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Studies on Properties of Knowledge Dependency and Reduction Algorithm in Tolerance Rough Set Model

Chen Wu, Lijuan Wang

***Abstract*—**Relation between tolerance class and indispensable attribute and knowledge dependency in rough set model with tolerance relation is explored. After giving definitions and concepts of knowledge dependency and knowledge dependency degree for incomplete information system in tolerance rough set model by distinguishing decision attribute containing missing attribute value or not, the result of maintaining reflectivity, transitivity, augmentation, decomposition law and merge law for complete knowledge dependency is proved. Knowledge dependency degrees (not complete knowledge dependency degrees) only satisfy some laws after transitivity, augmentation and decomposition operations. An algorithm to solve attribute reduction in an incomplete decision table is designed. The correctness is checked by an example.

***Keywords*—**Incomplete information system, rough set, tolerance relation, knowledge dependence, attribute reduction.

I. Introduction

ROUGH set model has been used to process imprecise or vague information to extract hidden and potential knowledge successively for many years. It was first suggested by Pawlak in 1982s [1]-[3]. For complete information systems, it establishes an indiscernibility relation (an equivalence relation) on a universe of objects and defines lower and upper approximations of a universe subset to form determinative rules and possible rules respectively. So it is widely applied in many areas such as business intelligence, classification, clustering, identification and etc. to solve decision, recognition or prediction problems [4], [5]. However, in incomplete information system (IIS) or incomplete decision table (IDT), because some object attribute values may miss for measurement error or data collecting limitations, such an indiscernibility relation cannot be constructed, owing to that the missing data (or null value) cannot be compared with other values. Fortunately, scientists have explored two main methods currently dispose such a case [6], [8]. One is to fill out missing values by appropriate values, e.g. means or frequent appeared values, to let the IIS or IDT be complete. This method is called indirect one. The other is to let the system or table remain unchanged but extend the equivalence relation to non-equivalence relation to process it. This method is called direct method. Unlike the indirect method, the direct method preserves the originality of the system and develops many new approaches to mine knowledge from the system. It has attracted interests from many scientists and obtained many meaningful research results. Unknown attribute value or null value has two different semantic meanings in IIS. One is that unknown attribute value is missing right now but it really exists. The other is that the value is absent and is prohibited to be compared with others.

Under the first semantic explanation, scientists have already suggested many approaches. The mainly processing ways are to expand the relation among objects. For example, in [4], a tolerance relation is put forward and according to tolerance relation, the tolerance class and generalized decision function are also formed to replace equivalence class and discernibility function respectively to acquire generalized decision rules. In [5], [6], non-symmetric similarity relation is suggested. In [7], limited tolerant relation is proposed for exerting stricter condition on tolerance relation. In [8], maximal consistent block technique for rule acquisition is explored. In [9], different approximations are discussed from information granules view and based on different coverings. In [10], a generalized rough set model with compatibility kernels is discussed. In [11], a variable precision rough set model is proposed. In [12], [13], algorithms to solve different upper and lower approximations are designed. In [14]-[16], multi- granular rough set models in IIS are suggested. In [17]-[19], some expanded rough set models are introduced and studied. In [20], an application of expanded rough set models to radar detection is researched.

Based on the first semantic explanation about missing data and compared to complete information system, the present paper mainly studies some special properties of dispensable/indispensable attribute, knowledge dependency, and knowledge dependency degree and etc. in IIS. It discusses relations between tolerance class and indispensable attribute, dispensable attribute, knowledge dependency in IIS. Depending on different situations, it newly defines the concepts of knowledge dependency degree for IIS [21]. It proves reflexivity, transitivity, augmentation, decomposition and merges laws about knowledge dependency in IIS. It also gives and proves that knowledge dependency degree for transferring, augmenting, decomposing satisfies some laws. An algorithm, based on knowledge dependency in IIS, to find attribute reduction is designed. The work is a contribution to discriminate some characters in IIS from complete information system in rough set model.

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Chen Wu is with the the School of Computer Science and Engineering, Jiangsu University of Science and Technology, Zhenjiang City, Jiangsu Province, China (corresponding author: +8613861390697; e-mail: wuchenzj@sina.com).

Lijuan Wang is with the the School of Computer Science and Engineering, Jiangsu University of Science and Technology, Zhenjiang City, Jiangsu Province,China (e-mail: yzlujing@sina.com).

II. Definitions and Concepts

Let $IIS=(U,AT=A\cup\{d\},V,f)$ be an IIS or decision table [4], where U is a finite non-empty set of objects, A is the finite non-empty set of condition attributes d is a decision attribute. AT is the entire attribute set. For $\forall a\in AT$, $a:V\rightarrow V_a$, where V_a is the value set of a. For an object, any attribute value may be missed. The missed value can be called null value (denoted by *). That is, for $\forall\ a\in AT$, we may have $*\in V_a$. $V=\cup V_a(a\in AT)$ is the set of all attribute values.

Definition 1. For $\forall B\subseteq AT$ in IIS, $T(B)=\{(x,y)|\ \forall a\in B\ (f(x,a)=f(y,a)\vee f(x,a)=*\vee f(y,a)=*)\}$ is called the tolerance relation on U. $T(a)$ is used to be as the short form of $T(B)=T(\{a\})$,i.e., $B=\{a\}$is a singleton set. Obviously, $T(B)$ is of reflexivity and symmetry on U. $S_B(x)=\{y|(x,y)\in T(B)\}$ is called the tolerance class referred to x as generator. $S_a(x)$ is used to be a short form of $S_B(x)=S_{\{a\}}(x)$, if $B=\{a\}$ is a singleton set. $U/T(B)=\{S_B(x)|x\in U\}$, U/B for short, is called a knowledge system or a complete cover on U.

Definition 2. For $\forall X\subseteq U$, $\forall B\subseteq AT$, $B\_(X)=\{y|y\in U, S_B(y)\subseteq X\}$ is the lower approximation of X; $B^{-}(X)=\{y|y\in U, S_B(y)\cap X\neq\varnothing\}$ is the upper approximation of X.

