Mean-Variance Optimization of Portfolios with Return of Premium Clauses in a DC Pension Plan with Multiple Contributors under Constant Elasticity of Variance Model

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Abstract**—**In this paper, mean-variance optimization of portfolios with the return of premium clauses in a defined contribution (DC) pension plan with multiple contributors under constant elasticity of variance (CEV) model is studied. The return clauses which permit death members to claim their accumulated wealth are considered, the remaining wealth is not equally distributed by the remaining members as in literature. We assume that before investment, the surplus which includes funds of members who died after retirement adds to the total wealth. Next, we consider investments in a risk-free asset and a risky asset to meet up the expected returns of the remaining members and obtain an optimized problem with the help of extended Hamilton Jacobi Bellman equation. We obtained the optimal investment strategies for the two assets and the efficient frontier of the members by using a stochastic optimal control technique. Furthermore, we studied the effect of the various parameters of the optimal investment strategies and the effect of the risk-averse level on the efficient frontier. We observed that the optimal investment strategy is the same as in literature, secondly, we observed that the surplus decreases the proportion of the wealth invested in the risky asset.

Keywords**—**DC pension fund, Hamilton Jacobi Bellman equation, optimal investment strategies, stochastic optimal control technique, return of premiums clauses, mean-variance utility.

I. Introduction

OPTIMAL portfolio selection is an interesting and important aspect of study in the field of mathematical finance and has grown over the years. It involves practical ways to determine how best investments can be done for optimal productivity with minimal risk. Recent publications have shown different methods of optimizing investment strategies and returns [1]-[7], [15]. A good number of researches have been published in this direction [7], [15], especially in DC Pension plan which requires members and their employers to contribute a certain proportion of their income to the pension scheme and the management of the pension scheme will in turn help to plan for the members' retirement.

Some research work in association with optimal investment strategies in the DC pension plan includes [1], where they studied optimal investment strategy for DC pension funds with the stochastic interest rate. Reference [2] investigated optimal investment for the DC pension fund whose interest rate was a Vasicek model. The optimal investment strategy of the DC pension fund with affine interest rate, which includes the Cox–Ingersoll–Ross model and Vasicek model, has been investigated (see for example [3]-[5]). Recently, CEV model has increasingly been used in the study of optimal investment strategies in a DC pension fund. Reference [6] studied CEV model and the Legendre transform-dual solution for annuity contracts. Since Geometric Brownian motion (GBM) can be considered as a special case of the CEV model, such work extended the research of [6] where they applied CEV model to derive a dual solution of a CRRA utility function via Legendre transform. Reference [7] extended [6] by obtaining solutions for investor with CRRA and CARA utility function. Reference [8] investigated an asset allocation problem under a loss-averse preference. Reference [9] considered stochastic salary income of a pension beneficiary and found the investment strategy that maximizes the expected power utility of the relationship between the terminal fund and the final salary. Reference [10] investigated a utility optimization problem for a DC pension plan with a stochastic salary income and a stochastic contribution process in a regime-switching economy. Reference [11] studied optimal portfolios for DC pension plans under a CEV model, [12] modeled pension fund with multiple contributors and obtained the explicit solution of the optimal investment strategy for CRRA and CARA using power transformation method. Reference [13] obtained explicit solution of the optimal investment strategies in DC pension fund with multiple contributions using the Legendre transformation method for CRRA and CARA. In [14], stochastic strategies of optimal investment for DC pension fund with multiple contributors were considered. The constant relative risk aversion (CRRA) and the constant absolute risk aversion (CARA) are some of the commonly used utility functions (see for example [1]-[4], [7]).

In recent times, there have been some publications on optimal investment strategy with return of premiums clauses. Reference [15] studied optimal investment strategy for a DC pension plan with the return of premiums clauses in a mean-

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variance framework, [16] extended the work of [15] by studying optimal time-consistent investment strategy for a DC pension with the return of premiums clauses and annuity contracts; in their work, they considered both the accumulation and distribution phases using Heston's stochastic volatility model. Reference [17] considered equilibrium investment strategy for DC pension plan with a default risk and return of premiums clauses under CEV model.

This paper focuses on optimizing the mean-variance of portfolios with the return of premium clauses in a DC pension plan with multiple contributors under CEV model. We consider a return clause which permits death members to withdraw their accumulated contributions. We assume that before investment, the surplus which includes funds of members who died after retirement adds to the total wealth. Next, we consider investments in a risk-free asset and a risky asset to meet up the expected returns of the remaining members and obtained an optimized problem with the help of Hamilton Jacobi Bellman equation. By using a stochastic optimal control technique, we obtain the optimal investment strategies for the two assets and the efficient frontier of the members. Also, the effect of the various parameters of the optimal investment strategies and the effect of the risk-averse level on the efficient frontier are studied as well.

