023	0.610092	4.6372E-5	0.0029	5.10189



(a) (b)

Fig. 2 Water free-surface using the conventional approach (a) and the proposed approach (b) at time t = 2 using different gridpoints

Note that the departure points $Y_{i+1/2}(t_n)$ do not coincide with the spatial position of a gridpoint and a cubic spline interpolation is used in this study.

Next we integrate (9) along the characteristics with respect to time and space over the time-space control domain $[t_n, t_{n+1}] \times [y_{i-1/2}, y_{i+1/2}]$ as

$$\mathbf{W}_i^{n+1} = \widetilde{\mathbf{W}}_i^n - \frac{\Delta t}{\Delta y}\left(\widetilde{\mathbf{F}}_{i+1/2}^n - \widetilde{\mathbf{F}}_{i-1/2}^n\right) + \frac{\Delta t}{\Delta y}\int_{y_{i-1/2}}^{y_{i+1/2}} \widetilde{\mathbf{S}}\, dy,$$

where $\widetilde{\mathbf{F}}_{i\pm\frac{1}{2}}^n = \mathbf{F}\left(\widetilde{\mathbf{W}}_{i\pm\frac{1}{2}}^n\right)$ are the numerical fluxes at $y = y_{i\pm\frac{1}{2}}$ and time $t = t_n$. The numerical construction of the fluxes can be carried out using any already existing procedure for conventional shallow water equations. Here we use the well-established Lax-Friedrichs method

$$\widetilde{\mathbf{F}}_{i+1/2} = \frac{\Delta y}{2\Delta t}\left(\widetilde{\mathbf{W}}_i - \widetilde{\mathbf{W}}_{i+1}\right) + \frac{1}{2}\left(\mathbf{F}\left(\widetilde{\mathbf{W}}_{i+1}\right) + \mathbf{F}\left(\widetilde{\mathbf{W}}_i\right)\right).$$

The discretization of the source term $\mathbf{S}(\mathbf{W})$ is reconstructed such that the discretization of the gradients and the source terms in (5) are well-balanced *i.e.*,

$$\mathbf{S}(\widetilde{\mathbf{W}}_i) = \begin{pmatrix} 0 \\ -g\dfrac{\widetilde{H}_{i-1} + 2\widetilde{H}_i + \widetilde{H}_{i+1}}{4}\dfrac{\widetilde{Z}_{i-1} - \widetilde{Z}_{i+1}}{2\Delta y} \end{pmatrix}.$$

Note that other numerical reconstructions of the numerical fluxes for the solution of (5) can easily be used without major conceptual modifications. It should also be noted that since explicit time stepping schemes are used in our solution procedure the time step must satisfy a stability condition in the same form as the canonical CFL condition.

IV. NUMERICAL RESULTS

In this section we examine the performance of the proposed techniques for dam-break problems over dry areas. We consider both dam-break flows over a flat bed and over a hump. For all considered examples we compare the results obtained using our approach to those obtained using the conventional method for which the shallow water equations are solved for the whole domain using the well-balanced Lax-Friedrichs scheme and the special wet/dry treatment studied in [2]. First we solve a dam-break in flat channel ($z(x) = 0$) with a length of 30 m and the dam is localized at the center of the channel. We perform a grid convergence for this test example and for this end we summarize in Table I the results obtained for both approaches at time $t = 1$. In this table we list the relative L^1-error in the water free-surface, the minimum time step Δt used in the simulation and the CPU time for each method. Note that a reference solution obtained on a very fine mesh with 12800 gridpoints is used as exact solution to calculate the errors in both methods. It is clear from the results presented in Table I that increasing the number of gridpoints results in a decrease of the relative L^1-errors in both approaches but a faster convergence can be observed for the