Definition 3. Let $B\subseteq AT$, $B\neq\varnothing$. If $T(B)=T(B\text{-}\{a\})$ for $a\in B$, then a is redundant or dispensable in B, otherwise a is indispensable. B is independent, if $\forall a\in B$ is indispensable in B, otherwise B is dependent.

Theorem 1. If $a\in B$ is redundant in B, $b\in B$ and for $\forall\ x\in U$ $S_a(x)=S_b(x)$, then a is also redundant in B.

Proof. Since $a\in B$ is redundant in B, i.e., $T(B)=T(B\text{-}\{a\})$, $\forall x\in U$, $S_{B\text{-}\{a\}}(x)=S_B(x)$. $S_B(x)=\cap S_e(x)\ (e\in B)=\cap S_e(x)\ (e\in B\text{-}\{a\})=\cap S_e(x)(e\in B\text{-}\{b\})=S_{B\text{-}\{b\}}(x)$. Therefore, $T(B)=T(B\text{-}\{b\})$. So b is also redundant in B.

Definition 4. Let $N\subseteq B\subseteq AT$. If $T(N)=T(B)$ and for any $M\subset N$ $T(M)\neq T(B)$, then N is called a reduction of B.

Theorem 2. Let $B\subseteq AT$, $a\in B$. If for $\forall\ x\in U$, $S_a(x)=\cup S_{B\text{-}\{a\}}(y)\ (y\in S_a(x))$, then a is redundant in B, i.e., $T(B\text{-}\{a\})=T(B)$.

Proof. Since $a\in B$, $T(B)\subseteq T(B\text{-}\{a\})$. Hence, the next thing is only to prove $T(B\text{-}\{a\})\subseteq T(B)$. Because $\forall\ (x,y)\in T(B\text{-}\{a\})$, $y\in S_{B\text{-}\{a\}}(y)\subseteq\cup S_{B\text{-}\{a\}}(y)\ (y\in S_a(x))=S_a(x)$. Therefore, $(x,y)\in T(a)$. It means that $T(B\text{-}\{a\})\subseteq T(a)$. So $T(B)=T(B\text{-}\{a\})\cap T(a)\supseteq T(B\text{-}\{a\})$. It follows that $T(B\text{-}\{a\})\subseteq T(B)$. So, $T(B\text{-}\{a\})=T(B)$.

Theorem 3. Let $B\subseteq AT$. If $T(B\text{-}\{a\})=T(B)$, then for $\forall x\in U$, $S_a(x)\subseteq\cup S_{B\text{-}\{a\}}(y)\ (y\in S_a(x))$.

Proof. Take any $z\in S_a(x)$. Since $z\in S_{B\text{-}\{a\}}(z)$, thus, $z\in\cup S_{B\text{-}\{a\}}(y)\ (y\in S_a(x))$. So, $S_a(x)\subseteq\cup S_{B\text{-}\{a\}}(y)\ (y\in S_a(x))$.

Theorem 4. Let $B\subseteq A$, $a\in B$ is indispensable in B if, and only if $\exists\, x,y\in U$, $y\in S_{B\text{-}\{a\}}(x)$, $y\notin S_a(x)$.

Proof. "$\Rightarrow$": It is clear that $B\text{-}\{a\}\subseteq B$, $T(B)\subseteq T(B\text{-}\{a\})$. a is indispensable in B, so $T(B\text{-}\{a\})\neq T(B)$. It follows that $T(B)\subset T(B\text{-}\{a\})$. Furthermore, $\exists\, x,y\in U$, $(x,y)\in T(B\text{-}\{a\})$, $(x,y)\notin T(B)$. It follows that $(x,y)\notin T(a)$, otherwise, $(x,y)\in T(B)$. Thus, $y\in S_{B\text{-}\{a\}}(x)$, $y\notin S_a(x)$.

"$\Leftarrow$": It is clear again that $B\text{-}\{a\}\subseteq B$, $T(B)\subseteq T(B\text{-}\{a\})$. If $a\in B$ is redundant in B, i.e. $T(B\text{-}\{a\})=T(B)$, then for $\forall\, x,y\in U$, $(x,y)\in T(B\text{-}\{a\})=T(B)$ means that $(x,y)\in T(a)$, $(x,y)\in T(B\text{-}\{a\})$. Therefore, $y\in S_{B\text{-}\{a\}}(x)$, $y\in S_a(x)$. It contradicts to the given condition. So $a\in B$ is indispensable in B.

Definition 5. Attribute set N is functionally dependent on attribute set B in IIS, denoted by $B\rightarrow N$, if and only if $T(B)\subseteq T(N)$. If attribute set N is functionally dependent on attribute set B, attribute set N is also called a derived one by attribute set B.

Theorem 5. If $T(B\text{-}\{a\})\subseteq T(B)$ for $a\in B$, then a is redundant in B.

Proof. It is clear that $B\text{-}\{a\}\subseteq B$, $T(B)\subseteq T(B\text{-}\{a\})$. Now $T(B\text{-}\{a\})\subseteq T(B)$, so $T(B\text{-}\{a\})=T(B)$, i.e. a is redundant in B.

Theorem 6. If $B\text{-}\{a\}\rightarrow B$ for $a\in B$, then a is redundant in B.

Proof. $B\text{-}\{a\}\rightarrow B$ means that $T(B\text{-}\{a\})\subseteq T(B)$ according to Definition 5. It is clear $B\text{-}\{a\}\subseteq B$, $T(B)\subseteq T(B\text{-}\{a\})$. Hence, $T(B\text{-}\{a\})=T(B)$, i.e. a is redundant in B.

Theorem 7. If $T(B\text{-}\{a\})=T(a)$ for $a\in B$, then a is redundant in B.

Proof. If $T(B\text{-}\{a\})=T(a)$, then $T(B)=T(B\text{-}\{a\})\cap T(a)=T(a)$. Hence, $T(B\text{-}\{a\})=T(B)$, i.e. a is redundant in B.

Theorem 8. If $B\text{-}\{a\}\rightarrow a$ and $a\rightarrow B\text{-}\{a\}$ for $a\in B$, then a is redundant in B.

