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Applying *p*-Balanced Energy Technique to Solve Liouville-Type Problems in Calculus

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***Abstract*—**We are interested in solving Liouville-type problems to explore constancy properties for maps or differential forms on Riemannian manifolds. Geometric structures on manifolds, the existence of constancy properties for maps or differential forms, and energy growth for maps or differential forms are intertwined. In this article, we concentrate on discovery of solutions to Liouville-type problems where manifolds are Euclidean spaces (i.e. flat Riemannian manifolds) and maps become real-valued functions. Liouville-type results of vanishing properties for functions are obtained. The original work in our research findings is to extend the q-energy for a function from finite in L^q space to infinite in non-L^q space by applying p-balanced technique where $q = p = 2$. Calculation skills such as Hölder's Inequality and Tests for Series have been used to evaluate limits and integrations for function energy. Calculation ideas and computational techniques for solving Liouville-type problems shown in this article, which are utilized in Euclidean spaces, can be universalized as a successful algorithm, which works for both maps and differential forms on Riemannian manifolds. This innovative algorithm has a far-reaching impact on research work of solving Liouville-type problems in the general settings involved with infinite energy. The p-balanced technique in this algorithm provides a clue to success on the road of q-energy extension from finite to infinite.

***Keywords*—**Differential Forms, Hölder Inequality, Liouville-type problems, p-balanced growth, p-harmonic maps, q-energy growth, tests for series.

I. Introduction

The study of Liouville-type problems in Differential Geometry is to discover constancy properties for maps or differential forms between the domain and the target on Riemannian manifolds. Existence of constancy properties is determined by geometric structures on manifolds and energy growth for maps or differential forms.

Liouville-type problems have been studied in two directions. One of the research directions is to study all kinds of manifolds equipped with various metric structures. Manifolds can be classified according to their curvature values. Flat manifolds such as Euclidean Spaces are manifolds with curvature values equal to zero. Non-flat manifolds are curved manifolds with non-zero curvature values varying from positive to negative. Mathematicians have been interested in investigating all possible manifold metric structures to assure existence of constancy properties for maps or differential forms. The other research direction is to study energy growth for maps or differential forms. Most research work was to study q-energy for maps or differential forms. Many Liouville-type results have been obtained by mathematicians in the context of finite q-energy for maps or differential forms in L^q space.

Liouville-type results are summarized as follows: In 1976, Schoen and Yau solved Liouville-type problems for harmonic maps and obtained the constancy property on manifolds with non-negative Ricci curvature values [1]. In 1981, Greene and Wu obtained Liouville-type results of vanishing property for harmonic 1-forms with finite q-energy in L^q space on non-negatively curved manifolds for $q > 1$ [2]. In 1994, Kawai discovered Liouville-type results for p-harmonic maps from p-parabolic manifolds to manifolds with non-positive curvature values when $p \geq 2$ [3]. In 1995, Cheung and Leung proved Liouville theorems for p-harmonic maps for $p \geq 2$ with finite q-energy in L^q space where $q = p - 1$ on the target space of Cartan-Hadamand manifolds [4]. In 2001, Zhang proved Liouville-type Theorems for closed and p-co-closed differential 1-form ($p > 1$) with finite q-energy in L^q space ($q > 0$) on positively curved manifolds [5]. In 2008, Pigola et al. explored Liouville-type results for p-harmonic maps on curved manifolds, provided the domain manifolds support a Sobolev–Poincaré Inequality [6].

In this article, we focus on solving Liouville-type problems where manifolds are Euclidean spaces and maps are real-valued functions. Our research goal is to break constraints of finite q-energy into compound energy inequalities to obtain Liouville-type results. In particular, we apply the p-balanced energy technique for functions to generalize q-energy from finite to infinite. More precisely, we apply Hölder's Inequality and Tests for Series to evaluate limits and integrations for function energy in the computational method. Liouville-type result of vanishing properties for functions is obtained.

The original work in our research findings as Liouville-type results is to extend the q-energy for a function from finite in L^q space to infinite in non-L^q space by applying p-balanced technique where $q = p = 2$. Computational methods and energy estimation techniques applied to functions in Euclidean spaces, which are presented in this article, can be generalized as a successful algorithm applied to maps or differential forms on Riemannian manifolds. This algorithm can play an important role in solving Liouville-type problems in the general settings with energy approaching to infinite. The p-balanced energy technique in this innovative algorithm has a far-reaching impact on research work of q-energy generalization from finite

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to infinite.