II. Mathematical Model of the Financial Market

We assume that the market is made up of risk-free asset (cash) and risky asset (stock). Let (Ω, F, P) be a complete probability space where Ω is a real space and P is a probability measure, $\{W_s(t), W_t(t)\}$ is a standard two dimensional motion, such that they are orthogonal to each other. $F$is the filtration and denotes the information generated by the Brownian motion $\{W_s(t), W_t(t)\}$.

Let $B_t(t)$and $S_t(t)$denote the prices of the risk-free asset and risky asset respectively and they are modelled as

$$\frac{dB_t(t)}{B_t(t)} = rdt, \tag{1}$$

$$\frac{dS_t(t)}{S_t(t)} = \alpha dt + K{S_t}^{\beta} dW_t. \tag{2}$$

where$r$ is the predetermined interest rate of the risk free asset, α is an expected instantaneous rate of return of the risky asset and satisfies the general condition $\alpha > r$. $K{S_t}^{\beta}$ is the instantaneous volatility, and β is the elasticity parameter and satisfies the general condition $\beta < 0$.

In DC pension fund system with multiple contributors, we assume:

(1) Payment is made by only those who have retired.
(2) Payment continues till the death of plan contributors.
(3) Death contributors are automatically deleted from the system.

From the above assumptions, the payment is a stochastic process. We assume the Brownian motion with drift as:

$$dC(t) = adt - bdW_s(t) \tag{3}$$

where a and b are positive constants and denote the amount given to the retired contributors and that which is due death contributors that are out of the system.

In a DC fund system, members remit certain proportion of their earning to the pension account every month; we assume that the number of contributors is constant and the contribution rate is modeled as:

$$dY = pdt \tag{4}$$

where $p = (1+\theta)a$ with safety loading $\theta > 0$, $p = (1+\theta)a$, with safetyloading $\theta > 0$. If there is no investment, the dynamics of the surplus is given by:

$$dR(t) = dY - dC(t) = \theta adt + bdW_s(t) = (p-a)dt + bdW_s(t) \tag{5}$$

Let q be the premium received at a given time, which is known, ω_0 represent the initial age of accumulation phase, T is the time frame of the accumulation phase such that $\omega_0 + T$ is the end age. The actuarial symbol $\delta_{\left(\frac{1}{i}\right),\omega_0+t}$ is the mortality rate from time $t$to $t+\frac{1}{i}$, tp is the premium accumulated at time t, $tp\delta_{\left(\frac{1}{i}\right),\omega_0+t}$ is the premium returned to the death members. Also, we assume that after return of premium to death members, the remaining accumulations are not shared equally unlike in [15].

Let φ represent the proportion of the wealth to be invested in risky assets and $\varphi_B = 1-\varphi$, the proportion to be invested in the risk free asset.

Considering the time interval $[t, t+\frac{1}{i}]$, the differential form associated with the fund size is given as:

$$X\left(t+\frac{1}{i}\right) = X(t)\left(\varphi\frac{S_{t+\frac{1}{i}}}{S_t} + (1-\varphi)\frac{B_{t+\frac{1}{i}}}{B_t}\right) + p\left(\frac{1}{i}\right) - tp\delta_{\left(\frac{1}{i}\right),\omega_0+t} + dR(t) \tag{6}$$

$$X\left(t+\frac{1}{i}\right) = X(t)\left(\varphi\left(\frac{S_{t+\frac{1}{i}}}{S_t} - \frac{S_t}{S_t} + \frac{S_t}{S_t}\right) + (1-\varphi)\left(\frac{B_{t+\frac{1}{i}}}{B_t} - \frac{B_t}{B_t} + \frac{B_t}{B_t}\right)\right) + p\left(\frac{1}{i}\right) - tp\delta_{\left(\frac{1}{i}\right),\omega_0+t} + dR(t) \tag{7}$$

$$X\left(t+\frac{1}{i}\right) = X(t)\left(\varphi + 1 - \varphi + \varphi\left(\frac{S_{t+\frac{1}{i}}}{S_t} - \frac{S_t}{S_t}\right) + (1-\varphi)\left(\frac{B_{t+\frac{1}{i}}}{B_t} - \frac{B_t}{B_t}\right)\right) + p\left(\frac{1}{i}\right) - tp\delta_{\left(\frac{1}{i}\right),\omega_0+t} + dR(t) \tag{8}$$

$$X\left(t+\frac{1}{i}\right) - X(t) = X(t)\left(\varphi\left(\frac{S_{t+\frac{1}{i}} - S_t}{S_t}\right) + (1-\varphi)\left(\frac{B_{t+\frac{1}{i}} - B_t}{B_t}\right)\right) + p\left(\frac{1}{i}\right) - tp\delta_{\left(\frac{1}{i}\right),\omega_0+t} + dR(t) \tag{9}$$

$$\delta_{\left(\frac{1}{i}\right),\omega_0+t} = 1 - \exp\{-\int_0^{\frac{1}{i}} \mu(\omega_0+t+s)ds\} = \mu(\omega_0+t)\frac{1}{i} + O\left(\frac{1}{i}\right) \tag{10}$$

$$i \to \infty, \delta_{\left(\frac{1}{i}\right),\omega_0+t} = \mu(\omega_0+t)dt,\ p\left(\frac{1}{i}\right) \to pdt,\ \frac{S_{t+\frac{1}{i}}-S_t}{S_t} \to \frac{dS_t(t)}{S_t(t)},\ \frac{B_{t+\frac{1}{i}}-B_t}{B_t} \to \frac{dB_t(t)}{B_t(t)} \tag{11}$$