Proof. $B\text{-}\{a\}\rightarrow a$ and $a\rightarrow B\text{-}\{a\}$ mean $T(B\text{-}\{a\})\subseteq T(a)$ and $T(a)\subseteq T(B\text{-}\{a\})$, that is $T(B\text{-}\{a\})=T(a)$. According to the above theorem, a is redundant in B. In complete information system, it holds that $B\rightarrow N$, $T(B\cup N)=T(B)$ and $POS_B(N)=U$ are equivalent for $B,N\subseteq AT$. But in IIS, this relationship might be always true. We only have the following results.

Theorem 9. For IIS, let $B,N\subseteq AT$, the following two expression are equivalent:

1. $B\rightarrow N$;
2. $T(B\cup N)=T(B)$.

Proof. (1)$\Rightarrow$(2): $B\rightarrow N$ means that $T(B)\subseteq T(N)$. Hence, $T(B\cup N)=T(B)\cap T(N)=T(B)$. So, $T(B\cup N)=T(B)$.

(2)$\Rightarrow$(1): $T(B\cup N)=T(B)$ means that $T(B\cup N)=T(B)\cap T(N)=T(B)$. It follows that $T(B)\subseteq T(N)$. That is. $B\rightarrow N$. Therefore, (1) and (2) are equivalent.

Theorem 10. Let $B\subseteq AT$, $a\in AT$. If $B\rightarrow a$, then $S_a(x)\subseteq\cup S_B(y)(\ y\in S_a(x))$ for $\forall x\in U$.

Proof. Take $\forall z\in S_a(x)$. $z\in S_B(z)\subseteq\cup S_B(y)(\ y\in S_a(x))$. So $S_a(x)\subseteq\cup S_B(y)(\ y\in S_a(x))$ for z is arbitrarily taken from $S_a(x)$.

Theorem 11. Let $B\subseteq AT$. If for $\forall x\in U$, $S_a(x)=\cup S_B(y)(\ y\in S_a(x))$, then $B\rightarrow a$.

Proof. Since $B\rightarrow a$ iff $T(B)\subseteq T(a)$, $T(B)\subseteq T(a)$ is only the target to be proved. For $\forall\ (x,y)\in T(B)$, $y\in S_B(x)$. Since $x\in S_a(x)$, $y\in S_B(x)\subseteq\cup S_B(y)\ (y\in S_a(x))=S_a(x), y\in S_a(x)$. Lt follows that $(x,y)\in T(a)$.Therefore, $T(B)\subseteq T(a)$.

Whether $a\in B$ or $a\notin B$, the conclusion in the above theorem is always valid.

Theorem 12. Let $a \in B \subseteq AT$. If $S_a(x)=\cup S_{B-\{a\}}(y)(\ y \in S_a(x))$ for $\forall\ x \in U$, then a is redundant in B.

Proof. According to Theorem 11, $B\text{-}\{a\} \rightarrow a$. It follows that $T(B\text{-}\{a\}) \subseteq T(a)$. It is clear that $T(B) \subseteq T(B\ \text{-}\{a\})$. Take $(x,y) \in T(B\text{-}\{a\})$. $y \in S_{B-\{a\}}(x)$. Since $x \in S_a(x)$, $y \in S_{B-\{a\}}(x) \subseteq \cup S_{B-\{a\}}(y)(y \in S_a(x))=S_a(x)$, i.e., $y \in S_a(x)$.So,$(x,y) \in T(a)$. Hence, $(x,y) \in T(B\text{-}\{a\})$ and $(x,y) \in T(a)$.Thus, $(x,y) \in T(B\text{-}\{a\}) \cap T(a) =T(B\ )$. It follows that $T(B\text{-}\{a\}) \subseteq T(B)$. So, $T(B\text{-}\{a\}) =T(B)$. That is, a is redundant in B. $B \rightarrow N$ does not mean that for any $X \in U/T(N)$, $B\_(X)=X$. $B \rightarrow a$ and $a \in B$ do not imply that a is redundant in B. That is, $B \rightarrow a$, $a \in B$ do not imply that $T(B\text{-}\{a\})=T(B)$. That is why only two equivalent expressions are obtained in Theorem 9.

Example 1. An IDT about cars is shown in Table I, where P, M, S, X represent Price, Mileage, Size and Max-Speed respectively. d =D is the decision attribute [4].

TABLE I
AN IDT DESCRIBED CARS

U	P	M	S	X	D
1	high	low	full	low	good
2	low	*	full	low	good
3	*	*	compact	low	poor
4	high	*	full	high	good
5	*	*	full	high	excellent
6	low	high	full	*	good

Let $A=\{P,M,S,X\}$,$B=\{P,S,X\}$. $T(A)=T(B)$. $S_A(1)= S_B(1)=\{1\}$, $S_A(2)=S_B(2)=\{2,6\}$, $S_A(3)=S_B(3)=\{3\}$, $S_A(4)= S_B(4)=\{4,5\}$, $S_A(5) = S_B(5)=\{4,5,6\}$, $S_A(6)= S_B(6)=\{2,5,6\}$. A is dependent on B. It can be verified that B is a reduction of A. But for any $X \in U/B$, $B\_(X)=X$ does not always hold. For instance, $B\_(\{2,6\})=\{2\} \neq \{2,6\}$, $B\_(\{4,5\}) =\{4\} \neq \{4,5\}$, $B\_(\{4,5,6\}) = \{4,5\} \neq \{4,5,6\}$, $B\_(\{2,5,6\})=\{2,6\} \neq \{2,5,6\}$. So $POS_A(B)= U$ may not be always true. Because B is a reduction of A, M is dispensable in A, $A \rightarrow M$.

In Table I, $S_M(1)=\{1,2,3,4,5\}$, $S_{A-\{M\}}(1)=\{1\}$,$S_{A-\{M\}}(2)=\{2,6\}$, $S_{A-\{M\}}(3)=\{3\}$, $S_{A-\{M\}}(4)= \{4,5\}$, $S_{A-\{M\}}(5)=\{4,5,6\}$, $S_M(1)=\{1,2,\ 3,4,5\} \neq \cup S_{A-\{M\}}(y)(y \in S_M(1))\ =\{1,2,3,4,5,6\}$. We only have $S_M(1)=\{1,2,3,4,5\} \subset \cup S_{A-\{M\}}(y)(y \in S_M(1))$.

