International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:11, No:9, 2017

$(\in,\in\vee q)$-Fuzzy Subalgebras and Fuzzy Ideals of BCI-Algebras with Operators

Yuli Hu, Shaoquan Sun

Abstract**—**The aim of this paper is to introduce the concepts of $(\in,\in\vee q)$-fuzzy subalgebras, $(\in,\in\vee q)$-fuzzy ideals and $(\in,\in\vee q)$-fuzzy quotient algebras of BCI-algebras with operators, and to investigate their basic properties.

Keywords**—**BCI-algebras with operators, $(\in,\in\vee q)$-fuzzy subalgebras, $(\in,\in\vee q)$-fuzzy ideals, $(\in,\in\vee q)$-fuzzy quotient algebras.

I. Introduction

THE fuzzy set is a generalization of the classical set and was used afterwards by several authors such as Imai [1], Iseki [2] and Xi [3], in various branches of mathematics. Particularly, in the area of fuzzy topology, after the introduction of fuzzy sets by Zadeh [15], much research has been carried out: the concept of fuzzy subalgebras and fuzzy ideals of BCK-algebras, and their some properties.

BCK-algebras and BCI-algebras are two important classes of logical algebras, which were introduced by Imai and Iseki [1], [2]. In 1991, Xi [3] applied the fuzzy sets to BCK-algebras and discussed some properties about fuzzy subalgebras and fuzzy ideals. From then on, fuzzy BCK/BCI-algebras have been widely investigated by some researchers. Jun et al. [4], [5] raised the notions of fuzzy positive implicative ideals and fuzzy commutative ideals of BCK-algebras. Ming and Ming [12] introduced the neighbourhood structure of a fuzzy point in 1980; Jun et al. [6] introduced the concept of $(\in,\in\vee q)$-ideals of BCI-algebras. In 1993, Zheng [7] defined operators in BCK-algebras and introduced the concept of BCI-algebras with operators and gave some isomorphism theorems of it. Then, Liu [9] introduced the university property of direct products of BCI-algebras. In 2002, Liu [8] introduced the notion of the fuzzy quotient algebras of BCI-algebras. In 2004, Jun [10] introduced the (α,β)-fuzzy ideals of BCK/BCI-algebras and established the characterizations of $(\in,\in\vee q)$-fuzzy ideals. Next, Pan [13] introduced fuzzy ideals of sub-algebra and fuzzy H-ideals of sub-algebra. In 2011, Liu and Sun [11] introduced the concept of generalized fuzzy ideals of BCI-algebra and investigated some basic properties. In 2017, we [14] also introduced the fuzzy subalgebras and fuzzy ideals of BCI-algebras with operators.

In this paper, we introduce the concepts of $(\in,\in\vee q)$-fuzzy subalgebras, $(\in,\in\vee q)$-fuzzy ideals and $(\in,\in\vee q)$-fuzzy quotient algebras of BCI-algebras with operators. Moreover, the basic properties were discussed and several results have been obtained.

II. Preliminaries

Some definitions and propositions were recalled which may be needed.

An algebra $\langle X;*,0\rangle$ of type (2,0) is called a BCI-algebra, if for all $x,y,z\in X$, it satisfies:

$(1)\big((x*y)*(x*z)\big)*(z*y)=0;$

$(2)\big(x*(x*y)\big)*y=0;$

$(3)\,x*x=0;$

$(4)\,x*y=0$ and $y*x=0$ imply $x=y.$

We can define $x*y=0$ if and only if $x\le y$, and the above conditions can be written as:

1. $(x*y)*(x*z)\le z*y;$
2. $x*(x*y)\le y;$
3. $x\le x;$
4. $x\le y$ and $y\le x$ imply $x=y.$

A BCI-algebra is called a BCK-algebra if it satisfies $0*x=0.$

Definition 1. [5] $\langle X;*,0\rangle$ is a BCI-algebra, a fuzzy subset A of X is called a fuzzy ideal of X if it satisfies:

$(1)A(0)\ge A(x),\forall x\in X,$

$(2)A(x)\ge A(x*y)\wedge A(y),\forall x,y\in X.$

Definition 2. [4] $\langle X;*,0\rangle$ is a BCI-algebra, a fuzzy subset A of X is called a fuzzy subalgebra of X if it satisfies:

$$A(x*y)\ge A(x)*A(y),\forall x,y\in X.$$

Definition 3. [12] $\langle X;*,0\rangle$ is a BCI-algebra, a fuzzy subset A of X of the form

$$A(y)=\begin{cases}t(\ne 0), y=x,\\ 0, y\ne x,\end{cases}$$

is said to be a fuzzy point with support x and value t, and is

Yuli Hu is with the College of Mathematics and Physics, Qingdao University of Science and Technology, Qingdao, China (phone: 178-06259520; e-mail:1198260194@qq.com).

Shaoquan Sun is with the College of Mathematics and Physics, Qingdao University of Science and Technology, Qingdao, China (phone: 185-61681686; e-mail: qdsunsaoquan@163.com).

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:11, No:9, 2017

denoted by x_t.

Definition 4. [12] If x_t is a fuzzy point, it is said to belong to (resp.be quasi-coincident with) a fuzzy subset A, written as $x_t \in A$ (resp. $x_t qA$) if $A(x) \geq t$ (resp. $A(x)+t>1$). If $x_t \in A$ or $x_t qA$, then we write $x_t \in \vee qA$. The symbol $\overline{\in \vee q}$ (resp. $\overline{\in}$ or $\overline{q}$) means $\in \vee q$ (resp. $\in$ or q) does not hold.

