# Invariant Characters of Tolerance Class and Reduction under Homomorphism in IIS 

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#### Abstract

Some invariant properties of incomplete information systems homomorphism are studied in this paper. Demand conditions of tolerance class, attribute reduction, indispensable attribute and dispensable attribute being invariant under homomorphism in incomplete information system are revealed and discussed. The existing condition of endohomomorphism on an incomplete information system is also explored. It establishes some theoretical foundations for further investigations on incomplete information systems in rough set theory, like in information systems.


Keywords-Attribute reduction, homomorphism, incomplete information system, rough set, tolerance relation.

## I. INTRODUCTION

ROUGH set theory is a useful mathematics tool for analyzing data. It was first proposed by Pawlak in 1982 [7]. Because it can process inconsistent, imprecise and incomplete information, it is successively applied in many fields such as pattern recognition, machine learning, decision making and data mining [8]. It attracts wide interests by various national scholars from all over the world. But traditional rough set model is based on complete information systems, i.e. all attribute values of each object in the given study universe are known. However, due to the data measuring error or the limitation condition in acquiring data, incomplete information system (IIS) (i.e. possibly some attribute values of objects are unknown) is always in front of us.

Recently there are two main approaches to deal with IIS. One is called direct approach in which related concepts in complete information system in rough set theory are appropriately extended to the case of IIS. The other is called indirect approach in which domain experts fill in missing data in IIS by some values such as mean value or frequent appearing value of the related attribute. Compared with the indirect, the direct one avoids the interference from expert subjective factor and is more objective. It has already attracted many experts' interests from different study fields. There are two different semantic explanations about unknown attribute values in IIS. One is that the unknown attribute is missing but it really exists. The other is

[^0]absent and is not allowed to be compared with other values.
In the direct approach area, such kind of researches has been done. Kryszkiewicz defines tolerance relation to process IIS based on the first semantic explanation. Based on the relation, knowledge reduction problem and etc. are deeply investigated by him [4]. Based on the second semantic explanation, Stefanowski suggests non-symmetric similarity relation [13]. Guoying proposes limited tolerant relation for the requiring condition of tolerance relation is relaxing [15]. With the granular points of view, Leung et al. introduce maximal consistent block technique for rule acquisition [5]. Wu et al. investigate information granules in general and complete covering [17]. Chen et al. discuss generalized model of rough set theory based on compatibility relation [1]. In order to deal with IIS under both semantic explanations for unknown attribute value simultaneously, Grzymala-Busse defines feature relation [2]. Many other scholars suggest some other methods. Anyway, building extended rough set models to study IIS has become a very important research topic.

The concept of homomorphism of complete information system is a powerful tool to study the relationship of complete information systems and is first put forward by Grzymala-Busse et al. [3]. In [3], the authors give the conditions of making an information system be selective. The endohomomorphism of complete information system is studied in [6]. An endohomomorphism complete information system based on attribute redundancy is also built in. Several meaningful results such as reduction preservation, core preservation etc. are obtained. Reference [12] discusses invariant characters of information systems under some homomorphisms. It reveals interdependence among object mapping, attribute mapping and value domain mapping. It also obtains some theorem results about invariant properties for upper and lower approximations in complete information system. Some other invariant properties are also explored in [14], [18]. Now experts even study multi-granular rough set models [9]-[11]. Acquiring knowledge from IIS from different granular views still remains as a hot topic [16], [19].

This paper studies some properties of IIS under the first semantic explanation, discusses conditions of invariant properties of tolerance class, attribute reduction, indispensable attribute and dispensable attribute in IIS under homomorphism. It also explores the condition of existing an endohomomorphism on an IIS. It lays a certain theoretical foundation of further studying IIS using rough set theory.

## II. BASIC CONCEPTS

Let $S=(U, A, V, f)$ be an IIS [3], where $U$ is a finite non-empty
set of objects, $A$ is a finite non-empty set of condition attributes and decision attributes. For any $a \in A T, a: V \rightarrow V_{a}$, where $V_{a}$ is called the value range of $a$. For each object, some attribute values are lost or missing, called null values. The set of all attribute values is denoted by $V=\cup V_{a}(a \in A T)$.
Definition 1. In IIS $S=(U, A, V, f)$, for any attribute subset $P \subseteq A T$, the tolerance relation on $U$ is defined as $\operatorname{SIM}(P)=\{(x, y) \mid \forall a \in P(f(x, a)=f(y, a) \vee f(x, a)=* \vee f(y, a)=*)\}$.

If $P$ is a singleton set, for example $P=\{a\}$, then $\operatorname{SIM}(P)=$ $\operatorname{SIM}(\{a\})$ is abbreviated to be $\operatorname{SIM}(a)$. Obviously, $\operatorname{SIM}(P)$ is reflexive and symmetry on $U$.
Definition 2. For any attribute subset $P \subseteq A T, S_{P}(x)=$ $\{y \mid(x, y) \in \operatorname{SIM}(P)\}$ is called a tolerance class generated by $x$ as generator. If $P$ is a singleton set, for example $P=\{a\}$, then $S_{P}(x)=S_{\{a ;}(x)$ is abbreviated by $S_{a}(x)$.
Definition 3. $U / \operatorname{SIM}(P)=\left\{S_{P}(x) \mid x \in U\right\}, U / P$ for short, a cover on $U$, is called a knowledge system.
Definition 4. For any $X \subseteq U, P \subseteq A T, P_{-}(X)=\{y \mid y \in U$, $\left.S_{P}(y) \subseteq X\right\}$ is called the lower approximation of $X$; $P-(X)=\left\{y \mid y \in U, S_{P}(y) \cap X \neq \varnothing\right\}$ is called the upper one.
Definition 5. Let $P \subseteq A T, P \neq \varnothing$. If $\operatorname{SIM}(P)=\operatorname{SIM}(P-\{a\})$ for $a \in P$, then $a$ is called dispensable or redundant in $P$, otherwise $a$ is called indispensable.