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Theorem 13. If $B \subseteq N$ and $B \rightarrow N$, then $POS_B(N)= \cup B\_(X)(X \in U/T(N))=U$.

Proof. It is clear that $POS_B(N) \subseteq U$. So it is needed only prove $U \subseteq POS_B(N)$. Since $B \subseteq N$ and $B \rightarrow N$, hence, $T(N) \subseteq T(B)$, $T(B) \subseteq T(N)$. Thus, $T(B)=T(N)$. So, $S_B(y)=S_N(y)$ for any $y \in U$. Take $\forall\ y \in U$ and $S_N(y) \in U/T(N)$. $y \in S_N(y) \subseteq \cup \{y \in U|S_M(y) =S_N(y) \subseteq S_N(y)\}(S_N(y) \in U/T(N))=POS_B(N)$. That is, y $\in POS_B(N)$. So, $U \subseteq POS_B(N)$. Therefore, $POS_B(N)= \cup B\_(X)(X \in U/T(N))=U$. $POS_B(N)= \cup B\_(X)(X \in U/T(N))=U$ does not imply $B \rightarrow N$.

Theorem 14. Let B, N, R, $Q \subseteq AT$. Then the following laws of reflexivity, transitivity, left merge, decomposition, pseudo transitivity, merge and augmentation hold:

1. if $N \subseteq B \subseteq AT$, then $B \rightarrow N$.
2. if $B \rightarrow N$ and $N \rightarrow R$, then $B \rightarrow R$.
3. if $B \rightarrow N$ and $N \rightarrow R$, then $B \cup N \rightarrow R$.
4. if $B \rightarrow N \cup R$, then $B \rightarrow N$ and $B \rightarrow R$.
5. if $B \rightarrow N$ and $N \cup R \rightarrow N$, then $B \cup R \rightarrow N$.
6. if $B \rightarrow N$ and $R \rightarrow N$, then $B \cup R \rightarrow N \cup N$.
7. if $B \rightarrow N$ and $B \subseteq R$, then $R \rightarrow N$.

Proof.

1. Since $N \subseteq B$, $T(\mathrm{B}) \subseteq T(\mathrm{N})$. Thus $B \rightarrow N$.
2. $T(B) \subseteq T(N)$ and $T(N) \subseteq T(R)$ imply that $T(B) \subseteq T(R)$.
3. From $T(B) \subseteq T(N)$ and $T(N) \subseteq T(R)$, $T(B \cup N)= T(B) \cap T(N) \subseteq T(N) \subseteq T(R)$.
4. $T(B) \subseteq T(N \cup R)=T(N) \cap T(R)$ means that $T(B) \subseteq T(N)$, $T(B) \subseteq T(R)$.
5. $T(B) \subseteq T(N)$ and $T(N \cup R)= T(Q)$ mean that $T(N) \cap T(R) \subseteq T(N)$, $T(B) \cap T(R) \subseteq T(N) \cap T(R) \subseteq T(N)$, i.e., $T(B \cup R) \subseteq T(N)$.
6. $T(B) \subseteq T(N)$ and $T(R) \subseteq T(N)$ imply that $T(B) \cap T(R) \subseteq T(N) \cap T(N)$,i.e., $T(B \cup R) \subseteq T(N \cup N)$.
7. $T(B) \subseteq T(N)$ and $T(R) \subseteq T(B)$ imply that $T(R) \subseteq T(N)$. That is, $R \rightarrow N$.

The decomposition law (4) can be equivalently rewritten as: (4') if $B \rightarrow N$ and $R \subseteq N$, then $B \rightarrow R$.

Proof. $T(B) \subseteq T(N)$ and $T(N) \subseteq T(R)$ imply that $T(B) \subseteq T(R)$.

The augmentation law (7) can be equivalently rewritten as: (7') if $B \rightarrow N$, $R \subseteq AT$, then $B \cup R \rightarrow N \cup R$.

Example 2. In Table I, let $N=P \cup M$, then $P \subseteq N$, $M \subseteq N$, $S_N(1)=S_N(4)=\{1,3,4,5\}$,$S_N(2)=S_N(6)=\{2,3,5,6\}$, $S_N(3)=S_N(5)=\{1,\ 2,\ldots,6\}$. So we know for $\forall\ x \in U$, $S_N\ (x) \subseteq S_P\ (x)$, $S_N\ (x) \subseteq S_M\ (x)$. So, $N \rightarrow P$, $N \rightarrow M$.

Theorem 15. $B \rightarrow N$ if, and only if for $\forall\ m \in B$ and $\forall\ n \in N$, $m \rightarrow n$.

Theorem 16. Let m, $n \in AT$. $m \rightarrow n$ if, and only if for $\forall\ x_1,x_2 \in U$, $x_1 \neq x_2$, $f(x_1,m)=f(x_2,m) \vee f(x_1,m)=* \ \vee f(x_2,m)=*$ then $f(x_1,n)=f(x_2,n) \vee f(x_1,n) =* \vee f(x_2,n) =*$.

Proof. $m \rightarrow n$ iff $T(\{m\}) \subseteq T(\{n\})$. Hence, if $m \rightarrow n$, $f(x_1,m)= m(x_2) \vee f(x_1,m)=* \vee f(x_2,m)=*$, i.e., $(x_1,x_2\ ) \in T(\{m\})$, then $f(x_1,n)=f(x_2,n) \vee f(x_1,n)=* \vee f(x_2,n)=*$,i.e.,$(x_1,x_2) \in T(\{n\})$. So, $m \rightarrow n$ implies $T(\{m\}) \subseteq T(\{n\})$. Conversely, if $f(x_1,m)=f(x_2,m) \vee f(x_1,m)=* \vee f(x_2,m)=*$, i.e., $(x_1,\ x_2) \in T(\{m\})$, from $T(\{m\}) \subseteq T(\{n\})$, $(x_1,\ x_2) \in T(\{n\})$, i.e., $f(x_1,n)=f(x_2,n) \vee f(x_1,n)=* \ \vee f(x_2,n) =*$, that is $m \rightarrow n$.

