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# A Study of Hamilton-Jacobi-Bellman Equation Systems Arising in Differential Game Models of Changing Society

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***Abstract*—This paper is concerned with a system of Hamilton-Jacobi-Bellman equations coupled with an autonomous dynamical system. The mathematical system arises in the differential game formulation of political economy models as an infinite-horizon continuous-time differential game with discounted instantaneous payoff rates and continuously and discretely varying state variables. The existence of a weak solution of the PDE system is proven and a computational scheme of approximate solution is developed for a class of such systems. A model of democratization is mathematically analyzed as an illustration of application.

***Keywords*—Differential games, Hamilton-Jacobi-Bellman equations, infinite horizon, political-economy models.

## I. Introduction

In this paper we study a system of semilinear first order partial differential equations in the form

$$\begin{aligned}\lambda w_{ji}(x) &= F(x)\cdot\nabla_x w_{ji}(x) + g_{ji}(x,\phi^*)\\ &\quad+\sum_{\mu=1}^{m} q_{j\mu}(x,\phi^*)\, w_{\mu i}\end{aligned} \tag{1}$$

for $j=1,\ldots,m$, $i=1,\ldots,n$, where

$$x\equiv(x_1(t),\ldots,x_d(t))\in\mathbb{R}^d$$

is subjected to the dynamical system

$$dx_k/dt = F_k(x_1,\ldots,x_d),\quad k=1,\ldots,d, \tag{2}$$

and $\phi^*\equiv\left(\phi^*_{ji}\right)$ is a solution of the maximization problem

$$\begin{aligned}\phi_i^* = \underset{\phi_{ji}\in X_{ji}\left(\phi^*_{j\hat{i}}\right)}{\arg\max}\ &\left\{g_{ji}\left(x,\phi_{ji},\phi^*_{j\hat{i}}\right)\right.\\ &\left.+\textstyle\sum_{\mu=1}^{m} q_{j\mu}\left(x,\phi_{ji},\phi^*_{j\hat{i}}\right)w_{\mu i}\right\}.\end{aligned} \tag{3}$$

Here, for each $j,i$, $\phi_{ji}$ is a (possibly mixed) strategy in a set of strategies $X_{ji}\left(\phi_{j\hat{i}}\right)$ which may depend on other players' strategies $\phi_{j\hat{i}}\equiv\left(\phi_{j1},\ldots,\phi_{ji-1},\phi_{ji+1},\ldots,\phi_{jn}\right)$. The main assumptions are that the autonomous dynamic system (2) has a global attractor $\bar{x}$ in a bounded domain $\Omega\subset\mathbb{R}^d$, and that the maximization problem (3) has a solution $\phi^*\equiv\phi^*(x,w)$ which is piecewise continuously differentiable in $(x,w)$. The goal of this paper is to prove the existence of a solution $(x(t), w^*(t), \phi^*(t))$ for any $x(0)\in\Omega$ under certain general conditions, and to develop a computation scheme for approximating solutions.

This system arises in infinite-horizon differential games with continuously varying state variables and discretely varying modes, where $i\in\{1,\ldots,n\}$ represents the players, $j\in\{1,\ldots,m\}$ represents the modes of the system, $x=(x_1,\ldots,x_d)\in\mathbb{R}^d$ represents the continuously varying state variables, $\phi=(\phi_1,\ldots,\phi_n)$ is the strategy profile of the players, $q_{j\mu}$ is the transition rate between two modes, $g_{ji}$ is the instantaneous payoff rate for player $i$ in mode $j$, $\lambda$ is the discount rate in the infinite-horizon accumulated payoff, and $w_{ji}$ is related to the value function for player $i$ in mode $j$. In the game-theoretic framework there is a set of players, each possessing a set of available strategies and uses them to gain the best benefit. In the meantime there is a number of state variables governed by a dynamical system defined by a system of differential equations, and the system switches among modes according to the rule of game. For example, in modeling the political changes in a society, players are major social groups, continuously-varying quantities are those that characterize the quantitative features of the society, such as the size and wealth of the population, rates of production, and incomes of social groups, and modes specify who is in power and whether the state is peaceful. Changes of these quantities are caused by the players' strategies. In an infinite-horizon game, players are concerned not only with their immediate benefits but also with their accumulated benefits in the entire future. A popular form is the discounted total payoff in the form