If each $a \in P$ is indispensable in $P$, then $P$ is called independent, otherwise $P$ is called dependent.
Theorem 1. If $a \in P$ is dispensable in $P, b \in P$ and for any $x \in U, S_{a}(x)=S_{b}(x)$, then $a$ is also dispensable in $P$.
Proof. Since $a \in P$ is dispensable in $P$, we have $\operatorname{SIM}(P)=\operatorname{SIM}(P-\{a\})$. For any $x \in U, \quad S_{P-\{a\}}(x)=S_{P}(x)$. $S_{P}(x)=S_{c}(x)(c \in P)=S_{c}(x) \quad(c \in P-\{a\})=S_{c}(x) \quad(c \in P-\{b\})$ $=S_{P-\{b\}}(x)$. It follows that $\operatorname{SIM}(P)=\operatorname{SIM}(P-\{b\})$. So $b$ is also dispensable in $P$.
Definition 6. Let $Q \subseteq P$. If $Q$ is independent and $\operatorname{SIM}(Q)=$ $\operatorname{SIM}(P)$, then $Q$ is called a reduction of $P$.
Definition 7. The set consisted of all indispensable attribute in $P$ is called the core of $P$, denoted by core $(P)$.

The relation of the core and reductions is $\operatorname{core}(P)=$ $\cap Q(Q \in \operatorname{red}(P))$, where $\operatorname{red}(P)$ is the collection of all reductions of $P$.
Theorem 2. Let $P \subseteq A$. If $a \in P$ and for any $x \in U$, $S_{a}(x)=\cup S_{P-\{a ;}(y)\left(y \in S_{a}(x)\right)$, then $\operatorname{SIM}(P-\{a\})=S I M(P)$, i.e. $a$ is dispensable in $P$.
Proof. Since $a \in P$, we must have $\operatorname{SIM}(P) \subseteq \operatorname{SIM}(P-\{a\})$. Therefore, we only need to prove $\operatorname{SIM}(P-\{a\}) \subseteq \operatorname{SIM}(P)$ under given conditions. Because for any $(x, y) \in \operatorname{SIM}(P-\{a\})$ we have $y \in S_{P-\{a\}}(y) \subseteq \cup S_{P-\{a\}}(y) \quad\left(y \in S_{a}(x)\right)=S_{a}(x)$. Thus, $(x, y)$ $\in \operatorname{SIM}(a)$. It follows that $\operatorname{SIM}(P-\{a\}) \subseteq \operatorname{SIM}(a)$. So $\operatorname{SIM}(P)=$ $\operatorname{SIM}(P-\{a\}) \cap \operatorname{SIM}(a) \supseteq \operatorname{SIM}(P-\{a\})$. That is $\operatorname{SIM}(P-\{a\})$ $\subseteq \operatorname{SIM}(P)$. Thus, $\operatorname{SIM}(P-\{a\})=\operatorname{SIM}(P)$.
Theorem 3. Let $P \subseteq A$. If $a \in P$ is dispensable in $P$, that is $\operatorname{SIM}(P-\{a\})=\operatorname{SIM}(P)$, Then $\forall x \in U, S_{a}(x) \subseteq \cup S_{P-\{a\}}(y)$ $\left(y \in S_{a}(x)\right)$.
Proof. For any $z \in S_{a}(x)$, since $z \in S_{P-\{a\}}(z)$, thus, $z \in \cup S_{P-\{a ;}(y)$ $\left(y \in S_{a}(x)\right)$. Because $z \in S_{a}(x)$ is arbitrarily chosen,
$S_{a}(x) \subseteq \cup S_{P-\{a\}}(y)\left(y \in S_{a}(x)\right)$ holds.
Theorem 4. Let $P \subseteq A$. $a \in P$ is dispensable in $P$ if, and only if there exist $x, y \in U$ such that $y \notin S_{a}(x), y \in S_{P-\{a\}}(x)$.
Proof. Since $a \in P$ is dispensable in $P, \operatorname{SIM}(P-\{a\}) \neq \operatorname{SIM}(P)$. Because $P-\{a\} \subseteq P, \operatorname{SIM}(P) \subseteq \operatorname{SIM}(P-\{a\})$. Therefore, we only have $\operatorname{SIM}(P) \subset \operatorname{SIM}(P-\{a\})$. Thus, there exists $(x, y) \in \operatorname{SIM}(P-\{a\}),(x, y) \notin \operatorname{SIM}(P)$. It must have $(x, y) \notin \operatorname{SIM}(a)$, otherwise, $(x, y) \in \operatorname{SIM}(P)$. Furthermore, $y \notin S_{a}(x), y \in S_{P-\{a\}}(x)$. Since $P-\{a\} \subseteq P, \operatorname{SIM}(P) \subseteq \operatorname{SIM}(P-\{a\})$. If $a \in P$ is dispensable in $P$, i.e. $\operatorname{SIM}(P-\{a\})=\operatorname{SIM}(P)$, then for $\forall(x, y) \in \operatorname{SIM}(P-\{a\})$ $=\operatorname{SIM}(P)$, we have $(x, y) \in \operatorname{SIM}(a),(x, y) \in \operatorname{SIM}(P-\{a\})$. Furthermore, $y \in S_{a}(x), y \in S_{P-\{a\}}(x)$. It contradicts to the given condition. So $a \in P$ is indispensable.
Suppose $S=(U, A, V, f)$ and $S^{\prime}=\left(U^{\prime}, A^{\prime}, V^{\prime}, f^{\prime}\right)$ are two IISs, $h_{o}: U \rightarrow U^{\prime}, h_{A}: A \rightarrow A^{\prime}, h_{D}: V \rightarrow V^{\prime}$, then $h=\left(h_{O}, h_{A}\right.$, $\left.h_{D}\right): S \rightarrow S^{\prime}$ is called a mapping from $S$ to $S^{\prime}$.
Definition 8. If for $\forall x \in U, \forall a \in A, h_{D}\left(f(x, a)=f^{\prime}\left(h_{O}(x), h_{A}\right.\right.$ (a)), then $h$ is called a homomorphism between $S$ and S'. If $S=S^{\prime}$, then $h$ is called an endohomomorphism.
Definition 9. Let $f: U \rightarrow U$ be a mapping on $U, D \subseteq U$ be a subset of $U$. If $f(D)=D$, then $D$ is called an invariant subset of $f$.