Definition 5. [10] $\langle X;*,0\rangle$ is a BCI-algebra, a fuzzy set A of X is called an $(\in,\in\vee q)$-fuzzy ideal of X if for all $t,r\in(0,1]$ and $x,y\in X$, it satisfies:

1. $x_t \in A \Rightarrow 0_t \in \vee qA$,
2. $(x*y)_t \in A$ and $y_r \in A \Rightarrow x_{t\wedge r} \in \vee qA$.

Definition 6. [10] A fuzzy set A is an $(\in,\in\vee q)$-fuzzy ideal of X if and only if it satisfies:

$(1)\, A(0) \geq A(x) \wedge 0.5, \forall x \in X$,

$(2)\, A(x) \geq A(x*y) \wedge A(y) \wedge 0.5, \forall x, y \in X$.

Definition 7. [7] $\langle X;*,0\rangle$ is a BCI-algebra, M is a non-empty set, if there exists a mapping $(m,x)\rightarrow mx$ from $M\times X$ to X which satisfies

$$m(x*y)=(mx)*(my), \forall x,y\in X, m\in M,$$

then M is called a left operator of X, X is called BCI-algebra with left operator M, or $M-$BCI-algebra for short.

Proposition 1. [6] Let $\langle X;*,0\rangle$ be a BCI-algebra, if A is an $(\in,\in\vee q)$-fuzzy ideal of it, and $x*y\leq z$, then

$$A(x) \geq A(y)\wedge A(z)\wedge 0.5, \forall x,y,z\in X.$$

Definition 8. [13] Let A and B be fuzzy sets of set X, then the direct product $A\times B$ of A and B is a fuzzy subset of $X\times X$, define $A\times B$ by

$$A\times B(x,y)=A(x)\wedge B(y), \forall x,y\in X.$$

Definition 9. [7] Let $\langle X;*,0\rangle$ and $\langle \overline{X};*,0\rangle$ be two $M-$BCI-algebras, if for all $x\in X, m\in M$, $f(mx)=mf(x)$, and f is a homomorphism from $\langle X;*,0\rangle$ to $\langle \overline{X};*,0\rangle$, then f is called a homomorphism with operators.

Definition 10. [13] $\langle X;*,0\rangle$ is an $M-$BCI-algebra, let B be a fuzzy set of X, and A be a fuzzy relation of B, if it satisfies:

$$A_B(x,y)=B(x)\wedge B(y), \forall x,y\in X,$$

then A is called a strong fuzzy relation of B.

Definition 11. [14] If $\langle X;*,0\rangle$ is an $M-$BCI-algebra, A is a non-empty subset of X, and $mx\in A$ for all $x\in A, m\in M$, then $\langle A;*,0\rangle$ is called a $M-$subalgebra of $\langle X;*,0\rangle$.

In this paper, X always means a $M-$BCI-algebra unless otherwise specified.

III. $(\in,\in\vee q)$-Fuzzy Subalgebras of BCI-Algebras with Operators

Definition 12. $\langle X;*,0\rangle$ is a BCI-algebra, a fuzzy set A of X is called a $M-(\in,\in\vee q)$-fuzzy subalgebra of X if for all $t,r\in(0,1]$ and $x,y\in X$, it satisfies:

1. $x_t\in A$ and $y_r\in A\Rightarrow (x*y)_{t\wedge r}\in\vee qA$,
2. $x_t\in A\Rightarrow (mx)_t\in\vee qA$.

Proposition 2. $\langle X;*,0\rangle$ is a BCI-algebra, a fuzzy set A of X is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X if and only if it satisfies:

$(1)\, A(x*y)\geq A(x)\wedge A(y)\wedge 0.5, \forall x,y\in X$,

$(2)\, A(mx)\geq A(x)\wedge 0.5, \forall x\in X$.

Proof. Suppose that A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X. (1) Let $x,y\in X$, suppose that $A(x)\wedge A(y)<0.5$, then $A(x*y)\geq A(x)\wedge A(y)$, if not, then we have $A(x*y)<t<A(x)\wedge A(y), \exists t\in(0,0.5)$; it follows that $x_t\in A$ and $y_t\in A$, but $(x*y)_{t\wedge t}=(x*y)_t\overline{\in\vee q}A$, which is a contradiction, then whenever $A(x)\wedge A(y)<0.5$. We have $A(x*y)\geq A(x)\wedge A(y)$. If $A(x)\wedge A(y)\geq 0.5$, then $(x)_{0.5}\in A$ and $y_{0.5}\in A$, which implies that $(x*y)_{0.5}=(x*y)_{0.5\wedge 0.5}\in\vee qA$, therefore $A(x*y)\geq 0.5$, because if $A(x*y)<0.5$, then $A(x*y)+0.5<0.5+0.5=1$, which is a contradiction, hence

$$A(x*y)\geq A(x)\wedge A(y)\wedge 0.5, \forall x,y\in X.$$

(2) Let $x\in X$ and assume that $A(x)<0.5$. If $A(mx)<A(x)$, then we have $A(mx)<t<A(x), \exists t\in(0,0.5)$, and we have $x_t\in A$ and $(mx)_t\overline{\in}A$, since $A(mx)+t<1$, we have $(mx)_t\overline{q}A$; it follows that $(mx)_t\overline{\in\vee q}A$, which is a contradiction, hence $A(mx)\geq A(x)$. Now if $A(x)\geq 0.5$, then $x_{0.5}\in A$, thus $(mx)_{0.5}\in\vee qA$, hence $A(mx)\geq 0.5$, otherwise $A(mx)+0.5<0.5+0.5=1$, which is a contradiction, consequently, $A(mx)\geq A(x)\wedge 0.5, \forall x\in X$.