$$U_i(t) = \mathbb{E}_i^t\int_t^\infty e^{-\lambda(\tau-t)}\Pi_i(\tau,x(\tau),\sigma(\tau),\phi(\tau))\,d\tau,$$

where $\lambda>0$ is a constant, $\Pi_i$ is the instantaneous rate of change of payoff, and $\mathbb{E}_i^t$ is the expectation operator conditional to the players' available information at time $t$. The expectation operator acts through the probability of the mode. Let $p_j(t)=\Pr(\sigma(t)=\sigma_j)$ be the projected probability of the system in mode $\sigma_j$ at time $t$ for $j=1,\ldots,m$, and let $g_{ji}(t,x,\phi)$ denote $\Pi_i(t,x,\sigma_j,\phi)$ for $i=1,\ldots,n$ and $j=1,\ldots,m$. Then the expectation of the instantaneous payoff rate is the sum

$$\begin{aligned}\mathbb{E}_i^t\Pi_i(t,x,\sigma,\phi) &= \sum_{j=1}^{m} p_j(t)\,g_{ji}(t,x,\phi)\\ &\equiv \langle p(t), g_i(t,x,\phi)\rangle,\end{aligned}$$

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where

$$\begin{aligned} p(t) &= (p_1(t), \ldots, p_m(t)), \\ g_i(t, x, \phi) &= (g_{ji}(t, x, \phi), \ldots, g_{mi}(t, x, \phi)), \end{aligned}$$

and $\langle \cdot, \cdot \rangle$ is the dot product in $\mathbb{R}^m$. In terms of these functions we can write the expected accumulated total discounted payoff for player $i$ as

$$U_i(t, \phi) = \int_t^\infty e^{-\lambda(\tau - t)} \langle p(\tau), g_i(\tau, x(\tau), \phi(\tau)) \rangle \, d\tau. \quad (4)$$

Typically, the continuously-varying dynamic state variable $x(t) = (x_1(t), \ldots, x_d(t))$ is governed by a system of differential equations in the form

$$dx_k/dt = \varphi_k(t, x(t), \sigma(t), \phi(t)), \quad k = 1, \ldots, d. \quad (5)$$

Let

$$f_{jk}(t, x, \phi) = \varphi_k(t, x, \sigma_j, \phi).$$

The equations can be written in the matrix form

$$dx/dt = p(t) F(t, x(t), \phi(t)) \quad (6)$$

where $F(t, x, u) = (f_{jk}(t, x, u))$ is an $m \times d$ matrix. The change of modes are characterized by the varying of probabilities $p_1(t)$, $\ldots$ , $p_j(t)$. Suppose the mode $\sigma(t)$ evolves as a continuous-time stochastic process

$$\begin{aligned} &\Pr(\sigma(t + \Delta t) = \sigma_\nu | \sigma(t) = \sigma_\mu, x(\tau), u(\tau), \tau \le t) \\ &\quad = \delta_{\mu\nu} + q_{\mu\nu} \Delta t + o(\Delta t), \end{aligned} \quad (7)$$

where $\delta_{\mu\nu}$ is the Kronecker delta, and $q_{\mu\nu}$, $\mu, \nu \in \{1, \ldots, m\}$, is the transition rate from state $\sigma_\mu$ to state $\sigma_\nu$. These quantities satisfy $q_{\mu\nu} \ge 0$ for $\mu \ne \nu$ and $\sum_{\nu=1}^m q_{\mu\nu} = 0$ for each $\mu$. In general $q_{\mu\nu}$ depends on $t$, $x$ and $u$. Thus $p(t)$ evolves according to the differential equation

$$dp/dt = p(t) Q(t, x(t), u(t)) \quad (8)$$

where $Q = (q_{\mu\nu})_{m \times m}$ is the transition matrix.