## III. Invariant Properties

Lemma 1. Let $S=(U, A, V, f)$ and $S^{\prime}=\left(U^{\prime}, A^{\prime}, V^{\prime}, f^{\prime}\right)$ be two IISs, $h=\left(h_{O}, h_{A}, h_{D}\right)$ be a homomorphism between IIS $S$ and $S^{\prime}$. If both $S$ and $S^{\prime}$ contain attribute value ${ }^{*}$, and $h_{D}\left({ }^{*}\right)=*$, then for any attribute subset $P \subseteq A$ and any object $x \in U$, we have $h_{O}\left(S_{P}(x)\right)$ $\subseteq S_{\mathrm{m}}\left(h_{O}(x)\right)$, where $m=h_{A}(P)$. Especially, if $h_{O}$ is surjective, $h_{D}$ is 1-to-1 corresponding injection and $h_{D}(*)=*$, then inverse conclusion also holds, i.e., $h_{O}\left(S_{P}(x)\right)=S_{m}\left(h_{O}(x)\right)$.
Proof. For $\forall y^{\prime} \in h_{O}\left(S_{P}(x)\right)$, there must have $\exists y \in S_{P}(x)$ such that $\forall h_{0}(y)=y$, and then $\forall a \in P, f(y, a)=f(x, a) \vee f(y$, $a)=* \vee f(x, a)=*$. From the definition of homomorphism, we have $y^{\prime}=h_{O}(y) \in S_{m}\left(h_{O}(x)\right.$ ). So $h_{O}\left(S_{P}(x)\right) \subseteq S_{m}\left(h_{O}(x)\right)$ for $y^{\prime} \in$ $h_{O}\left(S_{P}(x)\right)$ is arbitrarily chosen.
Now we prove that when $h_{O}$ is surjective, $h_{D}$ is one to one corresponding injection and $h_{D}\left({ }^{*}\right)=*, h_{O}\left(S_{P}(x)\right)=S_{m}\left(h_{O}(x)\right)$. Before $h_{O}\left(S_{P}(x)\right) \subseteq S_{m}\left(h_{O}(x)\right)$ is already proved, so now we need only to prove that $h_{O}\left(S_{P}(x)\right) \supseteq S_{m}\left(h_{O}(x)\right)$.

Take $y^{\prime} \in S_{m}\left(h_{O}(x)\right)$. Because $h_{O}$ is surjective, there exists $y \in U$ such that $h_{o}(y)=y^{\prime}$. By the definition of homomorphism, for any $a \in P$, we have $h_{D}(f(y, a))=f^{\prime}\left(h_{O}(y), h_{A}(a)\right)=f^{\prime}\left(y^{\prime}, h_{A}\right.$ (a)). $f^{\prime}\left(h_{O}(x), h_{A}(a)\right)=h_{D}(f(x, a))$.

Since $y^{\prime} \in S_{m}\left(h_{O}(x)\right)$, we have $f^{\prime}\left(y^{\prime}, h_{A}(a)\right)=f^{\prime}\left(h_{O}(x), h_{A}(a)\right)$ or $f^{\prime}\left(y^{\prime}, h_{A}(a)\right)=*$ or $f^{\prime}\left(h_{O}(x), h_{A}(a)\right)=*$. Furthermore, we obtain $y^{\prime} \in h_{O}\left(S_{P}(x)\right)$. So we always have $S_{m}\left(h_{O}(x)\right) \subseteq h_{O}\left(S_{P}(x)\right)$. Therefore, under the condition that $h_{O}$ is surjective, $h_{D}$ is one to one corresponding injection and $h_{D}(*)=*$, we have $h_{O}\left(S_{P}(x)\right)=$ $S_{m}\left(h_{O}(x)\right)$.

From the proof of Lemma 1, we can obtain the following conclusion: if $h_{O}$ is surjective, $h_{D}$ is one to one corresponding injection and $h_{D}\left({ }^{*}\right)=*, h_{A}$ is surjective, the number of tolerance classes in $S$ is equal to that in $S^{\prime}$.
Lemma 2. Let $S$ and $S^{\prime}$ be two IIS, $h_{O}: U \rightarrow U^{\prime}, h_{A}: A \rightarrow A^{\prime}$,
$h_{D}: V \rightarrow V^{\prime}, h=\left(h_{O}, h_{A}, h_{D}\right): S \rightarrow S^{\prime}$ is called a mapping from $S$ to $S^{\prime}$, and for $h, h_{O}$ be surjective, $h_{D}$ be 1- to-1 corresponding injection and $h_{D}\left({ }^{*}\right)=^{*}$ when both $S$ and $S^{\prime}$ contain $*$. If $a \in A$ is redundant in $A, A-\{a\}-h^{-1}\left(h_{A}(a)\right) \neq \varnothing$, then $h^{-1}\left(h_{A}(a)\right) \subseteq A$ is also redundant in $A$.
Proof. Take $q^{\prime} \in A^{\prime}$ and $q^{\prime}=h_{A}(a)$. For any $x \in U$ and any $b^{\prime} \in h$ $A^{-1}\left(q^{\prime}\right)$, from Lemma 1, $h_{O}\left(S_{a}(x)\right)=S_{q}\left(h_{O}(x)\right)=h_{O}\left(S_{b}(x)\right)$. We assert that $S_{a}(x)=S_{b}(x)$. Otherwise, there exists an element $x_{0} \in U$ such that $S_{a}\left(x_{0}\right) \neq S_{b}\left(x_{0}\right)$. Then there exists $x^{\prime} \in S_{a}\left(x_{0}\right)$, $x^{\prime} \notin S_{b}\left(x_{0}\right)$ or $x^{\prime \prime} \notin S_{a}\left(x_{0}\right), x^{\prime}, \in S_{b}\left(x_{0}\right)$ In the first case, $f\left(x^{\prime}, b\right) \neq f\left(x_{0}, b\right) \wedge f\left(x^{\prime}, b\right) \neq{ }^{*} \wedge f\left(x_{0}, b\right) \neq *, f\left(x^{\prime}, a\right)=f\left(x_{0}, a\right) \vee f\left(x^{\prime}\right.$, $a)=* \vee f\left(x_{0}, a\right)=*$.

