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Green Function and Eshelby Tensor Based on Mindlin's 2nd Gradient Model: An Explicit Study of Spherical Inclusion Case

A. Selmi, A. Bisharat

Abstract**—**Using Fourier transform and based on the Mindlin's 2nd gradient model that involves two length scale parameters, the Green's function, the Eshelby tensor, and the Eshelby-like tensor for a spherical inclusion are derived. It is proved that the Eshelby tensor consists of two parts; the classical Eshelby tensor and a gradient part including the length scale parameters which enable the interpretation of the size effect. When the strain gradient is not taken into account, the obtained Green's function and Eshelby tensor reduce to its analogue based on the classical elasticity. The Eshelby tensor in and outside the inclusion, the volume average of the gradient part and the Eshelby-like tensor are explicitly obtained. Unlike the classical Eshelby tensor, the results show that the components of the new Eshelby tensor vary with the position and the inclusion dimensions. It is demonstrated that the contribution of the gradient part should not be neglected.

Keywords**—**Eshelby tensor, Eshelby-like tensor, Green's function, Mindlin's 2nd gradient model, Spherical inclusion.

I. Introduction

THE size-dependency and scaling, in micro- and nano-structures such as thin films, quantum dots, plasticity, nanowires, nanotubes in addition to nanocomposite materials has acquired significant attention in recent times. Due to the lack of characteristic length parameters, the classical Eshelby tensor and local constitutive models are inadequate for mechanical applications at the micro- and nano-scale, since size-effects exhibited by particle-matrix composites and evidenced by experiments cannot be estimated [1]. This has motivated the studies on Eshelby-type inclusion problems using higher-order elasticity theories which contain microstructure dependent material length scale parameters and are therefore capable of explaining the size effect. The higher-order elasticity theories that have been used in examining the Eshelby inclusion problems include a micropolar theory, a microstretch theory and modified couple stress theory [2]-[6].

The work reported in [5] appears to be the only study that is based on the strain gradient elasticity with couple stresses and which involves two additional length scale parameters. In that work, for the spherical inclusion case, only the Eshelby tensor is derived in closed form.

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There is still a lack of studies on the Eshelby-type inclusion problems based on strain gradient elasticity theories involving two additional elastic constants. For the spherical inclusion, the explicit form of Eshelby tensor in- and outside the inclusion, the variation of Eshelby tensor and the volume average of its gradient part as a function of the two length parameters appear to be missing in the literature.

The objective of this paper is to tackle the aforementioned problems and therefore to afford a systematic study of various Eshelby type inclusion problems involving an inclusion embedded in an infinite or a finite homogeneous isotropic elastic body, applying a two-length scale-parameter strain gradient theory.

II. Green's Function

The linear constitutive equations for the stress and double stress quantities are obtained as in [7]:

$$\begin{cases} \sigma_{ij} = C_{ijhk}\varepsilon_{hk} \\ \tau_{ijk} = A_{ijklmn}\eta_{lmn} \end{cases} \tag{1}$$

The infinitesimal strain ε_{ij}, and the strain gradient η_{ijk} are, respectively, defined by:

$$\begin{cases} \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \\ \eta_{ijk} = u_{k,ij} \end{cases} \tag{2}$$

As shown in [7], the equilibrium equations have the form:

$$\sigma_{ij,i} - \tau_{kij,ki} + F_j = 0 \tag{3}$$

Substituting (1) and (2) into (3) yields the Navier-like displacement equations of equilibrium as

$$(\lambda + 2\mu)(1 - L_1^2\nabla^2)u_{p,pj} - \mu(1 - L_2^2\nabla^2)(u_{p,pj} - u_{j,pp}) + F_j = 0 \tag{4}$$

where L_1 and L_2 are two length scale parameters and λ and μ are the Lame' coefficients.

When the strain gradient effect is not taken into account considered (i.e. $L_1 = L_2 = 0$), (4) reduces to the Navier equation in classical elasticity.