II. Preliminary

In this section, we give definitions of *p*-harmonic maps, differential forms, *q*-energy and *p*-balanced energy for maps or differential forms respectively. We also recall Bochner-Weitzenbock Formula on manifolds, Hölder's Inequality, Cauchy-Schwarz Inequality, and Tests for Series in Calculus.

Let M be an n-dimensional complete non-compact Riemannian manifold with volume element dv and $B(x_0;r)$ (or $B(r)$) be a geodesic ball of radius r centered at a point x_0 on M.

Let $\mathcal{A}^k(\rho) = C(\Lambda^k T^*M \otimes V)$ be the space of smooth k-forms on M with values in the vector bundle $\rho : V \to M$. Let $d: \mathcal{A}^k(\rho) \to \mathcal{A}^{k+1}(\rho)$ be the exterior differential operator and $d^*: \mathcal{A}^k(\rho) \to \mathcal{A}^{k-1}(\rho)$ be the adjoint differential operator of d given by $d^* = -\sum_{j=1}^n i(e_j)\nabla_{e_j}$ where $\{e_j\}$ is a local orthonormal frame at $x \in M$, and $i(X)$ is the interior product by X given by $(i(X)\nu)(Y_1,\cdots,Y_{k-1}) = \nu(X,Y_1,\cdots,Y_{k-1})$ for any $X \in T_xM$, $\nu \in \mathcal{A}^k(\rho)$ and $Y_l \in T_xM$, $1 \le l \le k-1$. In particular, if $\nu \in \mathcal{A}^1(\rho)$, d^* is also defined by $d^*\nu = -\text{trace}\nabla\nu = -\text{div}\nu$. The Hodge Laplacian Δ is defined on the V-valued differential forms by $\Delta = -(dd^* + d^*d) : \mathcal{A}^k(V) \to \mathcal{A}^k(V)$. The norm of ν is denoted by $|\nu| = <\nu,\nu>^{\frac{1}{2}}$. More details can be found in [7].

Definition 1. A differential form ξ is said to be harmonic if $\Delta\xi = -(dd^* + d^*d)\xi = 0$, closed if $d\xi = 0$, co-closed if $d^*\xi = 0$.

Definition 2. A differential form ξ is said to be p-pseudo-co-closed (p > 1) if $d^*(|\xi|^{p-2}\xi) = 0$.

Definition 3. A function or a differential form f has finite q-energy (for q > 0) in L^q-space if $\int_M |f|^q \, dv < \infty$. Otherwise, f has infinite q-energy (for $q > 0$) in non-L^q-space if $\int_M |f|^q \, dv = \infty$.

The concept of *p*-balanced growth for $p > 1$ consists of 5 cases: *p*-finite growth, *p*-mild growth, *p*-obtuse growth, *p*-moderate growth, and *p*-small growth. A function or a differential form *f* is said to be with *p*-balanced growth provided *f* has one of "*p*-finite, *p*-mild, *p*-obtuse, *p*-moderate, and *p*-small" growth where *p* > 1. Otherwise, a function or a differential form *f* is said to be with *p*-imbalanced growth [8].

Definition 4. A function or a differential form f has p-finite growth if f satisfies

$$\liminf_{r\to\infty} \frac{1}{r^p} \int_{B(x_0;r)} |f|^q \, dv < \infty ,$$

and has p-infinite growth otherwise for p > 1 and q > 0.

A function or a differential form f has p-mild growth if there exists $x_0 \in M$, and a strictly increasing sequence of $\{r_j\}_0^\infty$ going to infinity, such that for every $l_0 > 0$, we have

$$\sum_{j=l_0}^{\infty} \left(\frac{(r_{j+1} - r_j)^p}{\int_{B(x_0;r_{j+1})\setminus B(x_0;r_j)} |f|^q \, dv} \right)^{\frac{1}{p-1}} = \infty ,$$

and has p-severe growth otherwise for p > 1 and q > 0.

A function or a differential form f has p-obtuse growth if there exists $x_0 \in M$ such that for every $a > 0$, we have

$$\int_a^\infty \left(\frac{1}{\int_{\partial B(x_0;r)} |f|^q \, ds} \right)^{\frac{1}{p-1}} dr = \infty ,$$

and has p-acute growth otherwise for p > 1 and q > 0.