Since $h_{D}$ is one to one corresponding injection and $h_{D}\left({ }^{*}\right)=*$, thus $* \neq h_{D}\left(f\left(x^{\prime}, b\right)\right) \neq h_{D}\left(f\left(x_{0}, b\right)\right) \neq * . * \neq f^{\prime}\left(h_{O}\left(x^{\prime}\right), q^{\prime}\right)=h_{D}$ $\left(f\left(x^{\prime}, b\right)\right) \neq h_{D} \quad\left(f\left(x_{0}, b\right)\right)=f^{\prime}\left(h_{O}\left(x_{0}\right), q^{\prime}\right) \neq{ }^{*}$. So, * $\neq h_{D}$ $\left(f\left(x^{\prime}, b\right)\right) \neq f^{\prime}\left(h_{O}\left(x_{0}\right), q^{\prime}\right) \neq$ * $^{*}$. Therefore, from $f\left(x^{\prime}, a\right)=f\left(x_{0}, a\right)$ $\vee f\left(x^{\prime}, a\right)=* \vee f\left(x_{0}, a\right)=*$, we obtain

1. If $f\left(x^{\prime}, a\right)=f\left(x_{0}, a\right) \neq *$, then we have $* \neq h_{D}\left(f\left(x^{\prime}, a\right)\right)=$ $h_{D}\left(f\left(x_{0}, a\right)\right) \neq{ }^{*}$, since $h_{D}$ is an one-to-one corresponding injection and $h_{D}\left({ }^{*}\right)=*$ when both $S$ and $S^{\prime}$ contain attribute value *. Therefore, $f^{\prime}\left(h_{O}\left(x^{\prime}\right), q^{\prime}\right)=h_{D}\left(f\left(x^{\prime}, a\right)\right)=h_{D}\left(f\left(x_{0}, a\right)\right)=$ $f^{\prime}\left(h_{O}\left(x_{0}\right), q^{\prime}\right) \neq \quad$. This contradicts to $f^{\prime}\left(h_{O}\left(x^{\prime}\right), q^{\prime}\right) \neq f^{\prime}\left(h_{O}\left(x_{0}\right), q^{\prime}\right)$.
2. When $f\left(x^{\prime}, a\right)=*, f\left(x_{0}, a\right) \neq *$, since $h_{D}$ is one-to-one corresponding injection and $h_{D}\left({ }^{*}\right)=^{*}$, then $f^{\prime}\left(h_{O}\left(x^{\prime}\right), q^{\prime}\right)=$ $h_{D}\left(f\left(x^{\prime}, a\right)\right)=h_{D}\left(^{*}\right)=*$. This contradicts to $f^{\prime}\left(h_{O}\left(x^{\prime}\right), q^{\prime}\right) \neq{ }^{*}$.
3. When $f\left(x^{\prime}, a\right) \neq{ }^{*}, f\left(x_{0}, a\right)=*$, since $h_{D}$ is one-to-one corresponding injection and $h_{D}\left(^{*}\right)=^{*}$, then $f^{\prime}\left(h_{O}\left(x_{0}\right), q^{\prime}\right)=$ $h_{D}\left(f\left(x_{0}, a\right)\right)=h_{D}\left({ }^{*}\right)={ }^{*}$. This contradicts to $f^{\prime}\left(h_{O}\left(x_{0}\right), q^{\prime}\right) \neq{ }^{*}$.
4. When $f\left(x^{\prime}, a\right)=*, f\left(x_{0}, a\right)=*$, for $h_{D}$ is 1-to-1 corresponding injection, $h_{D}\left({ }^{*}\right)=*$, then $f^{\prime}\left(h_{O}\left(x^{\prime}\right), q^{\prime}\right)=h_{D}\left(f\left(x^{\prime}, a\right)\right)=h_{D}\left({ }^{*}\right)=*$, $f^{\prime}\left(h_{O}\left(x_{0}\right), q^{\prime}\right)=h_{D}\left(f\left(x_{0}, a\right)\right)=h_{D}\left({ }^{*}\right)=*$. These contradict to $f^{\prime}\left(h_{O}\left(x^{\prime}\right), q^{\prime}\right) \neq *$ and $f^{\prime}\left(h_{O}\left(x_{0}\right), q^{\prime}\right) \neq *$ respectively.
That means $S_{a}(x)=S_{b}(x)$. In the other case, a contradiction can be also deduced. So the above assertion is true. Therefore, $a$ is redundant in $A$, and if $A-\{a\}-h^{-1}\left(h_{A}(a)\right) \neq \varnothing, b$ is also redundant in $A$. From that $b$ is arbitrarily chosen, we obtain all attributes in $h^{-1}\left(q^{\prime}\right)=h^{-1}\left(h_{A}(a)\right)$ are redundant in $A$.
Lemma 3. Let $h=\left(h_{O}, h_{A}, h_{D}\right)$ be a homomorphism between $S$ and $S^{\prime}$. If for $h$ we have: both $h_{O}$ and $h_{A}$ are surjective, $h_{D}$ is an one to one corresponding injection and $h_{D}(*)=*$ when both $S$ and $S^{\prime}$ contain ${ }^{*}$, then for any attribute subset $P \subseteq A$, all attributes in $P$ are redundant in $A$, then if $h_{A}(P) \subset A$, all attributes in $h_{A}(P)$ are redundant in $S^{\prime}$.
Proof. From the definition of redundancy, we just only need to prove that $\operatorname{SIM}\left(A^{\prime}\right)=\operatorname{SIM}\left(A^{\prime}-P^{\prime}\right)$, where $P^{\prime}=h_{A}(P)$. Because $A^{\prime}-P^{\prime} \subseteq A^{\prime}, \operatorname{SIM}\left(A^{\prime}\right) \subseteq \operatorname{SIM}\left(A^{\prime}-P^{\prime}\right)$. Now we are to prove the inverse inclusion is held, i.e. $\operatorname{SIM}\left(A^{\prime}-P^{\prime}\right) \subseteq \operatorname{SIM}\left(A^{\prime}\right)$. Let $x^{\prime}, y^{\prime} \in U^{\prime}$, satisfy $\left(x^{\prime}, y^{\prime}\right) \in \operatorname{SIM}\left(A^{\prime}-P^{\prime}\right)$, i.e., $y^{\prime} \in S_{A^{\prime}-P},\left(x^{\prime}\right)$. Since $h_{A}$ is surjective, $A^{\prime}=h_{A}(A)$. For $h_{O}$ is surjective, there exists $x \in U$ such that $h_{O}(x)=x^{\prime}$. From Lemma 1, denote $r=$ $A-h_{A}^{-1}\left(P^{\prime}\right), t=h_{A}(r)$, we have $y^{\prime} \in S_{A^{\prime}-P^{\prime}\left(x^{\prime}\right)}=S_{t}\left(x^{\prime}\right)=S_{t}\left(h_{O}(x)\right)=h_{O}$ $\left(S_{r}(x)\right)$. So, there exists $\mathrm{y} \in S_{r}(x)$ such that $h_{O}(y)=y$ ' and $(y, x) \in \operatorname{SIM}(r)$.