The Green's function is provided by the solution of (4) subject to the boundary conditions of, u_i, and their derivatives vanishing at infinity. The Green's function in the 2[nd] gradient model is obtained by applying Fourier transforms.

$$G_{ij}(x) = A(x)\delta_{ij} - B(x)_{,ij} \tag{5}$$

where

$$\begin{cases} A(x) = \dfrac{1}{4\pi\mu x}(1 - e^{\frac{-x}{L_2}}) \\ B(x) = \dfrac{1}{8\pi\mu x}(x + \dfrac{2L_2^2}{x} - \dfrac{2L_2^2}{x}e^{\frac{-x}{L_2}}) - \dfrac{1}{8\pi(\lambda+2\mu)}(x + \dfrac{2L_1^2}{x} - \dfrac{2L_1^2}{x}e^{\frac{-x}{L1}}) \end{cases}$$

Interpretation The Green's function is first obtained in terms of elementary functions by applying Fourier transforms, Its expression can be shown to be the same as that obtained by Zhang and Sharma using a different approach [5]. This Green's function can also be reduced to the Green's function in classical elasticity when the strain gradient effect is ignored.

III. Eshelby Tensor and Eshelby-Like Tensor

The Eshelby-type inclusion problem is determined analytically. The derived Eshelby tensor is the sum a classical part identical to the Eshelby tensor based on the classical elasticity theory and depending only on Poisson's ratio, and a gradient part involving the two length scale parameters and depending on the size of the inclusion additionally, thereby permitting the interpretation of the size effect. The gradient part vanishes when the strain gradient effect is not considered.

The classical part of Eshelby tensor:

$$S_{ijlm}^{C} = -\frac{1}{8\pi\mu}\left\{\Lambda_{,kj}\delta_{iq} + \Lambda_{,ki}\delta_{jq} - \frac{1}{2(1-\nu)}\Phi_{ijqk}\right\}C_{qklm}$$

The gradient part of Eshelby tensor:

$$S_{ijlm}^{G} = \frac{1}{8\pi\mu}\begin{Bmatrix} \Gamma_{,kj}^{2}\delta_{iq} + \Gamma_{,ki}^{2}\delta_{jq} + 2L_2^2(\Lambda - \Gamma^2)_{,ijqk} \\ -\dfrac{2\mu}{\lambda + 2\mu}L_1^2(\Lambda - \Gamma^1)_{,ijqk} \end{Bmatrix}C_{qklm}$$

The Eshelby like tensor:

$$T_{ijlmn} = \frac{1}{8\pi\mu}\begin{Bmatrix} (\Lambda - \Gamma^2)_{kpj}\delta_{ik} + (\Lambda - \Gamma^2)_{kpi}\delta_{jq} \\ -\left[\dfrac{1}{2(1-\nu)}\Phi + 2L_2^2(\Lambda - \Gamma^2) - \dfrac{2\mu}{\lambda + 2\mu}L_1^2(\Lambda - \Gamma^1)_{,ijpqk}\right] \end{Bmatrix}A_{kpqlmn}$$

where

$$\Phi(x) = <|x-y|>, \Lambda(x) = <\frac{1}{|x-y|}>, \Gamma(x) = <\frac{e^{\frac{|x-y|}{L}}}{|x-y|}>, \; L = L_1 \text{ or } L_2$$

and $< F(y) >$ is the volume integral, of a sufficiently smooth function $F(y)$ over the inclusion occupying region Ω, defined as: $< F(y) > = \iiint_\Omega F(y)dy$.

IV. Eshelby Tensor for a Spherical Inclusion

For the special case of spherical inclusion, the Eshelby tensor based on Mindlin's 2[nd] gradient model will be found by directly applying the general formula which is obtained in Section III.