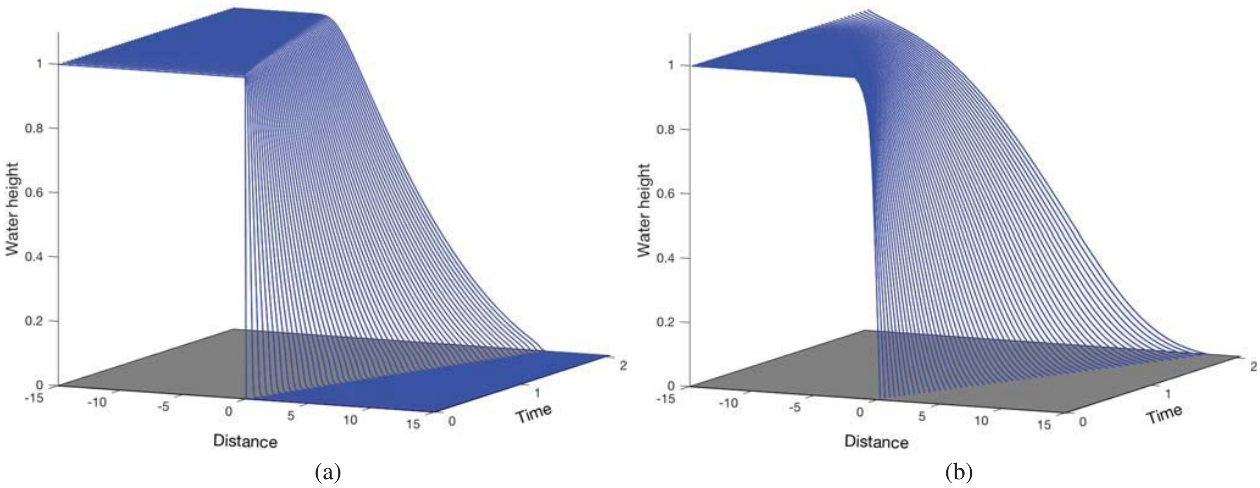


Fig. 3 Water free-surface in the space-time domain obtained using the conventional approach (a) and the proposed approach (b)

proposed approach compared to the conventional approach. In terms of time steps used in both simulations, a slightly larger time steps have been used for the proposed approach than for the conventional approach. However, a larger CPU times is required in the proposed approach compared to the conventional approach. This can be attributed to the additional method of characteristics used in the advective stage of the proposed approach.

In Fig. 2 we present the water free-surface obtained using the conventional approach and the proposed approach at time $t = 2$ using different gridpoints. As expected refining the mesh more accurate water free-surface profiles are obtained in both approaches but the proposed approach simulate more accurately this test example than the conventional approach. Similar conclusions have been drawn for results not presented here for the water discharge. Note that for coarse meshes numerical diffusion is more pronounced in both approaches compared to the refined meshes. Overall the flow features and the rarefaction waves are accurately captures by the proposed approach and no spurious oscillations have been detected as shown in the time-space solutions presented in Fig. 3. It is evident that the conventional approach exhibits a slower water fronts compared to those obtained using the proposed approach. The behavior of the wet/dry interface is also different in both approaches for example observe the time evolution of the interface in Fig. 3 for both approaches.

The second test example consists of a dam-break problem over a non-flat domain. Again the length of the channel is 30 m, the dam is localized at the center of the channel and the bottom is defined by

$$z(x) = \frac{1}{5} e^{\frac{(-x-15)^2}{20}}.$$

In Fig. 4 we present the results for the water surface obtained using the conventional and proposed approaches. For comparsion reason we also include a reference solution obtained on fined mesh with 12800 points along with the bed topography. As it can be seen, both the conventional and the proposed approach capture the flow structures associated with this dam-break problem. Compared to the reference solution the proposed approach produces the most accurate results. Similar features have observed for the water discharge. To further qualify the results we present in Fig. 5 the water free-surface profiles in the space-time domain. As in the previous dam-break problem, the conventional and the proposed approaches demonstrate differences in the water fronts and also in the wet/dry interfaces. A faster moving water fronts over the hump are detected in the conventional approach compared to the proposed approach. In sumamry, the proposed appoach is able to resolve dam-break problems over both flat and non-flat beds without introducing excessive numerical diffusion or nonphysical oscillations. In addition, the proposed approach does not require special treatment of the we/dry interface similar those techniques used for the conventional approaches.