A function or a differential form f has p-moderate growth if there exists $x_0 \in M$ and $F(r) \in \mathcal{F}$, such that

$$\limsup_{r\to\infty} \frac{1}{r^p F^{p-1}(r)} \int_{B(x_0;r)} |f|^q \, dv < \infty ,$$

where

$$\mathcal{F} = \{F : [a,\infty) \to (0,\infty) \mid \int_a^\infty \frac{dr}{rF(r)} = \infty \ \textit{for some}\ a \ge 0\} ,$$

(Notice that functions or differential forms in $\mathcal{F}$ are not necessarily monotone.) and has p-immoderate growth otherwise for p > 1 and q > 0.

A function or a differential form f has p-small growth if there exists $x_0 \in M$ such that for every $a > 0$, we have

$$\int_a^\infty \left(\frac{r}{\int_{B(x_0;r)} |f|^q \, dv} \right)^{\frac{1}{p-1}} dr = \infty ,$$

and has p-large growth otherwise for p > 1 and q > 0.

The above definitions of "*p*-finite, *p*-mild, *p*-obtuse, *p*-moderate, and *p*-small" and their counter-parts "*p*-infinite, *p*-severe, *p*-acute, *p*-immoderate, and *p*-large" growth depend on *q*, and *q* will be specified in the context in which the definition is used.

Here, it is obvious for us to observe that a function or a differential form *f* has the vanishing *p*-finite growth if *f* has finite *q*-energy, that is:

$$\liminf_{r\to\infty} \frac{1}{r^p} \int_{B(x_0;r)} |f|^q \, dv = 0 (< \infty) \text{ if } \int_M |f|^q \, dv < \infty .$$

Definition 5. The p-energy (p > 1) functional for a map u is given by $E_p(u) = \frac{1}{p}\int_M |du|^p \, dv$ where du denotes the differential of u.

Definition 6. A map u is said to be p-harmonic ($p > 1$) if it is a critical point of p-energy functional $E_p(u)$. Equivalently, u is p-harmonic if it is a solution to $\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$. u is said to be harmonic (i.e. p-harmonic for $p = 2$) if it is a solution to $\Delta u = \operatorname{div}(\nabla u) = 0$.

Lemma 1. (Bochner-Weitzenbock Formula) For any differential form ξ on M, the following identity holds:

$$\frac{1}{2}\Delta|\xi|^2 = <\Delta\xi, \xi> + |\nabla\xi|^2 + \mathcal{R}(\xi,\xi)$$

where $\mathcal{R}(\xi,\xi)$ denotes the Ricci curvature of M in the direction of ξ .

Lemma 2. (Hölder's Inequality) Let $p, q \in (1,\infty)$ with $1/p + 1/q = 1$. For any positive numbers $a = (a_1, a_2, \cdots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ in R^n , we have $\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}$. For any positive functions f and g, we have $\int fg dv \le \left(\int f^p dv\right)^{\frac{1}{p}} \left(\int g^q dv\right)^{\frac{1}{q}}$. In particular, for p=q=2, we have Cauchy-Schwarz Inequality as a special case of Hölder's Inequality: $\int fg dv \le \left(\int f^2 dv\right)^{\frac{1}{2}} \left(\int g^2 dv\right)^{\frac{1}{2}}$.

Next, let us recall Limit of the *n*-th Term of a Convergent Series and a test for a telescoping series in Calculus.

Theorem 1. If the series $\sum_{n=1}^{\infty} a_n$ converges, then the sequence $\{a_n\}$ converges to zero, that is:

$$\text{If } \sum_{n=1}^{\infty} a_n \text{ converges, then } \lim_{n\to\infty} a_n = 0 .$$

Theorem 2. A telescoping series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (b_n - b_{n+1})$ converges if and only if the sequence $\{b_n\}$ is convergent to a finite number L. Furthermore, the sum of the convergent telescoping series will approach to the value of $b_1 - L$, that is:

$$\sum_{n=1}^{\infty} (b_n - b_{n+1}) \text{ converges iff } \lim_{n\to\infty} b_n = L .$$

and the sum $S = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \lim_{n\to\infty}(b_1 - b_{n+1}) = b_1 - L$.

III. Result and Proof

In this section, we first give a statement of the Liouville Theorem for functions. After that, we present the detailed proof on how to obtain the vanishing property for a function as Liouville-type results of our research findings.

A. Result of Liouville Theorem for Functions

Theorem 3. Assume that two functions f and g satisfy the following two conditions where $0 \le f \le g$:

1. $\int_0^b f(t)g(t)dt \le \frac{\int_a^b f(t)g(t)dt}{b-a}$ for $\forall 0 < a < b$;
2. $\liminf_{r\to\infty} \frac{\int_0^r (g(t))^2 dt}{r^2} < \infty$ (that is, g has 2-finite growth for q = 2).