At any moment, $t$, each player chooses its strategy $\{\phi_i(\tau), \tau \ge t\}$ to maximize its accumulated payoff $U_i(t; \phi)$ defined by (4). The result is an equilibrium at which the inequality

$$U_i(t; \phi^*) \ge U_i\left(t; \left(\phi_{ji}, \phi^*_{j\hat{i}}\right)\right) \quad (9)$$

hold for all $i = 1, \ldots, n$, $\phi_{ji} \in X_{ji}\left(t, \phi^*_{j\hat{i}}\right)$ and $j = 1, \ldots, m$, where $X_{ji}\left(t, \phi^*_{j\hat{i}}\right)$ is the set of available strategies of player $i$ given that other players follow the strategy

$$\phi^*_{j\hat{i}} = \left(\phi^*_{j1}, \ldots, \phi^*_{ji-1}, \phi^*_{ji+1}, \ldots, \phi^*_{jn}\right).$$

Depending on the rule of game the equilibrium $\phi^*$ may be a Nash equilibrium, Stackelberg equilibrium, or of other types. The key to solve a differential game model is to determine the maximizing strategies $\phi^*_{ji}$. Once the strategies are chosen by all the players, the variables $x(\tau)$ and $p(\tau)$ are completely determined for $\tau \ge t$ by (6), (8) and their initial values $x(t)$, $p(t)$, respectively.

Equations (1) and (3) are derived from the Hamilton-Jacobi-Bellman equations formulation of the differential game. Define the value function $V_i(t, x, y)$ by

$$V_i(t, x, y) = \int_t^\infty e^{-\lambda\tau} \langle p(\tau), g_i(\tau, x(\tau), \phi^*(\tau)) \rangle \, d\tau$$

where $x(\tau)$, $p(\tau)$ are solutions of (6) and (8) with the initial values $x(t) = x$, $p(t) = y$, and $\phi^* = \left(\phi^*_{ji}\right)_{m \times d}$ is the maximizing strategy profile that satisfies (9). The Hamilton-Jacobi-Bellman equations for $V_i(t, x, y)$ is

$$\begin{aligned} &-\partial_t V_i(t, x, y) \\ &= \max_{\substack{\phi_{ji} \in X_{ji}\left(\phi^*_{j\hat{i}}\right), \\ j=1,\ldots,m}} \sum_{j=1}^m y_j \left\{ e^{-\lambda t} g_{ji}\left(t, x, \phi_{ji}, \phi^*_{j\hat{i}}\right) \right. \\ &\left. + F_j\left(t, x, \phi_{ji}, \phi^*_{j\hat{i}}\right) \partial_x V_i + Q_j\left(t, x, \phi_{ji}, \phi^*_{j\hat{i}}\right) \partial_y V_i \right\} \end{aligned} \quad (10)$$

where

$$\begin{aligned} \partial_x V_i &= (\partial_{x_1} V_i, \ldots, \partial_{x_d} V_i)^T, \\ \partial_y V_i &= (\partial_{y_1} V_i, \ldots, \partial_{y_m} V_i)^T \end{aligned}$$

are both column vectors, $F_j$ and $Q_j$ are the $j$th rows of matrices $F$ and $Q$, respectively. If the value function $V_i(t, x, y)$ can be solved together with the strategy profile $\phi^*$, the dynamical system (6)–(8) can be solved for any initial values. However, in general, the partial differential equation (10) is difficult to solve because it is highly nonlinear. The system (1)-(3) is a special case where the functions $\varphi_k$, $k = 1, \ldots, d$ in (5) are independent of the mode $\sigma$. In this case the variable $y$ can be eliminated from (10) since $x(\tau)$ is independent of $p(\tau)$ and the matrix $F(t, x, \phi)$ has identical rows. Furthermore, since System (8) is linear, its solution $p(\tau)$ is linear with respect to its initial value, $y$. Thus the value function $V_i(t, x, y)$ is also linear in $y$ and can be written in the form $V_i(t, x, y) = \langle y, W_i(t, x) \rangle$ where $W_i = \partial_y V_i$. Note that the components of $y$ are nonnegative since they are the probability distributions, (10) can be written in the form