In complete information system, $m \rightarrow n$ iff $f(x_1,m)=f(x_2,m)$ means $f(x_1,n)=f(x_1n)$. But here it is different. This is because there exist null values in IIS.

In complete information system, the definition of knowledge dependency degree in rough set model is given by using positive set. In IIS, if $POS_B(N)$ is defined by tolerance class instead of equivalence class, it will be as:

$$POS_B(N)=\cup_{X\in U/T(N)}\cup_{K\in U/T(B)}\{K \mid K\subseteq X\}$$

If the knowledge dependence degree k in $B \xrightarrow{k} N$ is still defined by $k=r_B(N)=|POS_B(N)|/|U|$, after analyzing that as long as there is an attribute $a\in N$ in attribute subset N and for $\forall x\in U$, $f(x,a)=*$, $S_a(x)=U$, it may have $POS_B(N)=U$, therefore k=1. N is completely dependent on B. This might be the wrong result. For example, because $f(P,1)=f(P,4)=high$, and $f(X,1)=low$, $f(X,4)=high$, i.e., $f(X,1)\neq f(X,4)$ for attribute P and attribute X in Table I, thus P is obviously not dependent on X. But according to the formula, it is obtained that $POS_B(X)=U$, therefore $k=r_B(X)=1$, P is completely dependent on X. That is not true. So knowledge dependency degree in IIS has to be newly defined. In the following, we are trying to give some definitions of it in two cases.

Case 1. B, $N\subseteq AT$, no attribute in N contains null value *. In this case, knowledge dependency degree k in $B \xrightarrow{k} N$ can be given by equivalence classes of N and tolerance classes of B.

Definition 6. For $\forall B, N\subseteq AT$, the extended dependence set of N on B is as:

$$POS_B(N)=\cup_{X\in U/IND(N)}\cup_{K\in U/T(B)}\{K \mid K\subseteq X\}$$

If $N=\varnothing$, then let $POS_B(N)=\varnothing$. In $B \xrightarrow{k} N$, the dependency degree k is calculated by $k=r_B(N)=k(B,N)=POS_B(N)/|U|$. Obviously, $0\le k\le 1$. If k=0, N is not dependent on B; If k=1, N is completely dependent on B, denoted by $B\rightarrow N$.

Case 2. There exists at least one attribute in both attribute subsets B and N contains null attribute value *.

Definition 7. In $B \xrightarrow{k} N$, the knowledge dependency degree is computed by

$$k=k(B,N)=\frac{1}{|U|}\sum_{x\in U}\frac{|S_B(x)\cap S_N(x)|}{|S_B(x)|}$$

It is obvious that $0\le k\le 1$. Because for any B, $N\subseteq AT$, we have $x\in S_B(x)\neq\varnothing$, and $x\in S_N(x)\neq\varnothing$, and $x\in S_B(x)\cap S_N(x)\neq\varnothing$, therefore, $0<k\le 1$ in general, k=0 only if $N=\varnothing$.

If N is dependent on B, i.e., $B\rightarrow N$ then the dependency degree k=1 in $B \xrightarrow{k} N$ by the computation formula.

Theorem 17. For $\forall B$, $N\subseteq AT$, $B\rightarrow N$ if, and only if for $\forall x\in U$, $S_B(x)\subseteq S_N(x)$.

Proof. If for $\forall x\in U$, $S_B(x)\subseteq S_N(x)$, then $S_B(x)\cap S_N(x)=S_B(x)$. Hence, k=1 and $B\rightarrow N$. Conversely, if $B \xrightarrow{k} N$, k=1, then $\sum_{x\in U}|S_B(x)\cap S_N(x)|\,/\,|S_B(x)|=|U|$. Hence, $S_B(x)\subseteq S_N(x)$.

Example 3. In Table I, for $\forall x\in U$, $S_M(x)\subseteq S_M(x)$. So, $M\rightarrow M$.

Example 4. In Table I, $S_P(1)=S_P(4)=\{1,3,4,5\}$, $S_P(2)=S_P(6)=\{2,3,5,6\}$, $S_P(3)=S_P(5)=\{1,2,3,4,5,6\}$; $S_M(1)=\{1,2,3,4,5\}$, $S_M(2)=S_M(3)=S_M(4)=S_M(5)=\{1,2,3,4,5,6\}$ $S_M(6)=\{2,3,4,5,6\}$. In order to avoid confusion, temporarily use Z to represent $S(Size)$, then $S_Z(1)=S_Z(2)=S_Z(4)=S_Z(5)=S_Z(6)=\{1,2,4,5,6\}$, $S_Z(3)=\{3\}$; $S_X(1)=S_X(2)=S_X(3)=\{1,2,3,6\}$, $S_X(4)=S_X(5)=\{4,5,6\}$, $S_X(6)=\{1,2,3,4,5,6\}$. So $P\rightarrow M$. $P,S,X\rightarrow M$. $P\xrightarrow{k_1}S$, $P\xrightarrow{k_2}X$, $X\xrightarrow{k_3}P$, $S\xrightarrow{k_4}X$, $X\xrightarrow{k_5}S$, where $k_1=121/180\approx0.67$, $k_2=53/72\approx0.74$, $k_3=55/72\approx0.93$, $k_4=11/15\approx0.73$, $k_5=55/144\approx0.76$.

IV. Properties of Knowledge Dependency and Knowledge Dependency Degree

Knowledge dependency and knowledge dependency degree satisfy some properties.

Theorem 18. Let (U,AT,V,f) be an IIS. For $\forall B,N,L\subseteq AT$, we obtain following results.

1. Let $B\rightarrow N$, $N\xrightarrow{k_1}L$, $B\xrightarrow{k_2}L$. If for $\forall x\in U$, $|S_B(x)\cap S_L(x)|/|S_B(x)|\le|S_N(x)\cap S_L(x)|/|S_N(x)|$, then $k_2\le k_1$.
2. If $B\xrightarrow{k_1}N$, $N\rightarrow L$, $B\xrightarrow{k_2}L$, then $k_2\ge k_1$.

Proof.