Then, we obtain the vanishing property for the function f, that is $f(t) \equiv 0$.

B. Proof

Since $r \to \infty$, there must exist an increasing sequence $\{r_j\}$ such that

$$r_{j+1} \ge 2r_j \text{ or } r_{j+1} - r_j \ge \frac{1}{2} r_{j+1} .$$

Letting $a = r_j$ and $b = r_{j+1}$, we can rewrite the first condition $\int_0^b f(t)g(t)dt \le \frac{\int_a^b f(t)g(t)dt}{b-a}$ as follows: $\int_0^{r_{j+1}} f(t)g(t)dt \le \frac{\int_{r_j}^{r_{j+1}} f(t)g(t)dt}{r_{j+1} - r_j}$.

Since $0 \le f \le g$, via Cauchy-Schwarz Inequality, we have:

$$\begin{aligned}\int_0^{r_{j+1}} (f(t))^2 dt &\le \int_0^{r_{j+1}} f(t)g(t)dt \le \frac{\int_{r_j}^{r_{j+1}} f(t)g(t)dt}{r_{j+1} - r_j} \\ &\le \frac{1}{r_{j+1} - r_j}\left(\int_{r_j}^{r_{j+1}} (f(t))^2 dt\right)^{\frac{1}{2}} \left(\int_{r_j}^{r_{j+1}} (g(t))^2 dt\right)^{\frac{1}{2}}\end{aligned} \quad (1)$$

For simplicity, we define the following notations:

$$\begin{aligned} Q_{j+1} &:= \int_0^{r_{j+1}} (f(t))^2 dt \\ A_j &:= \frac{1}{r_j^2}\int_0^{r_j} (g(t))^2 dt \end{aligned} .$$

We notice that the sequence $\{A_j\}$ is bounded since $\liminf_{j\to\infty} A_j = \liminf_{r\to\infty} \frac{\int_0^r (g(t))^2 dt}{r^2} < \infty$ from the second condition in assumption that g has 2-finite growth *for q = 2.* Then (1) becomes

$$Q_{j+1} \leq \frac{1}{r_{j+1}-r_j}\left(Q_{j+1}-Q_j\right)^{\frac{1}{2}}\left(r_{j+1}^2 A_{j+1}-r_j^2 A_j\right)^{\frac{1}{2}}.$$

By squaring on both sides, we have:

$$Q_{j+1}^2 \leq \frac{1}{(r_{j+1}-r_j)^2}\left(Q_{j+1}-Q_j\right)\left(r_{j+1}^2 A_{j+1}-r_j^2 A_j\right) \qquad (2)$$

This implies:

$$\begin{aligned} Q_{j+1} &\leq \frac{Q_{j+1}^2}{Q_{j+1}-Q_j} \\ &\leq \frac{1}{(r_{j+1}-r_j)^2}\left(r_{j+1}^2 A_{j+1}-r_j^2 A_j\right) \\ &\leq 4A_{j+1} \\ &\leq 4K \end{aligned} \qquad (3)$$

where we use the fact $r_{j+1}-r_j \geq \frac{1}{2}r_{j+1}$ and the bounded sequence $A_{j+1} \leq K$ (where K is a finite positive number).

We sum up (2) to get:

$$\sum_{j=1}^{N} Q_{j+1}^2 \leq 4K\sum_{j=1}^{N}\left(Q_{j+1}-Q_j\right) \leq 4K \cdot 4K = 16K^2 \ \ for \ \forall N > 1$$

where we apply $\frac{1}{(r_{j+1}-r_j)^2}\left(r_{j+1}^2 A_{j+1}-r_j^2 A_j\right) \leq 4A_{j+1} \leq 4K$ and $Q_{j+1} \leq 4K$ in (3).

Here, we prove that the series $\sum_{j=1}^{\infty} Q_{j+1}^2$ is convergent. By Theorem 1, we claim that $\lim_{j\to\infty} Q_{j+1} \to 0$, that is $\int_0^\infty (f(t))^2 dt = 0$.

Furthermore, we verify $f(t) \equiv 0$ if $\int_0^\infty (f(t))^2 dt = 0$. We obtain a Liouville-type result of the vanishing property for a function f in the context of p-finite growth as one case of p-balanced technique where $q = p = 2$.