Substituting (11) into (9) we have

$$dX(t) = X(t)\left(\varphi\left(\frac{dS_t(t)}{S_t(t)}\right) + (1-\varphi)\left(\frac{dB_t(t)}{B_t(t)}\right)\right) + pdt - tp\mu(\omega_0+t)dt + dR(t) \tag{12}$$

$$dX(t) = X(t)\left(\varphi\left(\alpha dt + KS_t{}^{\beta}dW_t\right) + (1-\varphi)(rdt)\right) + pdt - tp\mu(\omega_0+t)dt + (p-a)dt + bdW_s(t) \tag{13}$$

$$dX(t) = \left\{X(t)(\varphi(\alpha-r)+r) - a + 2p + \left(\frac{pt}{\omega-\omega_0-t}\right)\right\} + bdW_s(t) + \varphi X(t)KS_t{}^{\beta}dW_t\ \ X(0) = x_0 \tag{14}$$

$$dX(t) = \left\{X(t)(\varphi(\alpha-r)+r) + p\left(\frac{2\omega-2\omega_0-t}{\omega-\omega_0-t}\right) - a\right\} + bdW_s(t) + \varphi X(t)KS_t{}^{\beta}dW_t\ X(0) = x_0 \tag{15}$$

where

$$\mu(t) = \frac{1}{\omega-t} 0 \le t < \omega \tag{16}$$

$\mu(t)$is the force function and ω is the maximal age of the life table.

III. Methodology

Now considering the pension wealth and the volatility of the accumulation, the surviving members will want to maximize the fund size and at the same time minimize the volatility of the accumulated wealth. Hence we formulate the optimal investment problem under the mean-variance criterion as:

$$\sup_{\varphi}\{E_{t,x,s}X^{\varphi}(T) - Var_{t,x,s}X^{\varphi}(T)\} \tag{17}$$

Our interest here is to obtain the optimal investment strategies for both the risk-free and risky asset using the mean-variance utility function.

Applying variational inequality method in [15], [18]. The mean-variance control problem in (17) is equivalent to the following Markovian time inconsistent stochastic optimal control problem with value function $J(t,x,s)$

$$\left\{\begin{array}{c} N(t,x,\varphi) = E_{t,x}[X^{\varphi}(T)] - \frac{\gamma}{2}Var_{t,x}[X^{\varphi}(T)] \\ = E_{t,x}[X^{\varphi}(T)] - \frac{\gamma}{2}\left(E_{t,x}[X^{\varphi}(T)^2] - (E_{t,x}[X^{\varphi}(T)])^2\right) \\ M(t,x) = \sup_{\varphi} N(t,x,\varphi) \end{array}\right. \tag{18}$$

Following [15] the optimal investment strategy φ^* satisfies:

$$J(t,x,s) = \sup_{\varphi} I(t,x,s,\varphi^*) \tag{19}$$

where γ the risk-aversion coefficient of the members

Let $y^{\varphi}(t,x,s) = E_{t,x,s}[X^{\varphi}(T)], z^{\varphi}(t,x,s) = E_{t,x,s}[X^{\varphi}(T)^2]$ then $J(t,x,s) = \sup_{\varphi} f\big(t,x,y^{\varphi}(t,x,s),z^{\varphi}(t,x,s)\big)$ where

$$f(t,x,s,y,z) = y - \frac{\gamma}{2}(z-y^2) \tag{20}$$

Theorem 1 (verification theorem). If there exists three real functions F, G, H: [0,T]× $R \to R$ satisfying the following extended Hamilton Jacobi Bellman equation equations:

$$\left\{\begin{array}{c} \sup_{\rho}\left\{\begin{array}{c} F_t - f_t \\ +(F_x - f_x)\left[x(\varphi(\alpha-r)+r) + p\left(\frac{2\omega-2\omega_0-t}{\omega-\omega_0-t}\right) - a\right] \\ +(F_s - f_s)\alpha s + \frac{1}{2}(F_{xx} - U_{xx})(\varphi^2x^2k^2s^{2\beta} + b^2) \\ +\frac{1}{2}(F_{ss} - U_{ss})k^2s^{2\beta+2} + (F_{xs} - U_{xs})x\varphi k^2s^{2\beta+1} \end{array}\right\} = 0 \\ F(T,x,s) = f(T,x,s,x,x^2) \end{array}\right. \tag{21}$$

where:

$$U_{xx} = \gamma G_x^2, U_{xs} = \gamma G_xG_s, U_{ss} = \gamma G_s^2 \tag{22}$$