Conversely, assume that A satisfies condition (1), (2).

(1) Let $x,y\in X$ and $t_1,t_2\in(0,1]$ be such that $x_{t_1}\in A$ and $y_{t_2}\in A$, then $A(x)\geq t_1$ and $A(y)\geq t_2$. Suppose that $A(x*y)<t_1\wedge t_2$, if $A(x)\wedge A(y)<0.5$, then $A(x*y)\geq A(x)\wedge A(y)\wedge 0.5=A(x)\wedge A(y)\geq t_1\wedge t_2$, this is a contradiction, so we have $A(x)\wedge A(y)\geq 0.5$, it follows that

$$A(x*y)+t_1\wedge t_2>2A(x*y)\geq 2(A(x)\wedge A(y)\wedge 0.5)=1,$$

so that $(x*y)_{t_1\wedge t_2}\in\vee qA.$

(2) Let $x\in X$ and $t\in(0,1]$ be such that $x_t\in A,$ then we have $A(x)\geq t.$ Suppose that $A(mx)<t,$ if $A(x)<0.5,$ then $A(mx)\geq A(x)\wedge 0.5=A(x)\geq t,$ this is a contradiction, hence we know that $A(x)\geq 0.5,$ and we have

$$A(mx)+t>2A(mx)\geq 2(A(x)\wedge 0.5)=1,$$

then $(mx)_t\in\vee qA.$ Consequently, A is an $M-(\in,\in\vee q)$-fuzzy subalgebra.

Example 1. If A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X, then X_A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X, define X_A by

$$X_A:X\to[0,1],X_A(x)=\begin{cases}1,x\in A\\0,x\notin A.\end{cases}$$

Proof. (1) For all $x,y\in X,$ if $x,y\in A,$ then $x*y\in A,$ then we have

$$X_A(x*y)=1\geq X_A(x)\wedge X_A(y)\wedge 0.5,$$

if there exists at least one which does not belong to A between x and $y,$ for example $x\notin A,$ thus

$$X_A(x*y)\geq 0=X_A(x)\wedge X_A(y)\wedge 0.5.$$

(2) For all $x\in X$, $m\in M$, if $x\in A,$ then $mx\in A,$ therefore

$$X_A(mx)=1\geq X_A(x)\wedge 0.5,$$

if $x\notin A,$ then $X_A(mx)\geq 0=X_A(x)\wedge 0.5,$ therefore X_A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X.

Proposition 3. A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X if and only if A_t is an $M-$subalgebra of X, where A_t is a non-empty set, define X_A by

$$A_t=\{x\mid x\in X,A(x)\geq t\},\forall t\in[0,0.5].$$

Proof. Suppose A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X, A_t is a non-empty set, $t\in[0,0.5],$ then we have $A(x*y)\geq A(x)\wedge A(y)\wedge 0.5.$ If $x\in A_t,y\in A_t,$ then $A(x)\geq t,A(y)\geq t,$ thus

$$A(x*y)\geq A(x)\wedge A(y)\wedge 0.5\geq t,$$

then we have $x*y\in A_t.$ If A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X, then $A(mx)\geq A(x)\wedge 0.5\geq t,\forall x\in X,m\in M,$ then we have $mx\in A_t.$ Therefore A_t is an $M-$subalgebra of X. Conversely, suppose A_t is an $M-$subalgebra of X, then we have $x*y\in A_t$. Let $A(x)=t,$ then

$$A(x*y)\geq t=A(x)\geq A(x)\wedge A(y)\wedge 0.5.$$

If A_t is an $M-$subalgebra of X, then we have

$$A(mx)\geq t=A(x)\geq A(x)\wedge 0.5,\forall x\in X,m\in M,$$

therefore A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X.

Proposition 4. Suppose X,Y are $M-$BCI-algebras, f is a mapping from X to Y, if A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of the Y, then $f^{-1}(A)$ is a $M-(\in,\in\vee q)$-fuzzy subalgebra of X.

Proof. Let $y\in Y,$ suppose f is an epimorphism, and we have $y=f(x),\exists x\in X.$ If A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of Y, then we have

$$A(x*y)\geq A(x)\wedge A(y)\wedge 0.5,A(mx)\geq A(x)\wedge 0.5.$$

For all $x,y\in X,m\in M,$ we have

(1) $f^{-1}(A)(x*y)=A(f(x)*f(y))\geq A(f(x))\wedge A(f(Y))\wedge 0.5$
$=f^{-1}(A)(x)\wedge f^{-1}(A)(y)\wedge 0.5;$

(2) $f^{-1}(A)(mx)=A(f(mx))=A(mf(x))$
$\geq A(f(x))\wedge 0.5=f^{-1}(A)(x)\wedge 0.5.$

Then $f^{-1}(A)$ is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X.

IV. $(\in,\in\vee q)$- Fuzzy Ideals of BCI-algebras with Operators

Definition 13. $\langle X;*,0\rangle$ is a BCI-algebra, a fuzzy set A of X is called an $M-(\in,\in\vee q)$-fuzzy ideal of X if for all $t,r\in(0,1]$ and $x,y\in X$, it satisfies:

1. $x_t\in A\Rightarrow 0_t\in\vee qA,$
2. $(x*y)_t\in A$ and $y_r\in A\Rightarrow x_{t\wedge r}\in\vee qA,$
3. $x_t\in A\Rightarrow(mx)_t\in\vee qA.$

Proposition 5. [13] $\langle X;*,0\rangle$ is a BCI-algebra, a fuzzy set A is an $M-(\in,\in\vee q)$-fuzzy ideal of X if and only if it satisfies:

(1) $A(0)\geq A(x)\wedge 0.5,\forall x\in X,$
(2) $A(x)\geq A(x*y)\wedge A(y)\wedge 0.5,\forall x,y\in X,$
(3) $A(mx)\geq A(x)\wedge 0.5,\forall x\in X.$

Proof. Suppose that A is an $M-(\in,\in\vee q)$-fuzzy ideal of X.