Because $P$ is redundant in $S$, from Lemma $2, h_{A}^{-1}\left(P^{\prime}\right)$ is also
redundant in $A$, then $(x, y) \in \operatorname{SIM}(A)$. Thus, $\left(h_{O}(x)\right.$, $\left.h_{O}(y)\right) \in \operatorname{SIM}\left(h_{A}(A)\right)$, that is, $\left(x^{\prime}, y^{\prime}\right) \in \operatorname{SIM}\left(A^{\prime}\right)$. This means $\operatorname{SIM}\left(A^{\prime}-P^{\prime}\right) \subseteq \operatorname{SIM}\left(A^{\prime}\right)$.
Proposition. Let $h=\left(h_{O}, h_{A}, h_{D}\right)$ be a homomorphism between $S$ and $S^{\prime}$. If for $h$, we have that both $h_{O}$ and $h_{A}$ are one to one corresponding injection and $h_{D}(*)=*$ when both $S$ and $S^{\prime}$ contain ${ }^{*}$. If $P \subset A$ is redundant in $S, \forall a \in P$, $A-\{a\}-h_{A}^{-1}\left(h_{A}(a)\right) \neq \varnothing, h_{A}(P) \subset A^{\prime}$, then $h_{A}(P)$ is also redundant in $S^{\prime}$.
Proof. Let $A_{1}=h_{A}(A)$. Then $S_{1}=\left(U^{\prime}, A_{1}, V_{1}, f_{1}\right)$ is a subsystem of $S^{\prime}$, where $V_{1}=\cup V_{a},\left(a^{\prime} \in h_{A}(A)\right), f_{1}$ is the restriction of $f^{\prime}$ on $U^{\prime} \times A_{1}$. Thus, $h$ satisfies conditions given in Lemma 3 from $S$ to $S_{1}$. So $h_{A}(P)$ is redundant in $S_{1}$. Since $A_{1} \subseteq A^{\prime}, h_{A}(P)$ is redundant in $A^{\prime}$, that is, $h_{A}(P)$ is redundant in $S^{\prime}$.
Theorem 5. Let $S=(U, A, V, f)$ be an IIS. Suppose $P \subseteq A, A-P$ is redundant in $A$ and for each $a \in A-P$ there exists $a^{\prime} \in P$ such that $\operatorname{SIM}(a) \subseteq \operatorname{SIM}\left(a^{\prime}\right)$. Use $S^{\prime}$ to denote IIS $\left(U, A, V^{\prime}, f^{\prime}\right)$, where $V^{\prime}=V_{a}(a \in P), f^{\prime}$ is the restriction of $f$ on $U \times P$, then there exists a homomorphism $h$ from $S$ to $S$, and it is an endohomomorphism on $S$.
Proof. If $P=A$, then the theorem is obviously correct. So we assume that $P \subset A$. Define a mapping $h=\left(h_{O}, h_{A}, h_{D}\right): S \rightarrow S^{\prime}$ as follows: $h_{\mathrm{O}}=1_{U}$ is the identity mapping on $U . h_{A}(a)=a$, if $a \in P$; $h_{A}(a)=a^{\prime}$, if $a \in A-P, \operatorname{SIM}(a)=\operatorname{SIM}\left(a^{\prime}\right)$.

For $\forall x \in U$, let $h_{D}(f(x, a))=f(x, a)$, if $a \in P ; h_{D}(f(x, a))$ $=f\left(x, a^{\prime}\right)$, if $a \in A-P, a^{\prime}=h_{A}(a)$. Next, we explain the rationality of the definition of $h_{D}$. For any fixed $a \in A-P$, if $f(x, a)=f(y, a)$ or $f(x, a))=*$ or $f(y, a)=*(x, y \in U)$, then $f\left(x, a^{\prime}\right)=f\left(y, a^{\prime}\right)$ or $f\left(x, a^{\prime}\right)=*$ or $\left.f\left(y, a^{\prime}\right)\right)^{*}(x, y \in U)$, here, $a^{\prime}=h_{A}(a)$. In fact, notice that $\operatorname{SIM}(a)=\operatorname{SIM}\left(a^{\prime}\right)$ and $f(x, a)=f(y, a)$ or $f(x, a)=*$ or $f(y, a)=*$, then $(x, y) \in \operatorname{SIM}(a)$ implies $(x, y) \in \operatorname{SIM}\left(a^{\prime}\right)$. This means $f\left(x, a^{\prime}\right)=f\left(y, a^{\prime}\right)$ or $f\left(x, a^{\prime}\right)=*$ or $f\left(y, a^{\prime}\right)=*$. At this time, $h_{D}(f(x, a))$ $=f\left(x, a^{\prime}\right)=f\left(h_{O}(x), h_{A}(a)\right)$ (because $\left.h_{O}(x)=x\right)$. Again for any $x \in U$ and any $a \in A$, if $a \in P$, then $h_{D}(f(x, a))=f\left(x, a^{\prime}\right)=f\left(h_{O}(x), h_{A}(a)\right)$. Therefore, for any $x \in U$ and any $a \in A$, we have $h_{D}(f(x, a))=$ $f\left(h_{O}(x), h_{A}(a)\right)$. Thus, the definition of $h_{D}$ is rational. $h$ is a homomorphism from $S$ to $S^{\prime}$ and it is an endohomomorphism on $S$.