$$\left\{\begin{array}{c} \sup_{\rho}\left\{\begin{array}{c} G_t \\ +G_x\left[x(\varphi(\alpha-r)+r) + p\left(\frac{2\omega-2\omega_0-t}{\omega-\omega_0-t}\right) - a\right] \\ +G_s\alpha s + \frac{1}{2}G_{xx}(\varphi^2x^2k^2s^{2\beta} + b^2) \\ +\frac{1}{2}G_{ss}k^2s^{2\beta+2} + G_{xs}x\varphi k^2s^{2\beta+1} \end{array}\right\} = 0 \\ G(T,x,s) = x \end{array}\right. \tag{23}$$

$$\left\{\begin{array}{c} \sup_{\rho}\left\{\begin{array}{c} H_t \\ +H_x\left[x(\varphi(\alpha-r)+r) + p\left(\frac{2\omega-2\omega_0-t}{\omega-\omega_0-t}\right) - a\right] \\ +H_s\alpha s + \frac{1}{2}H_{xx}(\varphi^2x^2k^2s^{2\beta} + b^2) \\ +\frac{1}{2}H_{ss}k^2s^{2\beta+2} + H_{xs}x\varphi k^2s^{2\beta+1} \end{array}\right\} = 0 \\ H(T,x,s) = x^2 \end{array}\right. \tag{24}$$

Then $J(t,x,s) = F(t,x,s), y^{\varphi^*} = G(t,x,s), z^{\varphi^*} = H(t,x,s)$ for the optimal investment strategy φ^*.

Proof: The details of the proof can be found in [19]-[22]

IV. Main Result

A. Optimal Investment Strategy

Our focus now is to obtain the optimal investment strategy by solving (21), (23), (24).

Result 1. The optimal investment strategies for both assets is given as

$$\varphi_B{}^* = 1 - \varphi^*$$

$$\varphi^* = \frac{(\alpha-r)}{\gamma xk^2s^{2\beta}}e^{r(t-T)}\left[1 + \frac{(\alpha-r)}{r}\left(1 - e^{2r\beta(t-T)}\right)\right]$$

Proof. Recall that from (20),

$$f_t = f_x = f_{xx} = f_{xy} = f_{xz} = f_{yz} = f_{zz} = 0, f_y = 1 + \gamma y,$$

$$f_{yy} = \gamma, f_z = -\frac{\gamma}{2} \qquad (25)$$

Substituting (25) into (21) and differentiating (21) with respect to φ, we have φ as:

$$\varphi^* = -\left[\frac{(\alpha-r)F_x+k^2s^{2\beta+1}(F_{xs}-\gamma G_xG_s)}{(F_{xx}-\gamma G_x^2)xk^2s^{2\beta}}\right] \qquad (26)$$

Substituting (26) into (21) and (23) we have:

$$F_t + F_x\left[rx + p\left(\frac{2\omega-2\omega_0-t}{\omega-\omega_0-t}\right) - a\right] + \alpha sF_s + \frac{1}{2}(F_{xx} - \gamma G_x^2)b^2 + \frac{[\alpha-r]^2F_x^2}{2(F_{xx}-\gamma G_x^2)k^2s^{2\beta}} - \frac{1}{2}\left(\frac{F_{xs}-\gamma G_xG_s}{F_{xx}-\gamma G_x^2} - (F_{ss} - \gamma G_s^2)\right)k^2s^{2\beta+2} - \frac{s(\alpha-r)F_x(F_{xs}-\gamma G_xG_s)}{F_{xx}-\gamma G_x^2} = 0 \qquad (27)$$

$$G_t + G_x\left[rx + p\left(\frac{2\omega-2\omega_0-t}{\omega-\omega_0-t}\right) - a\right] + \alpha sG_s + \frac{1}{2}(G_{xx})b^2 - \frac{[\alpha-r]^2F_xG_x}{(F_{xx}-\gamma G_x^2)k^2s^{2\beta}} - (F_{xs} - \gamma G_xG_s)G_{xs}k^2s^{2\beta+2} - \frac{s(\alpha-r)(G_x(F_{xs}-\gamma G_xG_s)+G_{xs}F_x)}{F_{xx}-\gamma G_x^2} = 0 \qquad (28)$$

Assuming a solution for $F(t,x,s)$and $G(t,x,s)$ as follows:

$$\begin{cases} F(t,x,s) = A(t)x + \frac{B(t)s^{-2\beta}}{\gamma} + \frac{C(t)}{\gamma} \\ A(T) = 1, B(T) = 0, C(T) = 0 \\ G(t,x,s) = P(t)x + \frac{Q(t)s^{-2\beta}}{\gamma} + \frac{R(t)}{\gamma} \\ P(T) = 1, Q(T) = 0, R(T) = 0 \\ F_t = A_tx + \frac{B_ts^{-2\beta}}{\gamma} + \frac{C_t(t)}{\gamma}, F_x = A, F_{xx} = 0, \\ F_s = \frac{-2\beta B(t)s^{-2\beta-1}}{\gamma}, F_{ss} = \frac{2\beta(2\beta+1)B(t)s^{-2\beta-2}}{\gamma} \\ G_t = P_tx + \frac{Q_ts^{-2\beta}}{\gamma} + \frac{R_t(t)}{\gamma}, G_x = P, G_{xx} = 0, \\ G_s = \frac{-2\beta Q(t)s^{-2\beta-1}}{\gamma},\ F_s = \frac{2\beta(2\beta+1)Q(t)s^{-2\beta-2}}{\gamma} \end{cases} \qquad (29)$$