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:11, No:9, 2017

(1) Let $x \in X$ and assume that $A(x) < 0.5$. If $A(0) < A(x)$, then we have $A(0) < t < A(x), \exists t \in (0, 0.5)$, and we have $x_t \in A$ and $0_t \bar{\in} A$, since $A(0)+t<1$, we have $0_t \bar{q} A$, it follows that $0_t \overline{\in \vee q} A$, which is a contradiction, then $A(0) \geq A(x)$. Now if $A(x) \geq 0.5$, then $x_{0.5} \in A$, then we have $0_{0.5} \in \vee q A$, hence $A(0) \geq 0.5$, otherwise, $A(0)+0.5 < 0.5+0.5 = 1$, which is a contradiction, consequently,

$$A(0) \geq A(x) \wedge 0.5, \forall x \in X.$$

(2) Let $x, y \in X$ and suppose that $A(x*y) \wedge A(y) < 0.5$, then $A(x) \geq A(x*y) \wedge A(y)$, if not, then we have $A(x) < t < A(x*y) \wedge A(y), \exists t \in (0, 0.5)$, it follows that $(x*y)_t \in A$ and $y_t \in A$, but $x_{t \wedge t} = x_t \overline{\in \vee q} A$, which is a contradiction, hence whenever $A(x*y) \wedge A(y) < 0.5$, we have $A(x) \geq A(x*y) \wedge A(y)$. If $A(x*y) \wedge A(y) \geq 0.5$, then $(x*y)_{0.5} \in A$ and $y_{0.5} \in A$, which implies that $x_{0.5} = x_{0.5 \wedge 0.5} \in \vee q A$, therefore $A(x) \geq 0.5$, because if $A(x) < 0.5$, then $A(x)+0.5 < 0.5+0.5 = 1$, which is a contradiction, then

$$A(x) \geq A(x*y) \wedge A(y) \wedge 0.5, \forall x, y \in X.$$

(3) Let $x \in X$ and assume that $A(x) < 0.5$. If $A(mx) < A(x)$, then we have $A(mx) < t < A(x), \exists t \in (0, 0.5)$, and we have $x_t \in A$ and $(mx)_t \bar{\in} A$, since $A(mx)+t<1$, we have $(mx)_t \bar{q} A$, it follows that $(mx)_t \overline{\in \vee q} A$, which is a contradiction, then $A(mx) \geq A(x)$. Now if $A(x) \geq 0.5$, then $x_{0.5} \in A$, thus $(mx)_{0.5} \in \vee q A$, hence $A(mx) \geq 0.5$, otherwise $A(mx)+0.5 < 0.5+0.5 = 1$, which is a contradiction, consequently, $A(mx) \geq A(x) \wedge 0.5, \forall x \in X$. Conversely, suppose that A satisfies (1), (2), (3) of the Proposition 5, then we have

(1) Let $x \in X$ and $t \in (0,1]$ be such that $x_t \in A$, then we have $A(x) > t$, suppose that $A(0) < t$, if $A(x) < 0.5$, then $A(0) \geq A(x) \wedge 0.5 = A(x) \geq t$, which is a contradiction, then we know that $A(x) \geq 0.5$, and we have $A(0)+t > 2A(0) \geq 2(A(x) \wedge 0.5) = 1$, thus $0_t \in \vee q A$.

(2) Let $x, y \in X$ and $t_1, t_2 \in (0,1]$ be such that $(x*y)_{t_1} \in A$ and $y_{t_2} \in A$, then $A(x*y) \geq t_1$ and $A(y) \geq t_2$, suppose that $A(x) < t_1 \wedge t_2$, if $A(x*y) \wedge A(y) < 0.5$, then

$$A(x) \geq A(x*y) \wedge A(y) \wedge 0.5 = A(x*y) \wedge A(y) \geq t_1 \wedge t_2,$$

This is a contradiction, so we have $A(x*y) \wedge A(y) \geq 0.5$, it follows that

$$A(x) + t_1 \wedge t_2 > 2A(x) \geq 2\left(A(x*y) \wedge A(y) \wedge 0.5\right) = 1,$$

so that $x_{t_1 \wedge t_2} \in \vee q A$.

(3) Let $x \in X$ and $t \in (0,1]$ be such that $x_t \in A$, then $A(x) \geq t$, suppose that $A(mx) < t$, if $A(x) < 0.5$, then $A(mx) \geq A(x) \wedge 0.5 = A(x) \geq t$, which is a contradiction, then we know that $A(x) \geq 0.5$, and we have $A(mx) + t > 2A(mx) \geq 2(A(x) \wedge 0.5) = 1$, thus $(mx)_t \in \vee q A$. Consequently, A is an $M-(\in, \in \vee q)$-fuzzy ideal.

Example 2. If A is an $M-(\in, \in \vee q)$-fuzzy ideal of X, then X_A is an $M-(\in, \in \vee q)$-fuzzy ideal of X, define X_A by

$$X_A : X \to [0,1], X_A(x) = \begin{cases} 1, x \in A \\ 0, x \notin A. \end{cases}$$

Proof. (1) For all $x, y \in X$, if $x, y \in A$, then $x*y \in A$, thus

$$X_A(0) = 1 \geq X_A(x) \wedge 0.5,$$
$$X_A(x) = 1 \geq X_A(x*y) \wedge X_A(y) \wedge 0.5,$$

if there exists at least one between x and y which does not belong to A, for example $x \notin A$, thus

$$X_A(0) = 1 \geq X_A(x) \wedge 0.5,$$
$$X_A(x) \geq X_A(x*y) \wedge X_A(y) \wedge 0.5 = 0,$$

therefore X_A is a $(\in, \in \vee q)$-fuzzy ideal of X.