$$\begin{aligned} S_{ijlm}^{G} = {} & K_1^G(x)\delta_{ij}\delta_{lm} + K_2^G(x)(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) + K_3^G(x)\delta_{lm}x_i^0x_j^0 + K_4^G(x)\delta_{ij}x_l^0x_m^0 \\ & + K_5^G(x)(\delta_{il}x_j^0x_m^0 + \delta_{im}x_j^0x_l^0 + \delta_{jl}x_i^0x_m^0 + \delta_{jm}x_i^0x_l^0) + \delta_{jm} + K_6^G(x)(x_i^0x_j^0x_l^0x_m^0) \end{aligned}$$

where $x_\sim^0 = x_\sim/\|x\|$ is the component of the unit vector $x/\|x\|$, δ_{ij} is the Kronecker delta and

$$K_1^G = \frac{1}{2\pi(1-2\nu)}\left[\nu D_1\Gamma^2 - \nu L_2^2x^2D_3(\Gamma^2 - \Lambda) - (1+3\nu)L_2^2D_2(\Gamma^2 - \Lambda)\right] + \frac{\nu(1-2\nu)}{2(1-\nu)}L_1^2x^2D_3(\Gamma^1 - \Lambda) - \frac{(1+3\nu)(1-2\nu)}{2(1-\nu)}L_1^2D_2(\Gamma^1 - \Lambda)$$

$$K_2^G = \frac{1}{4\pi}\left[D_1\Gamma^2 - 2L_2^2D_2(\Gamma^2 - \Lambda) + \frac{(1-2\nu)}{1-\nu}L_1^2D_2(\Gamma^1 - \Lambda)\right]$$

$$K_3^G = \frac{x^2}{2\pi(1-2\nu)}\left[\nu D_2\Gamma^2 - \nu L_2^2x^2D_4(\Gamma^2 - \Lambda) - (1+5\nu)L_2^2D_3(\Gamma^2 - \Lambda)\right] + \frac{\nu(1-2\nu)}{2(1-\nu)}L_1^2x^2D_4(\Gamma^1 - \Lambda) - \frac{(1+5\nu)(1-2\nu)}{2(1-\nu)}L_1^2D_3(\Gamma^1 - \Lambda)$$

$$K_4^G = \frac{x^2}{2\pi}\left[-L_2^2D_3(\Gamma^2 - \Lambda) + \frac{(1-2\nu)}{2(1-\nu)}L_1^2D_3(\Gamma^1 - \Lambda)\right]$$

$$K_5^G = \frac{x^2}{8\pi}\left[D_2\Gamma^2 - 4L_2^2D_3(\Gamma^2 - \Lambda) + \frac{2(1-2\nu)}{1-\nu}L_1^2D_3(\Gamma^1 - \Lambda)\right]$$

$$K_6^G = \frac{x^4}{2\pi}\left[-L_2^2D_4(\Gamma^2 - \Lambda) + \frac{(1-2\nu)}{2(1-\nu)}L_1^2D_4(\Gamma^1 - \Lambda)\right]$$

The relations D_1, D_2, D_3 and D_4 for a sufficiently smooth function are given in [8].

The Eshelby like tensor is:

$$
\begin{aligned}
T_{ijlmn} = \frac{1}{8\pi\mu}\Big\{ & \left[x^2 x_j^0 D_3 P_2 + 5 x x_j^0 D_2 P_2\right]\left[\frac{a1}{2}(\delta_{il}\delta_{mn}+\delta_{im}\delta_{ln}) + 2a_3\delta_{in}\delta_{lm}\right] + \left[x^3 x_i^0 x_j^0 x_n^0 D_3 P_2 + x \langle \delta_{ij} x_n^0 \rangle_3 D_2 P_2\right]\left[2a_1\delta_{lm}\right] \\
& + \left[x^2 x_i^0 D_3 P_2 + 5 x x_i^0 D_2 P_2\right]\left[\frac{a_1}{2}(\delta_{jl}\delta_{mn}+\delta_{jm}\delta_{ln}) + 2a_3\delta_{jn}\delta_{lm}\right] + \left[x^3 x_i^0 x_j^0 x_l^0 D_3 P_2 + x \langle \delta_{ij} x_l^0 \rangle_3 D_2 P_2\right]\left[2a_2\delta_{mn}\right] \\
& + \left[x^3 x_i^0 x_j^0 x_m^0 D_3 P_2 + x \langle \delta_{ij} x_m^0 \rangle_3 D_2 P_2\right]\left[2a_2\delta_{ln}\right] + \left[x^3 x_j^0 x_l^0 x_m^0 D_3 P_2 + x \langle \delta_{jl} x_m^0 \rangle_3 D_2 P_2\right]\left[2a_4\delta_{in}\right] \\
& + \left[x^3 x_i^0 x_l^0 x_m^0 D_3 P_2 + x \langle \delta_{il} x_m^0 \rangle_3 D_2 P_2\right]\left[2a_4\delta_{jn}\right] + a_5\Big[x^3 D_3 P_2 (x_j^0 x_l^0 x_n^0 \delta_{im} + x_j^0 x_m^0 x_n^0 \delta_{il} + x_i^0 x_l^0 x_n^0 \delta_{jm} + x_i^0 x_m^0 x_n^0 \delta_{jl}) + \\
& x D_2 P_2 (\langle \delta_{jl} x_{n>_3}^0 \delta_{im} + \langle \delta_{jm} x_{n>_3}^0 \delta_{il} + \langle \delta_{il} x_{n>_3}^0 \delta_{im} + \langle \delta_{im} x_{n>_3}^0 \delta_{jl})\Big] - (\frac{a_1}{2}+a_2)\delta_{mn}\left[x^5 x_i^0 x_j^0 x_l^0 D_5 G + x^3(\langle \delta_{ij} x_l^0 \rangle_3 + 9 x_i^0 x_j^0 x_l^0) D_4 G + 7x \langle \delta_{ij} x_l^0 \rangle_3 D_3 G\right] \\
& -(\frac{a_1}{2}+a_2)\delta_{ln}\left[x^5 x_i^0 x_j^0 x_m^0 D_5 G + x^3(\langle \delta_{ij} x_m^0 \rangle_3 + 9 x_i^0 x_j^0 x_m^0) D_4 G + 7x \langle \delta_{ij} x_m^0 \rangle_3 D_3 G\right] - (a_1+2a_3)\delta_{lm}\left[x^5 x_i^0 x_j^0 x_n^0 D_5 G + x^3(\langle \delta_{ij} x_n^0 \rangle_3 + 9 x_i^0 x_j^0 x_n^0) D_4 G + 7x \langle \delta_{ij} x_n^0 \rangle_3 D_3 G\right] \\
& -2(a_4+a_5)\delta_{mn}\left[x^5 x_i^0 x_j^0 x_l^0 x_m^0 x_n^0 D_5 G + x^3(\langle \delta_{ij} x_l^0 x_m^0 x_n^0 \rangle_{10} D_4 G + x \langle \delta_{ij}\delta_{lm} x_n^0 \rangle_{15} D_3 G\right] \Big\}
\end{aligned}
$$

where a_i (i = 1,.., 5) are the five material constants defining the nonlocal isotropic behavior [7], and $\langle\ \rangle_3, \langle\ \rangle_{10}, \langle\ \rangle_{15}$ are defined in [8] and P_1, P_2, and G are defined in [9].

V. Numerical Results

For the purpose of illustration and using the expressions derived in the previous section, some numerical results are obtained and presented here to quantitatively estimate how the components of the newly obtained Eshelby tensor vary with position and inclusion size. The components of the gradient part of the Eshelby tensor at any x inside the spherical inclusion (radius R) along the x_1 axis (with $x_2 = 0 = x_3$) can be obtained. Here, only the component S_{1111}^G and its average are presented.

$$
S_{1111}^G = \frac{L_2 + R}{x^5} e^{-R/L_2} \left\{ \begin{aligned} & 2\left[(11-\nu)x^2 L_2^2 + 24 L_2^4\right]\sinh(\frac{x}{L_2}) \\ & + 2xL_2\left[(\nu-3)x^2 - 24L_2^2\right]\cosh(\frac{x}{L_2}) \\ & -(\nu-1)x^2 L_2\left[(x+L_2)e^{-x/L_2} + (x-L_2)e^{x/L_2}\right] \end{aligned} \right\}
$$