1. $B\rightarrow N$ means that for $\forall x\in U$, $S_B(x)\subseteq S_N(x)$. Hence $S_L(x)\cap S_B(x)\subseteq S_L(x)\cap S_N(x)$. Thus, $S_B(x)\cap S_L(x)\subseteq S_N(x)\cap S_L(x)$. $|S_B(x)\cap S_L(x)|\le|S_N(x)\cap S_L(x)|$, $|S_B(x)|\le|S_N(x)|$. If for $\forall x\in U$, $|S_B(x)\cap S_L(x)|/|S_P(x)|\le|S_N(x)\cap S_L(x)|/|S_N(x)|$, then $k_2\le k_1$.
2. $N\rightarrow L$ means that for $\forall x\in U$, $S_N(x)\subseteq S_L(x)$. Hence, $S_B(x)\cap S_N(x)\subseteq S_B(x)\cap S_L(x)$, $|S_B(x)\cap S_N(x)|/|S_B(x)|\le|S_B(x)\cap S_L(x)|/|S_B(x)|$. Thus, $k_2\ge k_1$.

Example 5.

1. In Table I, $X\xrightarrow{k_1}P$, $P\rightarrow M$, $X\xrightarrow{k_2}M$, where $k_1=55/72\approx0.93$, $k_2=55/72\approx0.93$, and $k_2\ge k_1$.
2. $P\rightarrow M$, $M\xrightarrow{k_1}X$, $P\xrightarrow{k_2}X$, where $k_1=59/90\approx0.66$, $k_2=19/24\approx0.79$, but $k_2\ge k_1$ because of the condition of the theorem is not satisfied. For example, $|S_P(2)\cap S_L(2)|/|S_P(2)|=3/4$, $|S_N(2)\cap S_L(2)|/|S_N(2)|=4/6$, and 3/4 is not less or equal to 4/6.

In Theorem 18, the transitivity of knowledge dependency and knowledge dependency degree are studied. The dependency degrees before and after transferring are compared. Theorem 18(i) shows that in certain case knowledge dependency degree may increase or not. Theorem 18(ii) shows that in the case the dependency degree increases after transferring.

Theorem 19. Let (U,AT,V,f) be an IIS. For $\forall B,N,L\subseteq AT$, we obtain following results.

1. Let $B\xrightarrow{k_1}N$ and $B\cup L\xrightarrow{k_2}N$. If for $\forall x\in U$, $|S_{B\cup L}(x)\cap S_N(x)|/|S_{B\cup L}(x)|\le|S_B(x)\cap S_N(x)|/|S_B(x)|$, then

$k_2 \le k_1$.

2. If $B \xrightarrow{k_1} N$ and $B \xrightarrow{k_2} N \cup L$, then $k_2 \le k_1$.

Proof. (i) $B \subseteq B \cup L$, $\forall x \in U$, $S_{B\cup L}(x) \subseteq S_B(x)$. Hence, $S_{B\cup L}(x) \cap S_N(x) \subseteq S_B(x) \cap S_N(x)$. $|S_{B\cup L}(x)| \le |S_B(x)|$, $|S_{B\cup L}(x) \cap S_N(x)| \le |S_B(x) \cap S_N(x)|$. Thus, if for $\forall x \in U$, $|S_{B\cup L}(x) \cap S_N(x)|/|S_{B\cup L}(x)| \le |S_B(x) \cap S_N(x)|/|S_B(x)|$, then $k_2 \le k_1$.

(ii) $N \subseteq N \cup L$, $\forall x \in U$, $S_{N\cup L}(x) \subseteq S_N(x)$. Hence, $S_B(x) \cap S_{N\cup L}(x) \subseteq S_B(x) \cap S_N(x)$. $|S_B(x) \cap S_{N\cup L}(x)|/|S_B(x)| \le |S_B(x) \cap S_N(x)|/|S_B(x)|$. Thus, $k_2 \le k_1$.

Example 6. (i) In Table I, let $B=\{X\} \cup \{S\}$. $S_B(1)=\{1,2,6\}$, $S_B(2)=\{1,2,6\}$, $S_B(3)=\{3\}$, $S_B(4)= S_B(5)=\{4,5,6\}$, $S_B(6)=\{1,2,4,5,6\}$, $P \xrightarrow{k_1} X$, $k_1= 19/24\approx 0.79$. $P \xrightarrow{k_2} B=\{X\} \cup \{S\}$, $k_2=4/9\approx 0.44$. $k_2 \le k_1$.

(ii) Let $L=\{P\} \cup \{S\}$. From Table I, $S_L(1)=\{1,4,5\}$, $S_L(2)= S_L(6)=\{2,5,6\}$, $S_L(3)=\{3\}$, $S_L(4)=\{1,4,5\}$, $S_L(5)=\{1,2,4,5,6\}$. In $P \xrightarrow{k_1} X$ and $L \xrightarrow{k_2} X$, $k_1=19/24 \approx 0.79$, $k_2=32/45\approx 0.71$, $k_2 \le k_1$. But it is not hold for $\forall x \in U$, $|S_{P\cup L}(x) \cap S_N(x)|/|S_{P\cup L}(x)| \le |S_P(x) \cap S_N(x)|/|S_P(x)|$. For example, $|S_{P\cup L}(3) \cap S_N(3)|/|S_{P\cup L}(3)| =1$, $|S_P(3) \cap S_N(3)| /|S_P(3)| =2/3$, but $1>2/3$.

Theorem 19 (i) shows that dependency degree of the same knowledge to be dependent on much knowledge becomes lesser if satisfying certain conditions. Theorem 19 (ii) shows that for the same dependent knowledge, the much the knowledge, the less the dependency degree.

Theorem 20. Let $B, N, L \subseteq AT$, $N \subseteq B$, $L \xrightarrow{k_1} B$, $L \xrightarrow{k_2} N$. Then $k_2 \ge k_1$.

Proof. $N \subseteq B$ means $T(B) \subseteq T(N)$. It follows that $B \to N$. Hence, for $\forall x \in U$, $S_B(x) \subseteq S_N(x)$, $S_L(x) \cap S_B(x) \subseteq S_L(x) \cap S_N(x)$, $|S_L(x) \cap S_B(x)| \le |S_L(x) \cap S_N(x)|$, $|S_L(x) \cap S_B(x)|/|S_L(x)| \le |S_L(x) \cap S_N(x)|/|S_L(x)|$. Thus, $k_2 \ge k_1$.