IV. Conclusions

The original work in our research findings is to explore and verify a new energy technology and an innovative algorithm as a successful way to extend q-energy from finite in L^q space to infinite in non-L^q space. Computational methods and algorithm utilized to functions for Liouville-type solutions in flat manifolds, which are presented in this article, can be successfully applied to maps or differential forms for Liouville-type solutions on curved manifolds.

V. Research Impact

Both p-balanced energy technique and this innovative algorithm have far-reaching research impact on solving Liouville-type problems in the general settings with infinite energy. As applications of this successful algorithm and the p-balanced energy technique, many Liouville-type results for maps and differential forms approaching infinite q-energy in non-L^q space have been achieved in Wu's research work [9]-[13].

Regarding research impacts on differential forms, we have explored the technique of p-balanced energy in this innovative algorithm to overcome difficulties of q-energy extension up to infinite [9]-[12]. Starting with the definition of p-balanced energy growth, we are interested in a harmonic form on a manifold with non-negative Ricci curvature [9], a closed and p-pseudo-co-closed differential 1-form on a curved manifold with the support of Sobolev-Poincaré Inequality in a mix of curvature signs [10], a closed and co-closed differential k-form on a complete non-compact manifold [11], a closed and p-pseudo-co-closed differential 1-form on a manifold with non-negative Ricci curvature [12].

In a summary, for any differential form ω on M and an appropriate range of m [9], [10], [12], we consider two non-negative functions $f = |\omega|^{m-1}|\nabla|\omega|^2|$ and $g = |\omega|^{m+1}$.

After that, we figure out that Bochner-Weitzenbock Formula applied for differential forms on manifolds has played a significant role in guaranteeing the first assumption of Theorem 3 to be satisfied. More precisely, we claim that Bochner-Weitzenbock Formula works as the foundation to establish (4), which initiates the process of this innovative logarithm listed as below:

Since $r \to \infty$, there must exist an increasing sequence $\{r_j\}$ such that

$$r_{j+1} \geq 2r_j \ \text{ or } \ r_{j+1}-r_j \geq \frac{1}{2}r_{j+1}.$$

Letting $a = r_j$ and $b = r_{j+1}$, we can rewrite the first condition $\int_0^b f(t)g(t)dt \leq \frac{\int_a^b f(t)g(t)dt}{b-a}$ or $\int_0^{r_{j+1}} f(t)g(t)dt \leq \frac{\int_{r_j}^{r_{j+1}} f(t)g(t)dt}{r_{j+1}-r_j}$ in terms of $f = |\omega|^{m-1}|\nabla|\omega|^2|$ and $g = |\omega|^{m+1}$. In addition,

$$\begin{aligned} \int_0^{r_{j+1}} (f(t))^2 dt &\leq \int_0^{r_{j+1}} f(t)g(t)dt \leq \frac{\int_{r_j}^{r_{j+1}} f(t)g(t)dt}{r_{j+1}-r_j} \\ &\leq \frac{1}{r_{j+1}-r_j}\left(\int_{r_j}^{r_{j+1}} (f(t))^2 dt\right)^{\frac{1}{2}}\left(\int_{r_j}^{r_{j+1}} (g(t))^2 dt\right)^{\frac{1}{2}} \end{aligned}$$

has been modified as

$$\begin{aligned} & C_1 \int_{B(r_{j+1})} \eta_j^2 |\omega|^{2m-2}|\nabla|\omega|^2|^2 \, dv \\ \leq \ & \frac{C_2}{r_{j+1}-r_j}\left(\int_{B(r_{j+1})\setminus B(r_j)} \eta_j^2 |\omega|^{2m-2}|\nabla|\omega|^2|^2 \, dv\right)^{\frac{1}{2}}\left(\int_{B(r_{j+1})\setminus B(r_j)} |\omega|^{2m+2} \, dv\right)^{\frac{1}{2}} \end{aligned} \qquad (4)$$

where C_1, C_2 are the positive constants and the test function of $\eta_j = \eta(x; r_j, r_{j+1})$ is a rotationally symmetric Lipschitz continuous function with the following properties:

a. $\eta_i \equiv 1$ on $B(x; r_j)$

b. $\eta_i \equiv 0$ off $B(x; r_{j+1})$

c. $0 \le \eta_i \le 1$ on $B(x; r_{j+1}) \setminus B(x; r_j)$

d. $|\nabla \eta_i| \le \dfrac{C}{r_{j+1} - r_j}$ a.e. on M for a positive constant C, which is independent with choices of $\{r_j\}$.