V. Conclusion

A novel method for solving shallow water flows over dry areas has been presented. The governing equations have been derived by introducing a new coordinate system using moving wet/dry fronts in shallow water flows. The method aims to track the point of interface between wet/dry region and then solve the problem up to that point only, where the mesh is updated at each time step based on the moving domain. The new governing equations have been reformulated in a hyperbolic system of conservation laws with a source term. As a numerical solver we proposed a combined method of characteristics and the well-known Lax-Friedrich finite volume method. The proposed method has been applied for solving dam-break problems on flat and non-flat dry beds. The obtained results indicate good shock resolution with high accuracy in smooth regions and without any nonphysical oscillations near the shock areas. Although we have restricted our numerical computations to the first order Lax-Friedrichs method, our next step will focus on extending these techniques to more complex shallow water flows over dray areas using high order finite volume methods.

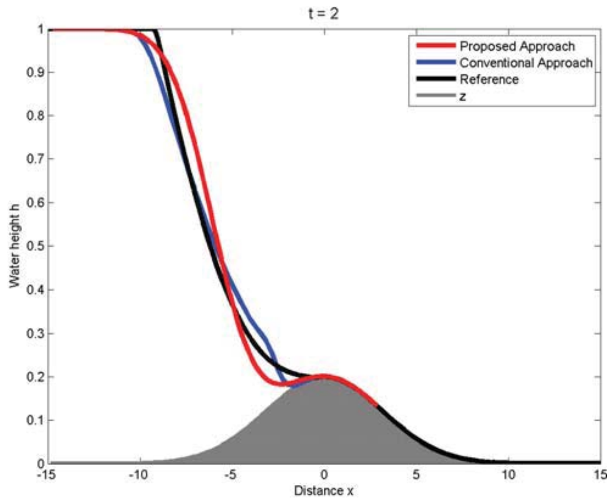
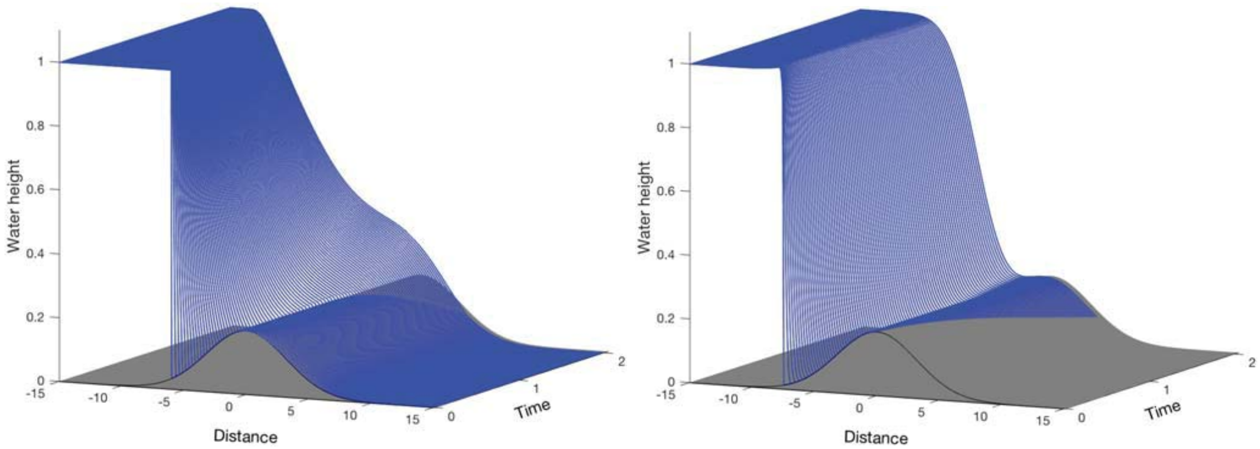


Fig. 4 Results obtained for the dam-break over a hump at time $t = 2$ using a mesh with 100 gridpoints



(a) (b)

Fig. 5 Water free-surface in the space-time domain obtained for the dam-break over a hump using the conventional approach (a) and the proposed approach (b) on a mesh with 100 gridpoints

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