$$\begin{aligned} &-\partial_t W_{ji}(t, x) = \max_{\phi_{ji} \in X_{ji}\left(\phi^*_{j\hat{i}}\right)} \left\{ e^{-\lambda t} g_{ji}\left(t, x, \phi_{ji}, \phi^*_{j\hat{i}}\right) \right. \\ &+ F\left(t, x, \phi_{ji}, \phi^*_{j\hat{i}}\right) \partial_x W_{ji}(t, x) \\ &\left. + Q_j\left(t, x, \phi_{ji}, \phi^*_{j\hat{i}}\right) W_i(t, x) \right\} \end{aligned} \quad (11)$$

for $j = 1, \ldots, m$, $i = 1, \ldots, n$. We next observe that in the autonomous case where functions $g_{ji}$, $F$, and $Q$ are independent of $t$, $U_i(t; \phi^*)$ defined in (4) is also independent of $t$. This means $e^{\lambda t} W_i(t, x)$ is independent of $t$. We denote it as $w_i(x)$. If, in addition, $F$ is independent of $\phi$, then by (11), the components of $w_i(x)$ satisfy the equations

$$\begin{aligned} &\lambda w_{ji}(x) = F(x) \cdot \partial_x w_{ji}(x) \\ &+ \max_{\phi_{ji} \in X_{ji}\left(\phi^*_{j\hat{i}}\right)} \left\{ g_{ji}\left(x, \phi_{ji}, \phi^*_{j\hat{i}}\right) + Q_j\left(x, \phi_{ji}, \phi^*_{j\hat{i}}\right) w_i(x) \right\}, \end{aligned} \quad (12)$$

for $j = 1, \ldots, m$, $i = 1, \ldots, n$, where "$\cdot$" is the dot product in $\mathbb{R}^d$. This is the same as (1) and (3), while (5) with $\varphi_k$ independent of $\sigma$ and $\phi$ is the same as (2).

Problem (1)–(3) is semilinear and therefore is easier to analyze then the general problem (6), (8) and (10). However, there is no results on the existence of solution and computation of the solution in the current literature. The Hamilton-Jacobi-Bellman equations for two-player zero sum games have been widely used in early works on differential games [6], [8], [9], [20]-[22]. However, the general $n$ equation system cannot be treated using the ordinary method of characteristics. There are two obstacles. One is that the boundary condition is not given on a non-characteristic manifold, but at an equilibrium point $\bar{x}$ which is a zero of $F$. This is because the values of $w_{ji}$ are unknown for $x \in \Omega$, and in the general case its value can be determined by solving (12) only at the equilibrium point $\bar{x}$ together with the maximization problem

$$\underset{\phi_i \in X_i\left(\phi^*_{-i}\right)}{\text{maximize}}\ g_{ji}\left(x, \phi_i, \phi^*_{-i}\right) + Q_j\left(x, \phi_i, \phi^*_{-i}\right) w_i\left(x\right), \quad (13)$$

for $i = 1, \ldots, n$. Another obstacle is that the solution $(x, w_i) \mapsto \phi^*$ of (13) is generally discontinuous. Thus the right-hand side of (12) is generally discontinuous on $(x, w_i)$. To overcome these difficulties, we use the Stable Manifold Theorem to obtain a unique solution in a small neighborhood of the equilibrium $(\bar{x}, \bar{w})$, and then use a fixed point approach to obtain the existence of the weak solution. This approach also leads to an approximation scheme for the solution.

The paper is organized as follows. In Section II, we prove the existence of a weak solution and develop a computation scheme for constructing approximate solutions for problem (1)–(3) under Hypothesis (H) below. In Section III, we use the results to analyze a democratization model proposed in [12] as an illustration of application. A numerical example is also given at the end of Section III to explain the general computation scheme. In Appendix, we give a proof of a technical lemma used in Section III.

## II. Existence and Computation of Solution

In this section, we prove the existence of solution to (1)-(3) and develop a computation scheme of the solution under the following conditions.