(2) For all $x \in X$, $m \in M$, if $x \in A$, then $mx \in A$, therefore $X_A(mx) = 1 \geq X_A(x) \wedge 0.5$. If $x \notin A$, then $X_A(mx) \geq 0 = X_A(x) \wedge 0.5$, therefore X_A is an $M-(\in, \in \vee q)$-fuzzy ideal of X.

Proposition 6. A is an $M-(\in, \in \vee q)$-fuzzy ideal of X if and only if A_t is an $M-$ideal of X, where A_t is non-empty set, define A_t by

$$A_t = \{x \mid x \in X, A(x) \geq t\}, \forall t \in [0, 0.5].$$

Proof. Suppose A is an $M-(\in, \in \vee q)$-fuzzy ideal of X, A_t is non-empty set, $t \in [0, 0.5]$, then we have $A(0) \geq A(x) \wedge 0.5 \geq t$, then we have $0 \in A_t$. If $x*y \in A_t, y \in A_t$, then $A(x*y) \geq t, A(y) \geq t$, thus $A(x) \geq A(x*y) \wedge A(y) \wedge 0.5 \geq t$, then we have $x \in A_t$. For all $x \in X$, $m \in M$, if A is an $M-(\in, \in \vee q)$-fuzzy ideal of X, hence $A(mx) \geq A(x) \wedge 0.5 \geq t$, thus $mx \in A_t$, therefore A_t is an $M-$ideal of X. Conversely, suppose A_t is an $M-$ideal of X, then we have $0\in A_t, A(0)\ge t$. Let $A(x)=t$, thus $x\in A_t$, we have $A(0)\ge t=A(x)$, suppose there is no $A(x)\ge A(x*y)\wedge A(y)\wedge 0.5$, then there exist $x_0, y_0\in X$, we have $A(x_0)<A(x_0*y_0)\wedge A(y_0)\wedge 0.5$, let $t_0=A(x_0*y_0)\wedge A(y_0)\wedge 0.5$, then $A(x_0)<t_0=A(x_0*y_0)\wedge A(y_0)\wedge 0.5$, if $x_0*y_0\in A_{t_0}, y_0\in A_{t_0}$, then we have $x_0\in A_{t_0}$, then $A(x_0)\ge t_0$, which is inconsistent with $A(x_0)<t_0=A(x_0*y_0)\wedge A(y_0)\wedge 0.5$, then we have $A(x)\ge A(x*y)\wedge A(y)\wedge 0.5$. If A_t is an $M-$ideal of X, then we have $A(mx)\ge t=t\wedge 0.5=A(x)\wedge 0.5, \forall x\in X, m\in M$, therefore A is an $M-(\in,\in\vee q)$-fuzzy ideal of X.

Proposition 7. Suppose X, Y are $M-$BCI-algebras, f is a mapping from X to Y, A is an $M-(\in,\in\vee q)$-fuzzy ideal of Y, then $f^{-1}(A)$ is an $M-(\in,\in\vee q)$- fuzzy ideal of X.

Proof. Let $y\in Y$, suppose f is an epimorphism, then we have $y=f(x), \exists x\in X$. If A is an $M-(\in,\in\vee q)$-fuzzy ideal of Y, then we have

$$\begin{aligned}&A(0)\ge A(x)\wedge 0.5,\\&A(x)\ge A(x*y)\wedge A(y)\wedge 0.5,\\&A(mx)\ge A(x)\wedge 0.5.\end{aligned}$$

For all $x, y\in X, m\in M$, we have

(1) $f^{-1}(A)(0)=A(f(0))=A(0)\ge A(f(x))\wedge 0.5=f^{-1}(A)(x)\wedge 0.5;$

(2) $\begin{aligned}&f^{-1}(A)(x)=A(f(x))\ge A(f(x)*f(y))\wedge A(f(y))\wedge 0.5\\&=A(f(x*y))\wedge A(f(y))\wedge 0.5=f^{-1}(A)(x*y)\wedge f^{-1}(A)(y)\wedge 0.5;\end{aligned}$

(3) $\begin{aligned}&f^{-1}(A)(mx)=A(f(mx))=A(mf(x))\\&\ge A(f(x))\wedge 0.5=f^{-1}(A)(x)\wedge 0.5.\end{aligned}$

Therefore $f^{-1}(A)$ is an $M-(\in,\in\vee q)$-fuzzy ideal of X.

V. $(\in,\in\vee q)$ - Fuzzy Quotient BCI-Algebras with Operators

Definition 14. Let A be an $M-(\in,\in\vee q)$-fuzzy ideal of X, for all $a\in X$, fuzzy set A_a on X defined as: $A_a: X\to[0,1]$

$$A_a(x)=A(a*x)\wedge A(x*a)\wedge 0.5, \forall x\in X.$$

Denote $X/A=\{A_a: a\in X\}$.