Generally, note that an arbitrary reduction of an IIS may not be able to determine a homomorphism from the later to the former. See the following example.
Example. An IIS $S$ is given in Table I.
TABLE I

| AN INCOMPLETE INFORMATION S SYSTEM |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $a$ |  |  |  | $b$ | $c$ |
| $x_{1}$ |  |  |  |  |  |

Table II is IIS $S^{\prime}$, a reduction of $S$. However, obviously, there does not exist a homomorphism from $S$ to $S^{\prime}$ or an endohomomorphism on $S$. This is because we cannot obtain an one-to-one corresponding mapping by specifying $h_{A}(0)=a$ or $h_{A}(0)=b$ where element 0 is in $V_{c}, a, b$ are in $V_{a}$ or $V_{b}$.

TABLE II
ANOTHER INCOMPLETE INFORMATION SYSTEM

|  | $a$ | $b$ |
| :---: | :---: | :---: |
| $x_{1}$ | 0 | 1 |
| $x_{2}$ | 1 | $*$ |
| $x_{3}$ | $*$ | 2 |
| $x_{4}$ | 2 | 0 |

Theorem 6. Let $S$ be an IIS, $h$ be an endohomomorphism on $S$, $h_{D}$ be one to one corresponding mapping and $h_{D}\left({ }^{*}\right)=*$, and for any $x \in U$ and any $a \in A$, tolerance class $S_{a}(x)$ generated by $x$ under attribute $a$ satisfies $\left|f\left(S_{a}(x), a\right)-\{*\}\right|=1$. If $h_{A}(A)$ is a $h_{A}$ -invariant subset, then $A-h_{A}(A)$ is redundant in $A$.
Proof. For convenience, let us denote $P=h_{A}(A)$. Take $a \in A-P$, as an any attribute from $A-P$. We assert that for any $x \in U$ we have $S_{a}(x)=\cup S_{P}(y)\left(y \in S_{a}(x)\right)$.

If not, i.e., $\exists x_{0} \in U, S_{a}\left(x_{0}\right) \subset \cup S_{P}(y)\left(y \in S_{a}\left(x_{0}\right)\right)$, then $\exists y_{0} \in S_{a}\left(x_{0}\right)$ such that $S_{P}\left(y_{0}\right) \nsubseteq S_{a}\left(x_{0}\right)$. In other words, $\exists y_{1} \in S_{P}\left(y_{0}\right)$ but $y_{1} \notin S_{a}\left(x_{0}\right)$. So, $f\left(x_{0}, a\right) \neq f\left(y_{1}, a\right) \wedge f\left(x_{0}, a\right)$ $\neq * \wedge f\left(y_{1}, a\right) \neq *$. By the definition of endohomomorphism and $h_{D}$ is one to one corresponding mapping and $h_{D}\left({ }^{*}\right)=*$, we have * $\neq h_{D}\left(f\left(x_{0}, a\right)\right) \neq h_{D}\left(f\left(y_{1}, a\right)\right) \neq$ *,i.e., ${ }^{*} \neq f\left(h_{O}\left(x_{0}\right), h_{A}(a)\right) \neq$ $f\left(h_{O}\left(y_{1}\right), h_{A}(a)\right) \neq *$. On the other hand, by Lemma 1 and $P$ is $h_{A}$ -invariant subset, i.e., $h_{A}(P)=P$, we have $h_{O}\left(S_{P}\left(y_{0}\right)\right) \subseteq S_{P}\left(h_{O}\right.$ $\left.\left(y_{0}\right)\right)$. Because $y_{1} \in S_{P}\left(y_{0}\right)$, thus $y_{1}, y_{0} \in S_{P}\left(y_{0}\right)$. Therefore, $h_{O}\left(y_{0}\right)$ and $h_{O}\left(y_{1}\right)$ both are in $S_{P}\left(h_{O}\left(y_{0}\right)\right)$. Since for any $a \in P h_{A}(a) \in P$, we have $f\left(h_{O}\left(y_{0}\right), h_{A}(a)\right)=f\left(h_{O}\left(y_{1}\right), h_{A}(a)\right)$ or $f\left(h_{O}\left(y_{0}\right), h_{A}(a)\right)=*$ or $f\left(h_{O}\left(y_{1}\right), h_{A}(a)\right)=*$. If $f\left(h_{O}\left(y_{1}\right), h_{A}(a)\right)=*$, then it is contradict to the above assumption. Thus it must have $f\left(h_{O}\left(y_{1}\right), h_{A}(a)\right) \neq{ }^{*}$. At this time