Substituting (29) into (26), (27) and (28), we have:

$$\varphi^* = -\left[\frac{(\alpha-r)A+2\beta QPk^2}{P\gamma xk^2s^{2\beta}}\right] \qquad (30)$$

$$A_t(t) + rA(t) = 0 \qquad (31)$$

$$B_t(t) + 2\alpha\beta B + \frac{(\alpha-r)^2A^2}{2P^2k^2} - 2\beta^2Q^2 + \frac{2(\alpha-r)\beta AQ}{P} = 0 \qquad (32)$$

$$\frac{C_t(t)}{\gamma} + A\left[p\left(\frac{2\omega-2\omega_0-t}{\omega-\omega_0-t}\right) - a\right] - \frac{1}{2}P^2b^2\gamma + \beta sPQk^2 + \frac{\beta(2\beta+1)B(t)k^2}{\gamma} = 0 \qquad (33)$$

$$P_t(t) + rP(t) = 0 \qquad (34)$$

$$Q_t(t) - 2r\beta Q + \frac{(\alpha-r)^2A}{Pk^2} = 0 \qquad (35)$$

$$\frac{R_t(t)}{\gamma} + P(t)\left[p\left(\frac{2\omega-2\omega_0-t}{\omega-\omega_0-t}\right) - a\right] = 0 \qquad (36)$$

Solving (31), (34), (35) we obtain:

$$A(t) = e^{r(T-t)} \qquad (37)$$

$$P(t) = e^{r(T-t)} \qquad (38)$$

$$Q(t) = \frac{(\alpha-r)^2}{2r\beta k^2}\{1 - e^{2r\beta(t-T)}\} \qquad (39)$$

Substituting (37), (38), (39) into (32), (33) and (36) we have:

$$B(t) = \frac{(\alpha-r)^2}{4\beta k^2}\left\{\left(1 - e^{2\alpha\beta(t-T)}\right)\left(\frac{(\alpha-r)^2}{\alpha rk^2} - \frac{2(\alpha-r)}{\alpha} - \frac{1}{\alpha}\right) + \left(\frac{\alpha-r-2rk^2}{rk^2}\right)(e^{2r\beta(t-T)} - e^{2\alpha\beta(t-T)})\right\} \qquad (40)$$

$$R(t) = \gamma\left(\frac{a}{r}\{1 - e^{r(T-t)}\} - p\int_t^T \frac{2\omega-2\omega_0-\tau}{\omega-\omega_0-\tau}e^{r(T-\tau)}d\tau\right) \qquad (41)$$

$$C(t) = \frac{(2\beta+1)(\alpha-r)^2}{8\beta}\left\{\left(1 - e^{2\alpha\beta(t-T)}\right)\left(\frac{rk^2+rk^2(\alpha-r)-(\alpha-r)^2}{\alpha^2rk^2}\right) + \left(\frac{\alpha-r-2rk^2}{rk^2}\right)(\frac{1}{r}(e^{2r\beta(t-T)} - 1) - \frac{1}{\alpha}(e^{2\alpha\beta(t-T)} - 1))\right\} + \frac{s(\alpha-r)^2}{2r^2}\left\{\frac{1}{2\beta-1}(e^{(2r\beta-r)(t-T)} - 1) - ((e^{r(T-t)} + 1)\right\} - b^2\gamma^2(e^{2r(T-t)} - 1) + \gamma\left(\frac{a}{r}\{1 - e^{r(T-t)}\} - p\int_t^T \frac{2\omega-2\omega_0-\tau}{\omega-\omega_0-\tau}e^{r(T-\tau)}d\tau\right) \qquad (42)$$

$$F(t,x,s) = xe^{r(T-t)} + \frac{s^{-2\beta}(\alpha-r)^2}{4\gamma r\beta k^4}\left\{\left(1 - e^{2\alpha\beta(t-T)}\right)\left(\frac{(\alpha-r)^2-2rk^2(\alpha-r)-rk^2}{\alpha}\right) + (\alpha - r - 2rk^2)(e^{2r\beta(t-T)} - e^{2\alpha\beta(t-T)})\right\} + \frac{(2\beta+1)(\alpha-r)^2}{8rk^2\beta\gamma}\left\{\left(1 - e^{2\alpha\beta(t-T)}\right)\left(\frac{rk^2+rk^2(\alpha-r)-(\alpha-r)^2}{\alpha^2}\right) + (\alpha - r - 2rk^2)(\frac{1}{r}(e^{2r\beta(t-T)} - 1) - \frac{1}{\alpha}(e^{2\alpha\beta(t-T)} - 1))\right\} + \frac{s(\alpha-r)^2e^{r(T-t)}}{2r^2\gamma}\left\{\frac{1}{2\beta-1}(e^{2r\beta(t-T)} - e^{r(t-T)}) - (1 - e^{r(t-T)})\right\} - \frac{b^2\gamma}{4r}(e^{2r(T-t)} - 1) + \left(\frac{a}{r}\{1 - e^{r(T-t)}\} - p\int_t^T \frac{2\omega-2\omega_0-\tau}{\omega-\omega_0-\tau}e^{r(T-\tau)}d\tau\right) \qquad (43)$$