Proposition 8. Let $A_a, A_b\in X/A$, then $A_a=A_b$ if and only if $A(a*b)\wedge A(b*a)\wedge 0.5=A(0)\wedge 0.5.$

Proof. Let $A_a=A_b$, then we have $A_a(b)=A_b(b)$, thus

$$A(a*b)\wedge A(b*a)\wedge 0.5=A(b*b)\wedge A(b*b)\wedge 0.5=A(0)\wedge 0.5,$$

that is $A(a*b)\wedge A(b*a)\wedge 0.5=A(0)\wedge 0.5.$ Conversely, suppose that $A(a*b)\wedge A(b*a)\wedge 0.5=A(0)\wedge 0.5.$ For all $x\in X$, since

$$(a*x)*(b*x)\le a*b, (x*a)*(x*b)\le b*a.$$

It follows from Proposition 1 that

$$\begin{aligned}&A(a*x)\ge A(b*x)\wedge A(a*b)\wedge 0.5,\\&A(x*a)\ge A(x*b)\wedge A(b*a)\wedge 0.5.\end{aligned}$$

Hence

$$\begin{aligned}A_a(x)&=A(a*x)\wedge A(x*a)\wedge 0.5\\&\ge A(b*x)\wedge A(x*b)\wedge A(a*b)\wedge A(b*a)\wedge 0.5\\&=A(b*x)\wedge A(x*b)\wedge A(0)\wedge 0.5\\&=A(b*x)\wedge A(x*b)\wedge 0.5=A_b(x),\end{aligned}$$

that is $A_a\ge A_b$. Similarly, for all $x\in X$, since

$$(b*x)*A(a*x)\le b*a, (x*b)*A(x*a)\le a*b.$$

It follows from Proposition 1 that

$$\begin{aligned}&A(b*x)\ge A(a*x)\wedge A(b*a)\wedge 0.5,\\&A(x*b)\ge A(x*a)\wedge A(a*b)\wedge 0.5.\end{aligned}$$

Hence

$$\begin{aligned}A_b(x)&=A(b*x)\wedge A(x*b)\wedge 0.5\\&\ge A(a*x)\wedge A(x*a)\wedge A(b*a)\wedge A(a*b)\wedge 0.5\\&=A(a*x)\wedge A(x*a)\wedge A(0)\wedge 0.5\\&=A(a*x)\wedge A(x*a)\wedge 0.5=A_a(x),\end{aligned}$$

that is $A_b\ge A_a$. Therefore, $A_a=A_b$. We complete the proof.

Proposition 9. Let $A_a=A_{a'}, A_b=A_{b'}$, then $A_{a*b}=A_{a'*b'}$.

Proof. Since

$$\begin{aligned}&((a*b)*(a'*b'))*(a*a')=((a*b)*(a*a'))*(a'*b')\\&\le(a'*b)*(a'*b')\le b'*b,\\&((a'*b')*(a*b))*(b*b')=((a'*b')*(b*b'))*(a*b)\\&\le(a'*b)*(a*b)\le a'*a.\end{aligned}$$

Hence

$$\begin{aligned}&A((a*b)*(a'*b'))\ge A(a*a')\wedge A(b'*b)\wedge 0.5,\\&A((a'*b')*(a*b))\ge A(b*b')\wedge A(a'*a)\wedge 0.5.\end{aligned}$$

Therefore

$$\begin{aligned}&A((a*b)*(a'*b'))\wedge A((a'*b')*(a*b))\wedge 0.5\\&=A(a*a')\wedge A(a'*a)\wedge 0.5\wedge A(b*b')\wedge A(b'*b)\wedge 0.5\wedge 0.5\\&=A(0)\wedge 0.5,\end{aligned}$$

it follows from Proposition 8. that $A_{a*b}=A_{a'*b'}$. We completed the proof.

Let A be an $M-(\in,\in\vee q)$-fuzzy ideal of X. The operation "*" of R/A is defined as: $\forall A_a, A_b \in R/A, A_a * A_b = A_{a*b}$. By Proposition 8, the above operation is reasonable.

Proposition 10. A is an $M-(\in,\in\vee q)$-fuzzy ideal of X, then $R/A=\{R/A;*,A_0\}$ is an $M-$BCI-algebra.

Proof. For all $A_x, A_y, A_z \in R/A$, we have

$$\left(\left(A_x * A_y\right)*\left(A_x * A_z\right)\right)*\left(A_z * A_y\right) = A_{((x*y)*(x*z))*(z*y)} = A_0;$$
$$\left(A_x *\left(A_x * A_y\right)\right)* A_y = A_{(x*(x*y))*y} = A_0;$$
$$A_x * A_x = A_{x*x} = A_0;$$

if $A_x * A_y = A_0, A_y * A_x = A_0$, then $A_{x*y} = A_0, A_{y*x} = A_0$, it follows from Proposition 8 that $A(x*y)=A(0), A(y*x)=A(0)$, hence $A(x*y)\wedge A(y*x)\wedge 0.5 = A(0)\wedge 0.5$, then we have $A_x = A_y$. Therefore $R/A=\{R/A;*,A_0\}$ is a BCI-algebra. For all $A_x \in R/A, m\in M$, we define $mA_x = A_{mx}$. Firstly, we verify that $mA_x = A_{mx}$ is reasonable. If $A_x = A_y$, then we verify $mA_x = mA_y$, that is to verify $A_{mx} = A_{my}$. We have

$$A(mx*my)\wedge 0.5 = A\left(m\left(x*y\right)\right)\wedge 0.5 \geq A\left(x*y\right)\wedge 0.5,$$
$$A(my*mx)\wedge 0.5 = A\left(m\left(y*x\right)\right)\wedge 0.5 \geq A\left(y*x\right)\wedge 0.5,$$

so we have

$$A(mx*my)\wedge A(my*mx)\wedge 0.5 \geq A(x*y)\wedge A(y*x)\wedge 0.5 = A(0)\wedge 0.5,$$

then $A(mx*my)\wedge A(my*mx)\wedge 0.5 = A(0)\wedge 0.5$, that is $A_{mx} = A_{my}$. In addition, for all $m\in M, A_x, A_y \in R/A$, we have

$$m\left(A_x * A_y\right) = mA_{x*y} = A_{m(x*y)}$$
$$= A_{(mx)*(my)} = A_{mx} * A_{my} = mA_x * mA_y.$$

Therefore $R/A=\{R/A;*,A_0\}$ is an $M-$BCI-algebra.