1. If $f\left(h_{O}\left(y_{0}\right), h_{A}(a)\right) \neq *$, then
i. If $f\left(h_{O}\left(x_{0}\right), h_{A}(a)\right)=*$, then $h_{D}\left(f\left(x_{0}, a\right)\right)=*, f\left(x_{0}, a\right)=*$. It contradicts to $f\left(x_{0}, a\right) \neq{ }^{*}$.
ii. If $f\left(h_{O}\left(x_{0}\right), h_{A}(a)\right) \neq{ }^{*}$, then because $y_{0} \in S_{a}\left(x_{0}\right), f\left(x_{0}, a\right)=$ $f\left(y_{0}, a\right) \vee f\left(x_{0}, a\right)=* \quad \vee f\left(y_{0}, a\right)=*$. But $\quad h_{D}\left(f\left(x_{0}, a\right)\right)=$ $f\left(h_{o}\left(x_{0}\right), h_{A}(a)\right) \neq{ }^{*} \Rightarrow$ implies $f\left(x_{0}, a\right) \neq{ }^{*}$; $h_{D}\left(f\left(y_{0}, a\right)\right)=$ $f\left(h_{O}\left(y_{0}\right), h_{A}(a)\right) \neq * \Rightarrow f\left(y_{0}, a\right) \neq{ }^{*}$. So it must have, $f\left(h_{O}\left(x_{0}\right)\right.$, $\left.h_{A}(a)\right)=f\left(h_{O}\left(y_{0}\right), h_{A}(a)\right) \neq{ }^{*}$. Thus $f\left(h_{O}\left(y_{1}\right), h_{A}(a)\right)=$ $f\left(h_{O}\left(y_{0}\right), h_{A}(a)\right)=f\left(h_{O}\left(x_{0}\right), h_{A}(a)\right) \neq{ }^{*}$. It is a contradiction.
2. If $f\left(h_{O}\left(y_{0}\right), h_{A}(a)\right)=*$, i.e., $h_{D}\left(f\left(y_{0}, a\right)\right)=f\left(h_{O}\left(y_{0}\right), h_{A}(a)\right)=*$ $\Rightarrow f\left(y_{0}, a\right)=*$, then
i. If $f\left(h_{O}\left(x_{0}\right), h_{A}(a)\right)=*$, then $h_{D}\left(f\left(x_{0}, a\right)\right)=f\left(h_{O}\left(x_{0}\right), h_{A}(a)\right)$ $=* \Rightarrow f\left(x_{0}, a\right)=*$. This contradicts to the assumption $f\left(x_{0}, a\right)$ $\neq *$ above.
ii. If $f\left(h_{O}\left(x_{0}\right), h_{A}(a)\right) \neq *^{*}$, then $h_{D}\left(f\left(x_{0}, a\right)\right)=f\left(h_{O}\left(x_{0}\right), h_{A}(a)\right)$ $\neq * \Rightarrow f\left(x_{0}, a\right) \neq{ }^{*}$. Since for any $a \in A$ and any $x \in U$, $\left|f\left(S_{a}(x), a\right)-\{*\}\right|=1$, according to the condition and $x_{0}, y_{0} \in S_{P}\left(x_{0}\right) \subseteq S_{a}\left(x_{0}\right), y_{0}, y_{1} \in S_{P}\left(y_{0}\right) \subseteq S_{a}\left(y_{0}\right), h_{A}(a) \in P$, therefore $x_{0} y_{1} \in S_{a}\left(y_{0}\right)$. Thus $f\left(h_{O}\left(y_{1}\right), \quad h_{A}(a)\right)=$ $\neq f\left(h_{O}\left(x_{0}\right), h_{A}(a)\right) \neq *$. This also contradicts to the above conclusion. So, $S_{a}(x)=\cup S_{P}(y)\left(y \in S_{a}(x)\right)$.
By the former theorem, we know that each attribute in $A-P$ is redundant in $A$.

The essence of homomorphism is clustering. The condition that $h_{D}$ is a one to one corresponding mapping and $h_{D}\left({ }^{*}\right)=*$ when both $S$ and $S^{\prime}$ contain * and $\left|f\left(S_{a}(x), a\right)-\{*\}\right|=1$ means that
the classification grade of each attribute deduced by $h$ does not decrease. In addition, if $h_{A}(A)$ is an $h_{A}$-invariant subset, then Theorem 6 ensures that $A-h_{A}(A)$ is also redundant in $A$.
Theorem 7. Let $S$ and $S^{\prime}$ be two IISs, $h=\left(h_{O}, h_{A}, h_{D}\right)$ be a homomorphism between IIS $S$ and $S^{\prime}$, satisfying that $h_{O}$ and $h_{A}$ are surjective, $h_{D}$ is one to one corresponding mapping, $h_{D}\left({ }^{*}\right)=*$ when both $S$ and $S^{\prime}$ contain *, and for any $x \in U$ and any $a \in A$, tolerance class $S_{a}(x)$ generated by $x$ under attribute $a$ satisfies $\left|f\left(S_{a}(x), a\right)-\{*\}\right|=1$. If $P \subseteq A$ is a reduction of $A$ in $S$, then $h_{A}(P) \subseteq A^{\prime}$ is also a reduction of $A^{\prime}$ in $S^{\prime}$.
Proof. Denote $h_{A}(P)$ by $P^{\prime}$. Firstly, we prove $S I M\left(P^{\prime}\right)=S I M\left(A^{\prime}\right)$ It is obvious that we should only prove $\operatorname{SIM}\left(P^{\prime}\right) \subseteq \operatorname{SIM}\left(A^{\prime}\right)$.

Let $\left(x^{\prime}, y^{\prime}\right) \in \operatorname{SIM}\left(P^{\prime}\right)$, i.e. $y^{\prime} \in S_{P}\left(x^{\prime}\right)$. Since $h_{O}$ is surjective, There exists $x \in U$ such that $h_{O}(x)=x^{\prime}$. Therefore according to Lemma 1, we have $S_{P}\left(x^{\prime}\right)=S_{P},\left(h_{O}(x)\right)=h_{O}\left(S_{P}(x)\right)$. Then there exists $y \in S_{P}(x)$ such that $h_{o}(y)=y^{\prime}$. So we have $(x, y) \in \operatorname{SIM}(P)$. Because $P$ is a reduction of $A$ in $S,(x, y) \in \operatorname{SIM}(P)$ implies $(x, y) \in \operatorname{SIM}(A)$. Thus, $\left(h_{O}(x), h_{O}(y)\right) \in \operatorname{SIM}\left(h_{A}(A)\right)$. For $h_{A}$ is surjective, we have ( $x^{\prime}, y^{\prime}$ ) $\in S I M\left(A^{\prime}\right)$. Furthermore, we obtain $\operatorname{SIM}\left(P^{\prime}\right) \subseteq \operatorname{SIM}\left(A^{\prime}\right)$. So $\operatorname{SIM}\left(P^{\prime}\right)=\operatorname{SIM}\left(A^{\prime}\right)$.
By Lemma 2, the remainder work is to prove that $P^{\prime}$ does not contain redundant attribute again. Conversely, we assume that $a^{\prime} \in P^{\prime}$ is redundant in $P^{\prime}$. Let $B$ is the set of reverse image of attribute $a^{\prime}$, included in $P$, under mapping $h_{A}$, i.e., $B=$ $h_{A}^{-1}\left(a^{\prime}\right) \cap P$.
Since $a^{\prime} \in P^{\prime}$ is redundant in $P^{\prime}$, for any $x^{\prime}, y^{\prime} \in U^{\prime}$ in $U^{\prime}$, if $y^{\prime} \in S_{A^{\prime}\left\{\left\{a^{\prime}\right\}\right.}\left(x^{\prime}\right)$ then $y^{\prime} \in S_{a^{\prime}}\left(x^{\prime}\right)$.