*Hypothesis (H):*

1) Functions $F_k$ for $k = 1, \ldots, d$ and $g_{ji}$, $q_{j\mu}$ for $i = 1, \ldots, n$ and $j, \mu = 1, \ldots, m$ are $C^1$ functions of their arguments.
2) $\Omega \subset \mathbb{R}^d$ is a bounded domain, with a $C^1$ boundary $\partial\Omega$. Also there is $\bar{x} \in \Omega$ such that it is the only solution to the equation $F(x) = \mathbf{0}$ and all the eigenvalues of the Jacobi matrix $D_x F(\bar{x})$ are negative or complex with negative real part. Furthermore $\bar{x}$ is the global attractor of the differential equation (2) in $\Omega$.
3) The solution $\phi^*$ of problem (3) exists and is piecewise constant in $D = \Omega \times \mathbb{R}^m$. Specifically, for each $i = 1, \ldots, n$ there are subdomains $\{D_{i1}, \ldots, D_{iN_i}\}$ such that each $D_{il}$ is open and connected, $D_{il} \cap D_{il'} = \varnothing$ whenever $l \neq l'$, $\overline{D} = \overline{\bigcup_{l=1}^{N} D_{il}}$, where the upper bar indicates the closure, and $\phi^*_i$ is constant in each $D_{il}$ for $l = 1, \ldots, N_i$.
4) Problem
$$\lambda \bar{w}_{ji} = g_{ji}\left(\bar{x}, \phi^*\left(\bar{x}, \bar{w}_i\right)\right) + \sum_{\mu=1}^{m} q_{j\mu}\left(\bar{x}, \phi^*\left(\bar{x}, \bar{w}_i\right)\right) \bar{w}_{\mu i}, \quad (14)$$
for $j = 1, \ldots, m$, $i = 1, \ldots, n$, has a solution $(\bar{w}_{ji})$ such that $(\bar{x}, \bar{w}_i) \in D_{il}$ for some $D_{il}$.

Since the system involves discontinuous functions, we solve it in the weak sense. A weak solution $\{w, \phi^*\}$ of (1)-(3) is defined by

*Definition 1:* We say $\{w, \phi\}$ is a weak solution of (1)–(3) if for each $i = 1, \ldots, n$, $w_i \in L^2(\Omega; \mathbb{R}^m)$, $F(x)\partial_x w_i \in L^2(\Omega; \mathbb{R}^m)$, and the equation

$$\int_\Omega \langle v, g_i\left(x, \phi^*\left(x, w_i\right)\right) + F\left(x\right)\partial_x w_i - \lambda w_i + Q\left(x, \phi^*\left(x, w_i\right)\right) w_i \rangle\, dx = 0$$

holds true for any $v \in L^2(\Omega; \mathbb{R}^m)$, and $w_i(\bar{x}) = \bar{w}_i$.

### A. Existence of Solution

The following theorem ensures the existence of a weak solution.

*Theorem 1:* Suppose (H) holds. Then Problem (1)–(3) has a solution $\{w^*_i(x), \phi^*(x)\}$ for any $x \in \Omega$, $i = 1, \ldots, n$.

*Proof:* We fix an $i \in \{1, \ldots, n\}$. Since $\phi^*(x, z)$ is piecewise constant in $\Omega \times \mathbb{R}^m$, for any $\varepsilon > 0$, we can construct a smooth approximation $\phi^*_\varepsilon$ of $\phi^*$ such that $\phi^*_\varepsilon$ is a $C^1$ function in $\Omega \times \mathbb{R}^m$ for each $\varepsilon > 0$, $\phi^*_\varepsilon(x, z) = \phi^*(x, z)$ in $D^\varepsilon_{il}$ for any $D_{il}$, where

$$D^\varepsilon_{il} = \left\{(x, w_i) \in D_{il},\ \text{dist}\left(\left(x, w_i\right), \partial D_{il}\right) > \varepsilon\right\},$$

and $\phi^*_\varepsilon$ is uniformly bounded. Let $g^*_{ji,\varepsilon}$ and $q^*_{ji,\varepsilon}$ denote the functions

$$g^*_{ji,\varepsilon}(x, z) = g_{ji}\left(x, \phi^*_\varepsilon(x, z)\right),\ q^*_{ji,\varepsilon}(x, z) = q_{ji}\left(x, \phi^*_\varepsilon(x, z)\right)$$