For simplicity, we re-define the following notations:

$$Q_{j+1} := \int_{B(r_{j+1})} \eta_j^2 \, |\omega|^{2m-2} |\nabla |\omega|^2|^2 \, dv$$

$$A_j := \frac{1}{r_j^2} \int_{B(r_j)} |\omega|^{2m+2} \, dv$$

We notice that the sequence $\{A_j\}$ is bounded since $\liminf_{j\to\infty} A_j = \liminf_{r_j\to\infty} \dfrac{\int_{B(r_j)} |\omega|^{2m+2}\, dv}{r_j^2} < \infty$ from the second condition in assumption that the differential form ω has 2-finite growth for $q = 2m+2$. Then (4) becomes

$$Q_{j+1} \le \frac{C_3}{r_{j+1} - r_j} \left(Q_{j+1} - Q_j\right)^{\frac{1}{2}} \left(r_{j+1}^2 A_{j+1} - r_j^2 A_j\right)^{\frac{1}{2}},$$

where C_3 is a positive constant.

As the same proof shown in Theorem 3, we prove that the series $\sum_{j=1}^{\infty} Q_{j+1}^2$ is convergent. By Theorem 1, we claim that $\lim_{j\to\infty} Q_{j+1} \to 0$, i.e. $\int_0^\infty (f(t))^2 dt = 0$. In the other words, we prove $\int_M (|\omega|^{m-1} \nabla |\omega|^2)^2 dv = 0.$ Based on the fact of $(|\omega|^{m-1} \nabla |\omega|^2)^2 = \dfrac{4}{(m+1)^2} |\nabla |\omega|^{m+1}|^2,$ we verify that $\nabla |\omega|^{m+1} \equiv 0$ on M and obtain the result of $|\omega| = constant$.

Furthermore, special manifold structures determined by curvature properties or Sobolev–Poincaré Inequality have ruled out the existence of differential forms equal to non-zero constants. Therefore, the existence of differential forms with the zero constant property, as Liouville-type results of vanishing properties for differential forms ω, has been achieved in the context of p-finite growth as one case of p-balanced energy technique where $p = 2$. The detailed proofs can be found [9], [10], [12].

In [11], we consider $f = \sqrt{|d\omega|^2 + |d^*\omega|^2}$ and $g = |\omega|$ for a differential form ω. The same algorithm has been initiated from the following inequality:

$$\int_{B(r_{j+1})} \eta_j^2 (|d\omega|^2 + |d^*\omega|^2) dv$$

$$\le \frac{C_4}{r_{j+1} - r_j} \left(\int_{B(r_{j+1}) \setminus B(r_j)} \eta_j^2 (|d\omega|^2 + |d^*\omega|^2) dv \right)^{\frac{1}{2}} \cdot \left(\int_{B(r_{j+1}) \setminus B(r_j)} |\omega|^2 \, dv \right)^{\frac{1}{2}}$$

along with the revised notations listed as below:

$$Q_{j+1} = \int_{B(r_{j+1})} \eta_j^2 (|d\omega|^2 + |d^*\omega|^2) dv$$

$$A_j = \frac{1}{r_j^2} \int_{B(r_j)} |\omega|^2 \, dv$$

for a positive constant C_4.

Regarding research impacts on maps, we have applied p-balanced energy technique and this innovative algorithm to p-harmonic maps to overcome difficulties of q-energy generalization leading to infinity [10]. Starting with a definition of p-balanced energy growth [10], we focus on the differential of a p-harmonic map u (i.e. du) on a manifold supported by Sobolev–Poincaré Inequality with the mixed curvature signs. Just as the same argument in this innovative logarithm by setting $\omega = du$, we obtain $du = 0$, which indicates u as constant. Liouville-type results of constancy property for a p-harmonic map have been obtained in the context of p-finite growth as one case of p-balanced technique where $p = 2$.

VI. Future Research Plans

Mathematicians have been studying Liouville-type problems to overcome the challenge of infinite q-energy for many years. Many effective research approaches and valuable results have been found. In this article, we only discover the p-finite growth as one case of the p-balanced energy technique to extend q-energy from finite to infinite. Actually, the remaining four cases of "p-mild, p-obtuse, p-moderate, and p-small" growth [5] in the p-balanced energy technique can be continuously explored as effective approaches to overcome difficulties of q-energy generalization leading to infinite q-energy. The logarithm in the context of p-balanced energy technique for cases of "p-mild, p-obtuse, p-moderate, and p-small" growth will be our follow-up research work. Furthermore, seeking successful algorithms with effective energy estimation techniques to solve Liouville-type problems in the general settings at all possible infinite energy situations will be our research interest in the future.

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