Example 7. In Table I, let $B=\{X\} \cup \{S\}$, then $X \subseteq B$, $P \xrightarrow{k_1} B$, $k_1=4/9\approx 0.44$; $P \xrightarrow{k_2} X$, $k_2=19/24\approx 0.79$, $k_2 \ge k_1$.

Theorem 20 expresses if attribute subset $N \subseteq B$, then the finer attribute subset N has a bigger knowledge dependency degree than B has on the same attribute subset L.

Theorem 21. Let $B,N,L \subseteq AT$, $B\cup L \xrightarrow{k_1} N$, $B \xrightarrow{k_2} N$, $L \xrightarrow{k_3} N$. If for $\forall x \in U$, $|S_{B\cup L}(x) \cap S_N(x)|/|S_{B\cup L}(x)| \le |S_B(x) \cap S_N(x)|/|S_B(x)|$, $|S_{B\cup L}(x) \cap S_N(x)|/|S_{B\cup L}(x)| \le |S_L(x) \cap S_N(x)|/|S_L(x)|$. Thus $\min\{k_2, k_3\} \ge k_1$.

Proof. $B \subseteq B \cup L$, $L \subseteq B \cup L$, so $S_{B\cup L}(x) \subseteq S_B(x)$, $S_{B\cup L}(x) \subseteq S_L(x)$. Hence $S_{B\cup L}(x) \cap S_N(x) \subseteq S_B(x) \cap S_N(x)$, $S_{B\cup L}(x) \cap S_N(x) \subseteq S_L(x) \cap S_N(x)$, $|S_{B\cup L}(x) \cap S_N(x)| \le |S_B(x) \cap S_N(x)|$, $|S_{B\cup L}(x) \cap S_N(x)| \le |S_L(x) \cap S_N(x)|$, $|S_{B\cup L}(x)| \le |S_B(x)|$, $|S_{B\cup L}(x)| \le |S_L(x)|$. If for $\forall x \in U$, $|S_{B\cup L}(x) \cap S_N(x)|/|S_{B\cup L}(x)| \le |S_B(x) \cap S_N(x)|/|S_B(x)|$, $|S_{B\cup L}(x) \cap S_N(x)|/|S_{B\cup L}(x)| \le |S_L(x) \cap S_N(x)|/|S_L(x)|$, then $k_1 \le k_2$, $k_1 \le k_3$, i.e., $min\{k_2, k_3\} \ge k_1$.

Example 8. In Table I, let $L=\{P\} \cup \{S\}$, we obtain $L \xrightarrow{k_1} X$, where $k_1\approx 0.711$. Before we have already obtained P $\xrightarrow{k_2} X$, $S \xrightarrow{k_3} X$, where $k_2\approx$ 0.736, $k_3\approx 0.733$. It is clear that $min\{k_2, k_3\} \ge k_1$.

Theorem 21 shows that knowledge dependency degree on partial knowledge is bigger than on entire knowledge. This means decomposition rule in IIS is still effective.

Theorem 22. Let $B, N, L \subseteq AT$, $B \to N$, $L \xrightarrow{k_1} N$, $L \xrightarrow{k_2} B$. Then $k_2 \le k_1$.

Proof. $B \to N$ implies that for $\forall x \in U$, $S_B(x) \subseteq S_N(x)$. Hence, $S_L(x) \cap S_B(x) \subseteq S_L(x) \cap S_N(x)$. Thus, $|S_L(x) \cap S_B(x)| \le |S_L(x) \cap S_N(x)|$, $|S_L(x) \cap S_B(x)| /|S_L(x)| \le |S_L(x) \cap S_N(x)|/|S_L(x)|$. Therefore, $k_2 \le k_1$.

Theorem 23. Let $B, N, L \subseteq AT$, $B \to N$, $B \xrightarrow{k_1} L$, $N \xrightarrow{k_2} L$. If for $\forall x \in U$, $|S_B(x) \cap S_L(x)| /|S_B(x)| \le |S_N(x) \cap S_L(x)|/|S_N(x)|$, then $k_2 \ge k_1$.

Proof. $B \to N$ means that for $\forall x \in U$, $S_B(x) \subseteq S_N(x)$. Hence, $S_L(x) \cap S_B(x) \subseteq S_L(x) \cap S_N(x)$, i.e., $S_B(x) \cap S_L(x) \subseteq S_N(x) \cap S_L(x)$. Thus, $|S_B(x) \cap S_L(x)| \le |S_N(x) \cap S_L(x)|$, $|S_B(x)| \le |S_N(x)|$. Furthermore, if for $\forall x \in U$, $|S_B(x) \cap S_L(x)|/|S_B(x)| \le |S_N(x) \cap S_L(x)|/|S_N(x)|$, then $k_2 \ge k_1$.

V. A Reduction Algorithm

For an IDT, if $T(A\text{-}\{a\} \cup \{d\}) \subseteq T(A \cup \{d\})$, i.e., $A\text{-}\{a\} \cup \{d\} \to A \cup \{d\}$, then a is redundant in $A \cup \{d\}$. If$A\text{-}\{a\} \to A$, i.e. $T(A\text{-}\{a\}) \subseteq T(A)$, then $A\text{-}\{a\} \cup \{d\} \to A \cup \{d\}$ or $T(A\text{-}\{a\} \cup \{d\}) \subseteq T(A \cup \{d\})$. That means $k(A\text{-}\{a\},A)=1$ implies $k(A\text{-}\{a\} \cup \{d\}, A \cup \{d\})$ =1. Based on such a fact, a new attribute reduction algorithm under the guidance of knowledge dependency degree about IDT can be designed as follows.

Algorithm. A reduction algorithm of IDT

Step 1. $B<=A$.

Step 2. For $\forall a \in B$, calculate $k(B\text{-}\{a\},B)$.