$$G(t,x,s) = xe^{r(T-t)} + \frac{s^{-2\beta}(\alpha-r)^2}{2r\beta\gamma k^2}\{1 - e^{2r\beta(t-T)}\} + \left(\frac{a}{r}\{1 - e^{r(T-t)}\} - p\int_t^T \frac{2\omega-2\omega_0-\tau}{\omega-\omega_0-\tau}e^{r(T-\tau)}d\tau\right) \qquad (44)$$

Substituting F_x, F_{xs}, G_x, G_s into (26) we have

$$\varphi^* = \frac{(\alpha-r)}{\gamma xk^2s^{2\beta}}e^{r(t-T)}\left[1 + \frac{(\alpha-r)}{r}\left(1 - e^{2r\beta(t-T)}\right)\right]$$

B. Efficient Frontier

Next, we compute the efficient frontier as follows

Result 2. The efficient frontier is given as

$$E_{t,x,s}[X^{\varphi^*}(T)] = xe^{r(T-t)} + \left(\begin{matrix}\frac{a}{r}\left\{1-e^{r(T-t)}\right\} - \\ p\int_t^T \frac{2\omega-2\omega_0-\tau}{\omega-\omega_0-\tau} e^{r(T-\tau)}d\tau\end{matrix}\right) + \frac{s^{-2\beta}(\alpha-r)}{2r\beta k^2}\left(1-e^{2r\beta(t-T)}\right)\sqrt{\frac{Var_{t,x,s}[X^{\varphi^*}(T)]+\frac{b^2}{2r}(1-e^{2r(T-t)})}{\left(\begin{matrix}\frac{s^{-2\beta}}{2r\beta k^4}\left\{\begin{matrix}\left(1-e^{2\alpha\beta(t-T)}\right)\left(\frac{(\alpha-r)^2-2rk^2(\alpha-r)-rk^2}{\alpha}\right)\\+(\alpha-r-2rk^2)\left(e^{2r\beta(t-T)}-e^{2\alpha\beta(t-T)}\right)\end{matrix}\right\}\\+\frac{(2\beta+1)}{4rk^2\beta}\left\{\begin{matrix}\left(e^{2\alpha\beta(t-T)}-1\right)\left(\frac{rk^2+rk^2(\alpha-r)-(\alpha-r)^2}{\alpha^2}\right)\\+(\alpha-r-2rk^2)(\frac{1}{r}(e^{2r\beta(t-T)}-1)-\frac{1}{\alpha}(e^{2\alpha\beta(t-T)}-1))\end{matrix}\right\}\\\frac{se^{r(T-t)}}{r^2}\left\{\left(1-e^{r(t-T)}\right)-\frac{1}{2\beta-1}\left(e^{2r\beta(t-T)}-e^{r(t-T)}\right)\right\}\end{matrix}\right)}}$$

Proof. Recall that

$$Var_{t,x}[X^{\varphi^*}(T)] = E_{t,x}[X^{\varphi^*}(T)^2] - (E_{t,x}[X^{\varphi^*}(T)])^2$$
$$Var_{t,x,s}[X^{\varphi^*}(T)] = \frac{2}{\gamma}(G(t,x,s) - F(t,x,s))$$

Substituting (43) and (44) for $F(t,x,s)$ and $G(t,x,s)$ in the above equation, we have

$$Var_{t,x,s}[X^{\varphi^*}(T)] = \left[\frac{s^{-2\beta}(\alpha-r)^2}{2\gamma^2 r\beta k^4}\left\{\left(1-e^{2\alpha\beta(t-T)}\right)\left(\frac{(\alpha-r)^2-2rk^2(\alpha-r)-rk^2}{\alpha}\right) + (\alpha - r - 2rk^2)(e^{2r\beta(t-T)} - e^{2\alpha\beta(t-T)})\right\} + \frac{(2\beta+1)(\alpha-r)^2}{4rk^2\beta\gamma^2}\left\{\left(e^{2\alpha\beta(t-T)} - 1\right)\left(\frac{rk^2+rk^2(\alpha-r)-(\alpha-r)^2}{\alpha^2}\right) + (\alpha - r - 2rk^2)(\frac{1}{r}(e^{2r\beta(t-T)} - 1) - \frac{1}{\alpha}(e^{2\alpha\beta(t-T)} - 1))\right\} + \frac{s(\alpha-r)^2e^{r(T-t)}}{r^2\gamma^2}\left\{\left(1-e^{r(t-T)}\right) - \frac{1}{2\beta-1}(e^{2r\beta(t-T)} - e^{r(t-T)})\right\} + \frac{b^2}{2r}\{e^{2r(T-t)} - 1\}\right] \quad (45)$$