$$
\begin{aligned}
\left\langle S_{1111}^G \right\rangle_V = & \begin{bmatrix} \dfrac{1+3\nu}{10(1-\nu)}(\dfrac{L_1}{R})^3\left[1-(\dfrac{R}{L_1})^2-(1+\dfrac{R}{L_1})^2 e^{\frac{-2R}{L_1}}\right] \\ -\dfrac{1}{5}(\dfrac{L_2}{R})^3\left[1-(\dfrac{R}{L_2})^2-(1+\dfrac{R}{L_2})^2 e^{\frac{-2R}{L2}}\right] \end{bmatrix} \delta_{ij}\delta_{lm} \\
& + \begin{bmatrix} \dfrac{1-2\nu}{10(1-\nu)}(\dfrac{L_1}{R})^3\left[1-(\dfrac{R}{L_1})^2-(1+\dfrac{R}{L_1})^2 e^{\frac{-2R}{L_1}}\right] \\ -\dfrac{3}{10}(\dfrac{L_2}{R})^3\left[1-(\dfrac{R}{L_2})^2-(1+\dfrac{R}{L_2})^2 e^{\frac{-2R}{L2}}\right] \end{bmatrix} (\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})
\end{aligned}
$$

The following relation between L_1 and L_2 can be derived:

$$
L_1 = f\sqrt{\frac{1-2\nu}{2(1-\nu)}L_2}
$$

The Poisson's ratio v and the length scale parameter L_2 are assumed to be respectively 0.3 and17.6 μm.

It is seen from Figs. 1 and 2 that S_{1111} varies with x (the position) and depends on R (the inclusion size), unlike the classical part S_{1111}^C which is a constant (i.e., independent of both x and R. Fig. 3 shows that $\langle S_{1111} \rangle_V$ is indeed varying with R: the smaller R, the smaller $\langle S_{1111} \rangle_V$. Similar trends are observed for the other components of the Eshelby tensor.

VI. Conclusion

Using the Mindlin's 2nd gradient model and based on the Green's function, the general form of the non-classical Eshelby tensor is elicited. This later is written as the sum of two terms; the classical Eshelby tensor and the gradient part which depends on the two additional length parameters and varies with the position in-and outside the inclusion. A so called Eshelby-like tensor liking the eigen strain gradient to the induced strain is deduced in this study. For the spherical inclusion problem, the classical Eshelby tensor, the gradient part and its volume average are explicitly obtained by employing the developed general form of the non-classical Eshelby tensor. Numerical results reveal that the components of the new Eshelby tensor vary with both the position and the inclusion size, thereby capturing the size effect at the micron scale. The components of the averaged Eshelby tensor are found to decrease as the inclusion radius decreases, and these components are observed to approach from below the values of the corresponding components of the Eshelby tensor based on classical elasticity when the inclusion size is large enough.

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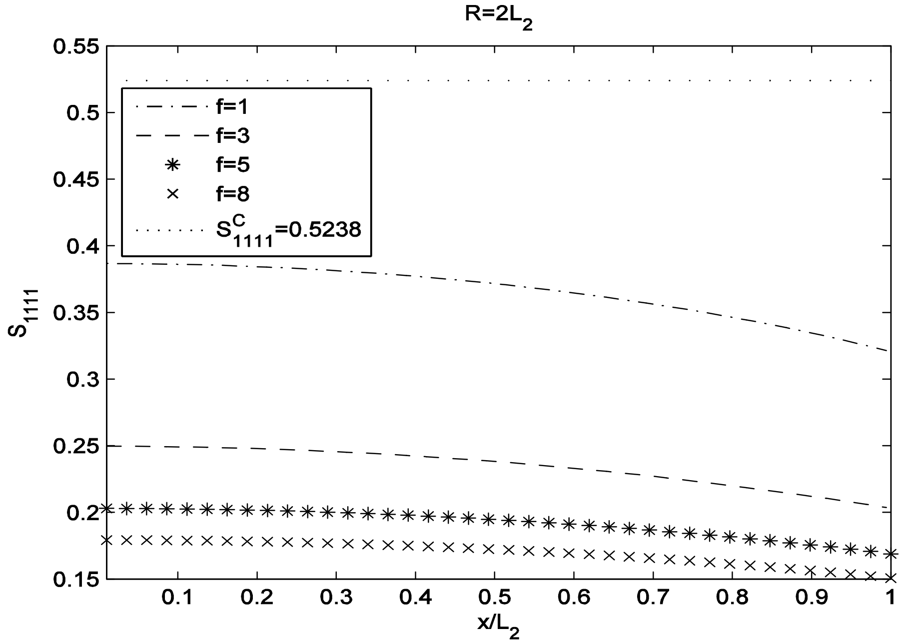