for $(x, z) \in \Omega \times \mathbb{R}^m$. Then the equation

$$\lambda z_{ji} = g^*_{ji,\varepsilon}\left(\bar{x}, z_i\right) + \sum_{\mu=1}^{m} q^*_{j\mu,\varepsilon}\left(\bar{x}, z_i\right) z_{\mu i}$$

has the same solution $\bar{w}_{ji}$ if $\varepsilon$ is sufficiently small such that $(\bar{x}, \bar{w}_i) \in D^\varepsilon_{il}$ for each $i$. Let $x_0 \in \overline{\Omega}$ and let $x(s)$ be the solution of (2) with the initial value $x(0) = x_0 \in \Omega$. The characteristic equations for (1) with $g_{ji}(x, \phi^*(x, w_i))$ and $q_{j\mu}(x, \phi^*(x, w_i))$ replaced by $g^*_{ji,\varepsilon}(x, w_i)$ and $q^*_{ji,\varepsilon}(x, w_i)$, respectively, are

$$\begin{aligned} & dx/ds = F(x), \quad x(0) = x_0; \\ & dz_{ji}/ds = \lambda z_{ji} - g^*_{ji,\varepsilon}\left(x(s), z_i\right) \\ & \qquad - \textstyle\sum_{\mu=1}^{m} q^*_{j\mu,\varepsilon}\left(x(s), z_i\right) z_{\mu i}, \\ & \lim_{s\to\infty} z_{ji}(s) = \bar{w}_{ji}. \end{aligned} \quad (15)$$

We show that this problem has a unique solution.

We first observe that the equation for $x$ is independent of $z_{ji}$. Since $F \in C^1(\Omega)$, there is a unique solution for any $x_0 \in \Omega$. Also, since $\bar{x}$ is the global attractor of (2) in $\overline{\Omega}$, it follows that $\lim_{s\to\infty} x(s) = \bar{x}$. We show that (15) has a solution in a neighborhood of $(\bar{x}, \bar{w})$. Note that system (15) is autonomous and has $(\bar{x}, \bar{w}_i)$ as an isolated equilibrium. Since $\phi^*_\varepsilon$ is constant in $D^\varepsilon_{il}$, the Jacobian matrix $J_{i,\varepsilon}(\bar{x}, \bar{w}_i) = (A_{kl})$ of the functions on the right-hand sides of the differential equations in (15) at the equilibrium $(\bar{x}, \bar{w}_i)$ has the entries

$$\begin{aligned} A_{11} &= D_x F(\bar{x}), \quad A_{12} = 0, \\ A_{21} &= -D_x g^*_{i,\varepsilon}\left(\bar{x}, \bar{w}_i\right) - D_x Q^*_\varepsilon\left(\bar{x}, \bar{w}_i\right) \bar{w}_i, \\ A_{22} &= \lambda I - Q^*_\varepsilon\left(\bar{x}, \bar{w}_i\right), \end{aligned}$$