Definition 15. Let μ be an $M-(\in,\in\vee q)$-fuzzy subalgebra of X, and A be an $M-(\in,\in\vee q)$-fuzzy ideal of X, we define a fuzzy set of X/A as follows:

$$\mu/A: X/A \to [0,1],\quad \mu/A\left(A_t\right) = \sup_{A_x = A_t} \mu\left(x\right)\wedge 0.5, \forall A_t \in X/A.$$

Proposition 11. μ/A is an $M-(\in,\in\vee q)$-fuzzy subalgebea of X/A.

Proof. For all $A_x, A_y \in X/A$, we have

$$\mu/A\left(A_x * A_y\right) = \mu/A\left(A_{x*y}\right) = \sup_{A_z = A_{x*y}} \mu\left(z\right)\wedge 0.5$$
$$\geq \sup_{A_s = A_x, A_t = A_y} \mu\left(s*t\right)\wedge 0.5 \geq \sup_{A_s = A_x, A_t = A_y} \mu\left(s\right)\wedge\mu\left(t\right)\wedge 0.5$$
$$= \sup_{A_s = A_x,} \mu\left(s\right)\wedge \sup_{A_t = A_y,} \mu\left(t\right)\wedge 0.5$$
$$= \mu/A\left(A_x\right)\wedge \mu/A\left(A_y\right)\wedge 0.5.$$

For all $m\in M, A_x \in R/A$, we have

$$\mu/A\left(A_{mx}\right) = \sup_{A_{mz} = A_{mx}} \mu\left(mz\right)\wedge 0.5$$
$$\geq \sup_{A_z = A_x} \mu\left(z\right)\wedge 0.5 = \mu/A\left(A_x\right)\wedge 0.5.$$

Therefore μ/A is an $M-\left(\in,\in\vee q\right)$-fuzzy subalgebra of X/A.

VI. Direct Products of $(\in,\in\vee q)$ - Fuzzy Ideals of BCI-algebras with Operators

Proposition 12. Suppose A and B are $M-(\in,\in\vee q)$-fuzzy ideals of X, then $A\times B$ is an $M-(\in,\in\vee q)$-fuzzy ideal of $X\times X$.

Proof. (1) Let $(x,y)\in X\times X$, then

$$A\times B\left(0,0\right) = A\left(0\right)\wedge B\left(0\right) \geq A\left(x\right)\wedge 0.5\wedge B\left(y\right)\wedge 0.5$$
$$= A\left(x\right)\wedge B\left(y\right)\wedge 0.5 = A\times B\left(x,y\right)\wedge 0.5,$$

then $A\times B(0,0)\geq A\times B(x,y)\wedge 0.5, \forall (x,y)\in X\times X$;

(2) For all $(x_1,x_2),(y_1,y_2)\in X\times X$, we have

$$A\times B\left(\left(x_1,x_2\right)*\left(y_1,y_2\right)\right)\wedge A\times B\left(y_1,y_2\right)\wedge 0.5$$
$$= A\times B\left(x_1 * y_1, x_2 * y_2\right)\wedge A\times B\left(y_1,y_2\right)\wedge 0.5$$
$$= \left(A\left(x_1 * y_1\right)\wedge B\left(x_2 * y_2\right)\right)\wedge A\left(y_1\right)\wedge B\left(y_2\right)\wedge 0.5$$
$$= \left(A\left(x_1 * y_1\right)\wedge A\left(y_1\right)\right)\wedge\left(B\left(x_2 * y_2\right)\wedge B\left(y_2\right)\right)\wedge 0.5$$
$$\leq A\left(x_1\right)\wedge B\left(x_2\right) = A\times B\left(x_1,x_2\right),$$

then for all $(x_1,x_2),(y_1,y_2)\in X\times X$, we have

$$A\times B\left(x_1,x_2\right)\geq A\times B\left(\left(x_1,x_2\right)*\left(y_1,y_2\right)\right)\wedge A\times B\left(y_1,y_2\right)\wedge 0.5;$$

(3) For all $\left(x,y\right)\in X\times X$, we have

$$A\times B\left(m\left(x,y\right)\right) = A\times B\left(mx,my\right) = A\left(mx\right)\wedge B\left(my\right)$$
$$\geq A\left(x\right)\wedge 0.5\wedge B\left(y\right)\wedge 0.5 = A\left(x\right)\wedge B\left(y\right)\wedge 0.5$$
$$= A\times B\left(x,y\right)\wedge 0.5,$$

then we have

$$A\times B\big(m(x,y)\big)\geq A\times B(x,y)\wedge 0.5,\forall(x,y)\in X\times X.$$

Therefore $A\times B$ is an $M-(\in,\in\vee q)$-fuzzy ideal of $X\times X$.

Proposition 13. Suppose A and B are fuzzy sets of X, if $A\times B$ is an $M-(\in,\in\vee q)$-fuzzy ideal of $X\times X$, then A or B is an $M-(\in,\in\vee q)$-fuzzy ideal of X.