$$\frac{1}{\gamma} = \frac{1}{(\alpha-r)}\sqrt{\frac{Var_{t,x,s}[X^{\varphi^*}(T)]+\frac{b^2}{2r}(1-e^{2r(T-t)})}{\left(\begin{matrix}\frac{s^{-2\beta}}{2r\beta k^4}\left\{\begin{matrix}\left(1-e^{2\alpha\beta(t-T)}\right)\left(\frac{(\alpha-r)^2-2rk^2(\alpha-r)-rk^2}{\alpha}\right)\\+(\alpha-r-2rk^2)\left(e^{2r\beta(t-T)}-e^{2\alpha\beta(t-T)}\right)\end{matrix}\right\}\\+\frac{(2\beta+1)}{4rk^2\beta}\left\{\begin{matrix}\left(e^{2\alpha\beta(t-T)}-1\right)\left(\frac{rk^2+rk^2(\alpha-r)-(\alpha-r)^2}{\alpha^2}\right)\\+(\alpha-r-2rk^2)(\frac{1}{r}(e^{2r\beta(t-T)}-1)-\frac{1}{\alpha}(e^{2\alpha\beta(t-T)}-1))\end{matrix}\right\}\\\frac{se^{r(T-t)}}{r^2}\left\{\left(1-e^{r(t-T)}\right)-\frac{1}{2\beta-1}\left(e^{2r\beta(t-T)}-e^{r(t-T)}\right)\right\}\end{matrix}\right)}} \quad (46)$$

$$E_{t,x,s}[X^{\varphi^*}(T)] = G(t,x,s) = xe^{r(T-t)} + + \left(\frac{a}{r}\left\{1-e^{r(T-t)}\right\} - p\int_t^T \frac{2\omega-2\omega_0-\tau}{\omega-\omega_0-\tau} e^{r(T-\tau)}d\tau\right) + \frac{1}{\gamma}\left(\frac{s^{-2\beta}(\alpha-r)^2}{2r\beta\gamma k^2}\left\{1-e^{2r\beta(t-T)}\right\}\right) \quad (47)$$

Substituting (46) into (47) we obtain the efficient frontier:

$$E_{t,x,s}[X^{\varphi^*}(T)] = xe^{r(T-t)} + \left(\begin{matrix}\frac{a}{r}\left\{1-e^{r(T-t)}\right\} - \\ p\int_t^T \frac{2\omega-2\omega_0-\tau}{\omega-\omega_0-\tau} e^{r(T-\tau)}d\tau\end{matrix}\right) + \frac{s^{-2\beta}(\alpha-r)}{2r\beta k^2}\left(1-e^{2r\beta(t-T)}\right)\sqrt{\frac{Var_{t,x,s}[X^{\varphi^*}(T)]+\frac{b^2}{2r}(1-e^{2r(T-t)})}{\left(\begin{matrix}\frac{s^{-2\beta}}{2r\beta k^4}\left\{\begin{matrix}\left(1-e^{2\alpha\beta(t-T)}\right)\left(\frac{(\alpha-r)^2-2rk^2(\alpha-r)-rk^2}{\alpha}\right)\\+(\alpha-r-2rk^2)\left(e^{2r\beta(t-T)}-e^{2\alpha\beta(t-T)}\right)\end{matrix}\right\}\\+\frac{(2\beta+1)}{4rk^2\beta}\left\{\begin{matrix}\left(e^{2\alpha\beta(t-T)}-1\right)\left(\frac{rk^2+rk^2(\alpha-r)-(\alpha-r)^2}{\alpha^2}\right)\\+(\alpha-r-2rk^2)(\frac{1}{r}(e^{2r\beta(t-T)}-1)-\frac{1}{\alpha}(e^{2\alpha\beta(t-T)}-1))\end{matrix}\right\}\\\frac{se^{r(T-t)}}{r^2}\left\{\left(1-e^{r(t-T)}\right)-\frac{1}{2\beta-1}\left(e^{2r\beta(t-T)}-e^{r(t-T)}\right)\right\}\end{matrix}\right)}} \quad (48)$$

Lemma1. Suppose $(\alpha - r) > 0, x > 0, k^2s^{2\beta} > 0, e^{r(t-T)} > 0, \left[1 + \frac{(\alpha-r)}{r}\left(1 - e^{2r\beta(t-T)}\right)\right] > 0, 0 < r < 1, \beta < 0$ then $\frac{d\varphi^*}{d\gamma} < 0$

Proof.