Because $B$ is not redundant in $P, \operatorname{SIM}(P) \neq \operatorname{SIM}(P-B)$. But for $P-B \subseteq P, \operatorname{SIM}(P) \subset \operatorname{SIM}(P-B)$. Thus, there exist $x, y \in U$ such that $(x, y) \in \operatorname{SIM}(P-B),(x, y) \notin \operatorname{SIM}(P)$, i.e., $y \in S_{P-B}(x)$, $y \notin S_{B}(x)$. It follows that there exists $b \in B$ such that * $\neq f(y, b) \neq f(x, b) \neq{ }^{*} . h_{D}$ is one to one corresponding mapping and $h_{D}\left({ }^{*}\right)=*$, so $* \neq h_{D}(f(y, b)) \neq h_{D}(f(x, b)) \neq *$. That is to say, $\left.\left.* \neq f\left(h_{O}(y), a^{\prime}\right)\right) \neq f\left(h_{O}(x), a^{\prime}\right)\right) \neq$ * $^{*}$, i.e., $h_{O}(y) \notin S_{a^{\prime}}\left(h_{O}(x)\right)$. However, from $y \in S_{P-B}(x)$ we have $h_{O}(y) \in h_{O}\left(S_{P-B}(x)\right)=$ $S_{P^{\prime},\left\{a^{\prime}\right\}}(x), h_{O}\left(S_{B}(x)\right)=S_{a^{\prime}}\left(h_{O}(x)\right)$. This contradicts to that $a^{\prime} \in P^{\prime}$ is redundant in $P^{\prime}$ and for any $x^{\prime}, y^{\prime} \in U^{\prime}$ in $U^{\prime}$, if $y^{\prime} \in S_{P^{\prime}\left\{\left\{a^{\prime}\right\}\right.}\left(x^{\prime}\right)$ then $y^{\prime} \in S_{a^{\prime}}\left(x^{\prime}\right)$. This contradiction shows that there is no redundant attribute in $h_{A}(P)$.
Theorem 8. Let $S$ be an IIS, $h=\left(h_{O}, h_{A}, h_{D}\right)$ be an endohomomorphism from $S$ to $S$ ' satisfying: $h_{O}$ and $h_{A}$ be surjective, $h_{D}$ be one to one corresponding mapping and $h_{D}(*)=*$, and for any $x \in U$ and any $a \in A$, tolerance class $S_{a}(x)$ generated by $x$ under attribute $a$ satisfies $\left|f\left(S_{a}(x), a\right)-\{*\}\right|=1$. Then core $\left(S^{\prime}\right)=h_{A}(\operatorname{core}(S)$.
Proof. Let $a \in A$ be indispensable in $S$. We are to prove $h_{A}(a)$ is indispensable in S'. Denote $a^{\prime}=h_{A}(a)$. Because $a$ is indispensable in $S$, there exist $x, y \in U$ such that $y \notin S_{a}(x)$, $y \in S_{A-\{a\}}(x)$. However, for $S_{a}(x) \subseteq \cup S_{A-\{a ;}(y)\left(y \in S_{a}(x)\right)$ and if $S_{a}(x)=\cup S_{A-\{a ;}(y) \quad\left(y \in S_{a}(x)\right)$ then $a$ is dispensable in $S$, therefore, it only follows that $S_{a}(x) \subset \cup S_{A-\{a\}}(y)\left(y \in S_{a}(x)\right)$. Because $t=A-h^{-1}{ }_{A}\left(a^{\prime}\right) \subseteq A-\{a\}$, thus $S_{A-\{a\}}(x) \subseteq S_{t}(x)$. But $y \in S_{A-\{a\}}(x)$, so $y \in S_{t}(x)$. Furthermore, $f(y, t)=f(x, t) \vee f(y, t)=*$ $\vee f(x, t)=*$. Because $y \notin S_{a}(x), * \neq f(x, a) \neq f(y, a) \neq *$.
Since $h_{D}$ is a one-to-one corresponding mapping and
$h_{D}\left({ }^{*}\right)=*$, it infers that $* \neq f(x, a) \neq f(y, a) \neq *, * \neq h_{D}(f(x, a)) \neq$ $h_{D}(f(y, a)) \neq{ }^{*}, * \neq f\left(h_{O}(x), h_{A}(a)\right) \neq f\left(h_{O}(y), h_{A}(a)\right) \neq{ }^{*}$.

$$
\begin{aligned}
& f(x, t)=f(y, t) \vee f(x, t)=* \vee f(y, t)=* \\
& \Rightarrow h_{D}(f(x, t))=h_{D}(f(y, t)) \vee h_{D}(f(x, t))=* \vee h_{D}(f(f, t))=* \\
& \Rightarrow f^{\prime}\left(h_{O}(x), A^{\prime}-\left\{a^{\prime}\right\}\right)=f^{\prime}\left(h_{O}(y), A^{\prime}-\left\{a^{\prime}\right\}\right) \vee f^{\prime}\left(h_{O}(x), A^{\prime}-\left\{a^{\prime}\right\}\right) \\
&=* \vee f^{\prime}\left(h_{O}(y), A^{\prime}-\left\{a^{\prime}\right\}\right)=* .
\end{aligned}
$$

That is $h_{O}(y) \in S_{t}\left(h_{O}(x)\right), h_{O}(y) \notin S_{a^{\prime}}\left(h_{O}(x)\right)$. Thus, $a^{\prime}=h_{O}(a)$ is indispensable in $S$ '.

Because attribute in $A^{\prime}$ can be divided into two types, one is the image of indispensable attribute of $A$, another is the image of redundant attributes of $A$. According to Lemma 3, under the condition of the theorem, the image of redundant attribute is also redundant, so the indispensable attributes of $S^{\prime}$ are all images of indispensable attributes in $S$ and therefore $\operatorname{core}\left(S^{\prime}\right)=h_{A}(\operatorname{core}(S)$.

## IV. Conclusions

The present paper studies on some invariant properties in IISs. It finds out preserving invariance conditions respectively for tolerance classes in tolerance relation, attribute reduction, indispensable attribute and dispensable attribute under homomorphism in IISs. It also discusses the condition of endohomomorphism. It obtains several meaningful results. It lays a certain theoretic foundations for engaging researching IIS modeling under rough set theory.

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