Proof. Suppose A and B are $M-(\in,\in\vee q)$-fuzzy ideals of X, then for all $(x_1,x_2),(y_1,y_2)\in X\times X$, we have

$$\begin{aligned}A\times B(x_1,x_2)&\geq A\times B\big((x_1,x_2)*(y_1,y_2)\big)\wedge A\times B(y_1,y_2)\wedge 0.5\\&=A\times B\big((x_1*y_1),(x_2*y_2)\big)\wedge A\times B(y_1,y_2)\wedge 0.5,\end{aligned}$$

if $x_1=y_1=0,$ then

$$A\times B(0,x_2)\geq A\times B(0,x_2*y_2)\wedge A\times B(0,y_2)\wedge 0.5,$$

then we have

$$A\times B(0,x)=A(0)\wedge B(x)=B(x),$$

thus $B(x_2)\geq B(x_2*y_2)\wedge B(y_2)\wedge 0.5.$ If $A\times B$ is an $M-(\in,\in\vee q)$-fuzzy ideal of X, then

$$A\times B\big(m(x,y)\big)\geq A\times B(x,y)\wedge 0.5,\forall(x,y)\in X\times X,$$

let $x=0$, then

$$\begin{aligned}&A\times B\big(m(x,y)\big)=A\times B(mx,my)=A(mx)\wedge B(my)=B(my)\\&\geq A(x)\wedge B(y)\wedge 0.5=A(0)\wedge B(y)\wedge 0.5\\&=B(y)\wedge 0.5,\end{aligned}$$

then we have

$$B(my)\geq B(y)\wedge 0.5,\forall y\in X,m\in M.$$

Therefore B is an $M-(\in,\in\vee q)$- fuzzy ideal of X.

Proposition 14. If B is a fuzzy set, A is a strong fuzzy relation A_B of B, then B is an $M-(\in,\in\vee q)$-fuzzy ideal of X if and only if A_B is an $M-(\in,\in\vee q)$- fuzzy ideal of $X\times X$.

Proof. If B is an $M-(\in,\in\vee q)$-fuzzy ideals of X, then for all $(x,y)\in X\times X,$ we have

$$\begin{aligned}A_B(0,0)&=B(0)\wedge B(0)\geq B(x)\wedge 0.5\wedge B(y)\wedge 0.5\\&=A_B(x,y)\wedge 0.5;\end{aligned}$$

for all $(x_1,x_2),(y_1,y_2)\in X\times X$, we have

$$\begin{aligned}A_B(x_1,x_2)&=B(x_1)\wedge B(x_2)\\&\geq\big(B(x_1*y_1)\wedge B(y_1)\wedge 0.5\big)\wedge\big(B(x_2*y_2)\wedge B(y_2)\wedge 0.5\big)\\&=\big(B(x_1*y_1)\wedge B(x_2*y_2)\big)\wedge\big(B(y_1)\wedge B(y_2)\big)\wedge 0.5\\&=A_B(x_1*y_1,x_2*y_2)\wedge A_B(y_1,y_2)\wedge 0.5\\&=A_B\big((x_1,x_2)*(y_1,y_2)\big)\wedge A_B(y_1,y_2)\wedge 0.5;\end{aligned}$$

for all $(x,y)\in X\times X,$ we have

$$\begin{aligned}&A_B\big(m(x,y)\big)=A_B(mx,my)=B(mx)\wedge B(my)\\&\geq B(x)\wedge 0.5\wedge B(y)\wedge 0.5=A_B(x,y)\wedge 0.5.\end{aligned}$$

Therefore A_B is an $M-(\in,\in\vee q)$-fuzzy ideal of $X\times X$. Conversely, suppose A_B is an $M-(\in,\in\vee q)$-fuzzy ideal of $X\times X$, for all $(x_1,x_2)\in X\times X$, we have

$$B(0)\wedge B(0)=A_B(0,0)\geq A_B(x,x)\wedge 0.5=B(x)\wedge B(x)\wedge 0.5,$$

for all $(x_1,x_2),(y_1,y_2)\in X\times X$, we have

$$\begin{aligned}B(x_1)\wedge B(x_2)&=A_B(x_1,x_2)\\&\geq A_B\big((x_1,x_2)*(y_1,y_2)\big)\wedge A_B(y_1,y_2)\wedge 0.5\\&=A_B(x_1*y_1,x_2*y_2)\wedge A_B(y_1,y_2)\wedge 0.5\\&=\big(B(x_1*y_1)\wedge B(x_2*y_2)\big)\wedge\big(B(y_1)\wedge B(y_2)\big)\wedge 0.5\\&=\big(B(x_1*y_1)\wedge B(y_1)\big)\wedge\big(B(x_2*y_2)\wedge B(y_2)\big)\wedge 0.5,\end{aligned}$$

let $x_2=y_2=0,$ then

$$B(x_1)\wedge B(0)\geq\big(B(x_1*y_1)\wedge B(y_1)\big)\wedge B(0)\wedge 0.5,$$

if A_B is an $M-(\in,\in\vee q)$-fuzzy ideal of $X\times X$, then

$$A_B\big(m(x,y)\big)\geq A_B(x,y),\forall x,y\in X\times X,m\in M,$$

We have

$$B(mx)\wedge B(my)=A_B(mx,my)\geq A_B(x,y)\wedge 0.5=B(x)\wedge B(y)\wedge 0.5,$$

if $x=0,$ then

$$B(0)\wedge B(my)=A_B(0,my)\geq A_B(0,y)\wedge 0.5=B(0)\wedge B(y)\wedge 0.5,$$

namely, $B(my)\geq B(y)\wedge 0.5.$ Therefore B is an $M-(\in,\in\vee q)$-fuzzy ideal of X.

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