$$\varphi^* = \frac{(\alpha-r)}{\gamma xk^2s^{2\beta}}e^{r(t-T)}\left[1 + \frac{(\alpha-r)}{r}\left(1 - e^{2r\beta(t-T)}\right)\right]$$
$$\frac{d\varphi^*}{d\gamma} = -\frac{(\alpha-r)}{\gamma^2xk^2s^{2\beta}}e^{r(t-T)}\left[1 + \frac{(\alpha-r)}{r}\left(1 - e^{2r\beta(t-T)}\right)\right]$$

Since $\frac{(\alpha-r)}{\gamma^2xk^2s^{2\beta}}e^{r(t-T)} > 0$ *and* $\left[1 + \frac{(\alpha-r)}{r}\left(1 - e^{2r\beta(t-T)}\right)\right] > 0$

$$\frac{d\varphi^*}{d\gamma} < 0$$

Lemma2. Suppose $(\alpha - r) > 0, x > 0, k^2s^{2\beta} > 0, e^{r(t-T)} > 0, \left[1 + \frac{(\alpha-r)}{r}\left(1 - e^{2r\beta(t-T)}\right)\right] > 0, 0 < r < 1, \beta < 0$ then $\frac{d\varphi^*}{dx} < 0$

Proof.

$$\varphi^* = \frac{(\alpha-r)}{\gamma xk^2s^{2\beta}}e^{r(t-T)}\left[1 + \frac{(\alpha-r)}{r}\left(1 - e^{2r\beta(t-T)}\right)\right]$$
$$\frac{d\varphi^*}{dx} = -\frac{(\alpha-r)}{x^2\gamma k^2s^{2\beta}}e^{r(t-T)}\left[1 + \frac{(\alpha-r)}{r}\left(1 - e^{2r\beta(t-T)}\right)\right]$$

Since $\frac{(\alpha-r)}{x^2\gamma k^2s^{2\beta}}e^{r(t-T)} > 0$ *and* $\left[1 + \frac{(\alpha-r)}{r}\left(1 - e^{2r\beta(t-T)}\right)\right] > 0$

$$\frac{d\varphi^*}{dx} < 0$$

Lemma3. Suppose $(\alpha - r) > 0, x > 0, k^2s^{2\beta} > 0, e^{r(t-T)} > 0, \left[1 + \frac{(\alpha-r)}{r}\left(1 - e^{2r\beta(t-T)}\right)\right] > 0, 0 < r < 1, \beta < 0$ then $\frac{d\varphi^*}{dt} > 0$

Proof.

$$\varphi^* = \frac{(\alpha-r)}{\gamma xk^2s^{2\beta}}e^{r(t-T)}\left[1 + \frac{(\alpha-r)}{r}\left(1 - e^{2r\beta(t-T)}\right)\right]$$
$$\frac{d\varphi^*}{dt} = e^{r(t-T)}\left[-(\alpha - r)\beta e^{2r\beta(t-T)} - (\alpha - r)e^{2r\beta(t-T)} + \alpha\right]$$

Since $\beta < 0, \alpha > 0, -(\alpha - r)\beta e^{2r\beta(t-T)} + \alpha > (\alpha - r)e^{2r\beta(t-T)}$, hence $\frac{d\varphi^*}{dt} > 0$.

V. Discussion

Lemma 1 and 2 show that the proportion of the wealth invested in the risky asset increases as the risk-averse level and initial wealth decrease. The implication here is that if members have high-risk averse level, they will prefer to invest in a risk-free asset and that will reduce the proportion of the wealth to be invested in the risky asset. Similarly, when the initial wealth is high, members prefer to invest in risk-free asset to reduce risk but if the initial wealth is low, members will take the risk and invest in the risky asset to increase their final wealth.

Lemma 3 shows that as time increases, the optimal investment strategy increases as well, meaning that as time goes on and there is return of premium, the fund manager will increase the proportion of the wealth to be invested in the risky asset to increase the overall pension wealth to satisfy the remaining members still alive.

From (46) we observed that the surplus from the death members after retirement increases the initial wealth of the pension fund and as such increases the risk-averse level, this implies that the surplus decreases the optimal investment strategy. This is in agreement with Lemma 2. Also from (45) and (48), we observed that the higher the risk-averse level, the lower the variance and the expected return of the members and vice versa.

VI. Conclusion

The optimal investment strategy for a DC pension scheme with multiple contributors and with the return of premium clauses under CEV model is studied using mean-variance utility function. We consider the return clause which permits members to claim their accumulated contributions after death. We also consider that before the investment, the surplus includes funds from members who died after retirement. In addition, we considered the investment in a risk-free asset and a risky asset to achieve the expected returns of the remaining members and we obtained an optimized problem with the help of Hamilton's equation Jacobi Bellman. By using a stochastic optimal control technique, we obtain the optimal investment strategies for the two assets and the efficient frontier of the members. Furthermore, we studied the effect of the various parameters of the optimal investment strategies and the effect of the risk-averse level on the efficient frontier. We observed that the optimal investment strategy is the same in [7], secondly, we observed that the surplus decreases the proportion of the wealth invested in the risky asset.

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