Step 3. Find a such that $k(B\text{-}\{a\}, B)=max\{k(B\text{-}\{b\}, B)|b \in B\}$. If $k(B\text{-}\{a\},B)=1$, then a is redundant in B. Set $B \Leftarrow B\text{-}\{a\}$. If there are many attributes satisfying $k(B\text{-}\{a\},B)=1$ simultaneously, take any one of them as a redundant attribute and delete it from B. Go to Step 2.

Step 4. B is a reduction of A, output B, end the algorithm.

Example 9. For the IDT shown in Table I, according to the reduction algorithm, let B=A at first, we find that attribute M satisfies $k(B\text{-}\{M\},B)=1$, so M is a redundant attribute and can be erased from the attribute set B. Any other attribute $b \in B\text{-}\{M\}$ have not such a degree $k(B\text{-}\{b\},B)=1$, i,e., all other attributes are indispensable. So the unique reduction $\{P, S, X\}$ is finally found, see Table II.

TABLE II
A REDUCTION OF TABLE I

U	P	S	X	D
1	high	full	low	good
2	low	full	low	good
3	*	compact	low	poor
4	high	full	high	good
5	*	full	high	excellent
6	low	full	*	good

VI. CONCLUSIONS

Using tolerance rough set model, the present paper first studies the characters of indispensable attribute and dispensable attribute in attribute set. Then it discusses the relationships between tolerance class and indispensable attribute and dispensable attribute and knowledge dependency, obtains several necessary and/or sufficient theorems. Next we discussed how to define knowledge dependence degree in two cases –one is that no decision attribute takes missing value; the another is that at least one decision attribute takes a missing value. We define a knowledge dependence degree computation form for the latter case. We prove that knowledge dependency satisfies reflexivity, transitivity, augmentation, and decomposition law in IIS. But that knowledge dependence degree satisfies laws of transitivity, augmentation, and decomposition should satisfies some special conditions. According to the research results in the present paper, an algorithm solving attribute reduction of IDT by using dependency degree is designed. Through an example, the correctness of the reduction algorithm is consolidated.

REFERENCES

[1] Pawlak.S. Rough set theory and its applications to data analysis. *Journal of Cybernetics and Systems*. 1998,(29):661-688
[2] Pawlak.Z. Rough sets and intelligent data analysis. *Journal of Information Sciences*. 2002,(147): 1-12
[3] Pawlak Z. Rough Sets. Theoretical Aspects of Reasoning about Data. *Dordecht: Kluwer Academic Publishers*.1991
[4] Kryszkiewicz. M. Rough Set Approach to Incomplete Information Systems. *Information Sciences*. 1998,(112):39-49
[5] Jerzy. Stefanowski. Incomplete information tables and rough classification.*Journal of Computational Intelligence*. 2001,(17):545-566
[6] Jerzy. Stefanowski,Tsoukias. A. On the extension of rough sets under incomplete information. *In: N Zhong, A. Tsoukias, S. Ohsugaeds. Proc of the 7th Int'l Workshop on New Directions in Rough Sets. Data Mining, and Granular-Soft Computing*. Berlin: Springer-Verlag, 1999,73-81
[7] Wang Guoyin. Extension of rough set under incomplete information systems. *Journal of Computer Research and Development*, 2002, Vol. 39, no.10, pp. 1238-1243
[8] J.Y. Leung, D. Y.Li, "Maximal consistent block technique for rule acquisition in incomplete information systems", *Information Sciences*, 2003, Vol. 153, pp.86-106
[9] C. Wu, X.B. Yang, "Information Granules in General and Complete Covering", *Proceedings of the 2005 IEEE International Conference on Granular Computing*, 2005, pp. 675-678
[10] W.L. Chen, J.X. Cheng, C.J. Zhang, "A Generalized Model of Rough Set Theory Based on Compatibility Relation", *Journal of computer engineering and applications*, 2004, Vol.16, No.4, pp.26-28
[11] J.S. Mi, W.Z. Wu and W.X. Zhang, "Approaches to Knowledge Reduction Based on Variable Precision Rough Set Model", *Information Sciences*, 2004, Vol.159, No. 3-4, pp.255-272
[12] C.W, X.Hu, Z. Li, X. Zhou, P. Achananuparp, "Algorithms for Different Approximations in Incomplete Information Systems with Maximal Compatible Classes as Primitive Granules". *Proc. of IEEE International Conference on Granular Computing, GrC* 2007, San Jose, California, USA, 2007, pp.169-174
[13] W. Xu, G.H. Zhang, W.X. Zhang, "Lower-approximation Distribution Reduct and Rules in Incomplete Information System", *Journal of Xi'an institute of Technology*(*in Chinese*),2004, Vol.24, No.4, pp.386-390
[14] Y. H. Qian, J.Y. Liang, Y.Y. Yao, "MGRS: A multi-granulation rough set", *Information Sciences*, 2010,Vol.180,No.6,pp.949–970
[15] Y.H. Qian, J.Y. Liang, C.Y. Dang, "Incomplete multi-granulation rough set", *IEEE Transactions on Systems, Man and Cybernetics*, Part A, 2010,Vol.40,No.2,pp.420-431
[16] Y.H. Qian, J.Y. Liang, W. Wei, "PesTistic rough decision", in: Second International Workshop on Rough Sets Theory, Zhoushan, China, 2010,pp. 440–449
[17] Deng Fang'an, Xu Yang. Rough Set Approach to Data Reasoning for Incomplete Information Systems. *Computer Engineering and Applications*, 2004,30,51-53
[18] Zhang Wenxiu, Liang Jiye. Rough Set Theory and Approaches. *Beijing*: *Science Press*, 2001:206-212
[19] Liang Jiye, Shi Zhongzhi. The information entropy, rough entropy and knowledge granulation in rough set theory. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 2004,12 (1):37-46
[20] Qi Zongfeng, Han Shan, Li Jianxun. Applications of Generalized Rough Set Theory in Evaluation Index System of Radar Anti-Jamming Performance. *Journal of Shanghai Jiaotong University*(*science*), 2016, 21(2):151 -158
[21] Hu Dan, Li Hongxing. The Measurement of Dependent Degree of Knowledge. *Journal of Beijing Normal University*(*Natural Science*), 2004, 40(3),320-325