Fig. 1 S_{1111} along a radial direction of the spherical inclusion for R= 2 L

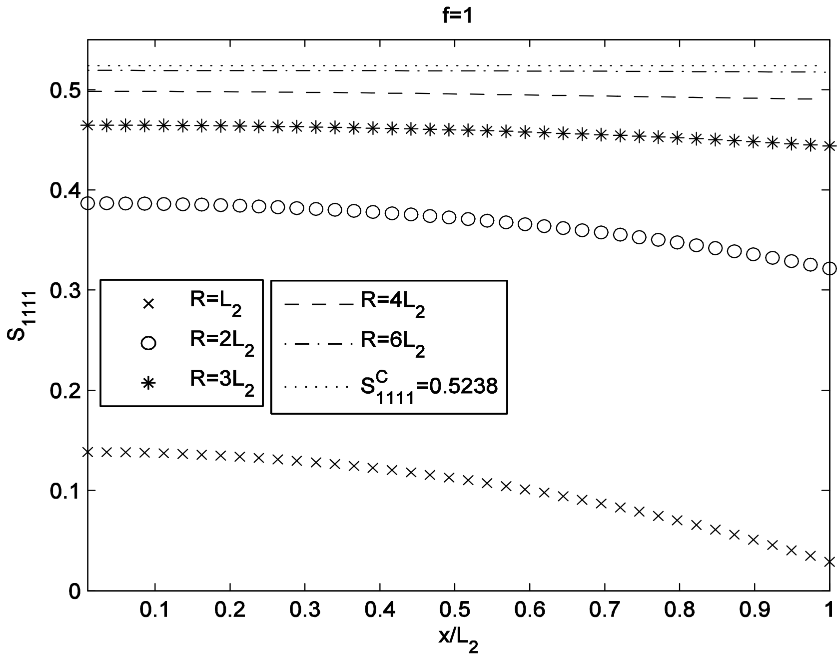

Fig. 2 S_{1111} along a radial direction of the spherical inclusion for f= 1

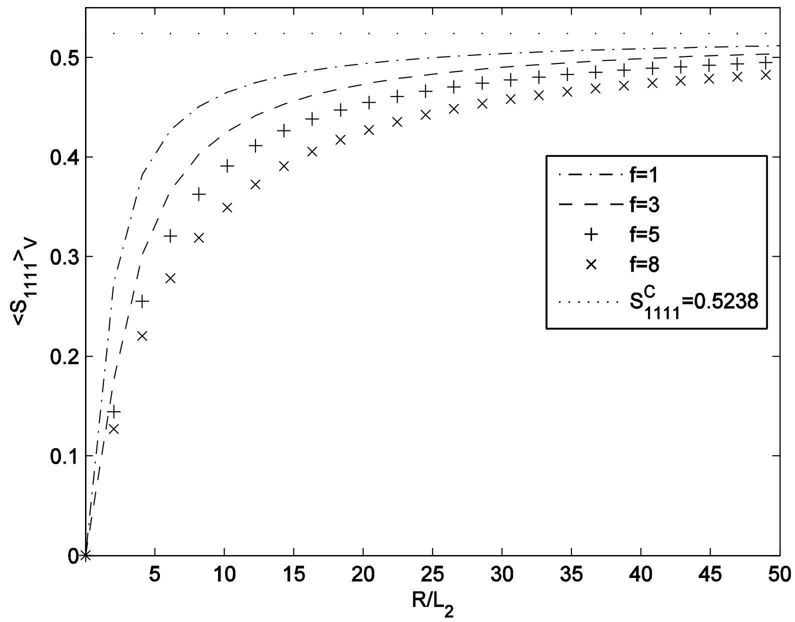

Fig. 3 Variation of $< S_{1111}>_V$ with the inclusion radius

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