where $g^*_{i,\varepsilon}(x,w_i) = \left(g^*_{ji,\varepsilon}(x,w_i)\right)_{j=1}^m$, $Q^*_\varepsilon(x,w_i) = \left(q^*_{j\mu,\varepsilon}(x,w_i)\right)_{m\times m}$, and $I$ is the $m\times m$ identity matrix. Clearly the eigenvalues of $J_{i,\varepsilon}(\bar{x},\bar{w}_i)$ are the eigenvalues of $D_xF(\bar{x})$ and the eigenvalues of $\lambda I - Q^*_\varepsilon(\bar{x},\bar{w}_i)$ combined. Note that the off-diagonal entries of $Q^*_\varepsilon(\bar{x},\bar{w}_i)$ are all nonnegative, and the sum of each row of $Q^*_\varepsilon(\bar{x},\bar{w}_i)$ is zero. Therefore by the Perron–Frobenius Theorem the largest eigenvalue of $Q^*_\varepsilon(\bar{x},\bar{w}_i)$ is zero and all other eigenvalues are either negative or complex with negative real part. So, since $\lambda > 0$, the eigenvalues of $\lambda I - Q^*_\varepsilon(\bar{x},\bar{w}_i)$ are all positive. On the other hand, by assumption the eigenvalues of $D_xF(\bar{x})$ are all negative or complex with negative real part. Therefore, none of the eigenvalues of $J(\bar{x},\bar{x}_i)$ is zero, and by the Stable Manifold Theorem (cf. e.g. [19, Section 2.7]), the stable manifold near $(\bar{x},\bar{w}_i)$ has the dimension $d$. This means that there is a $d$-dimensional stable manifold such that any trajectory on the manifold remains on the manifold and converges to $(\bar{x},\bar{w}_i)$. By the uniqueness of solution, there is exactly one trajectory whose $x$-components reaches $x_0$. Let $\{(x(s), z_{i,\varepsilon}(s)) : s\ge 0\}$ denote this trajectory. Then there is $T>0$ such that $(x(s), z_{i,\varepsilon}(s)) \in D^\varepsilon_{il}$ for $s>T$. In particular, we can find a neighborhood in the form $N_\delta(\bar{x})\times N_\delta(\bar{w}_i)$ where

$$\begin{aligned} N_\delta(\bar{x}) &= \{x\in\Omega: \quad |x-\bar{x}|_{\mathbb{R}^d}<\delta\}, \\ N_\delta(\bar{w}_i) &= \{w_i\in\mathbb{R}^m: \quad |w_i-\bar{w}_i|_{\mathbb{R}^m}<\delta\} \end{aligned}$$

such that $\bar{N}_\delta(\bar{x})\times\bar{N}_\delta(\bar{w}_i)\subset D^\varepsilon_{i,l}$ and choose $T$ so that $x(T)\in\partial N_\delta(\bar{x})$. Since $\phi^*_{i,\varepsilon}=\phi^*_i$ in $D^\varepsilon_{il}$, if $\varepsilon$ is sufficiently small, $z_{i,\varepsilon}(s)$ is independent of $\varepsilon$ for $s\ge T$. We denote the trajectory by $(x(s), z_i(s))$ for $s>T$.

We next extend the solution of (15) for $s<T$. Consider the terminal value problem

$$\begin{aligned} &dz_{ji,\varepsilon}/ds = \lambda z_{ji,\varepsilon} - g^*_{ji,\varepsilon}(x,z_i) \\ &\qquad - \textstyle\sum_{\mu=1}^m q^*_{j\mu,\varepsilon}(x(s), z_{i,\varepsilon}) z_{\mu i,\varepsilon}, \quad s\in[0,T); \\ &z_{ji,\varepsilon}(T) = z_{ji}(T), \quad j=1,\ldots,m, \quad i=1,\ldots,n. \end{aligned} \tag{16}$$

Since the right-hand side of (16) is continuously differentiable in $(x,z)$, the solution exists and is unique on the interval $[0,T]$. We denote the solution by $z_{ji,\varepsilon}(s)$. We show that the functions $z_{ji,\varepsilon}$ is uniformly bounded and equicontinuous on $[0,T]$. Using a change of variable $\tau=T-s$ and $Z_{i,\varepsilon}(\tau) = (z_{ji,\varepsilon}(T-\tau))_{j=1}^m$, Problem (16) is equivalent to the integral equation

$$Z_{i,\varepsilon}(\tau) = z_i(T) + \int_0^\tau H_{i,\varepsilon}(\xi, Z_{i,\varepsilon}(\xi))\,d\xi \quad \text{for } \tau\in[0,T]$$

where

$$\begin{aligned} H_{i,\varepsilon}(\xi, Z_{i,\varepsilon}) &= -\lambda Z_{i,\varepsilon} + g^*_{i,\varepsilon}(x(\xi), Z_{i,\varepsilon}) \\ &\quad + Q^*_\varepsilon(x(\xi), Z_{i,\varepsilon}) Z_{i,\varepsilon}. \end{aligned}$$

Since the functions $g_i(x,\phi)$ and $Q(x,\phi)$ are bounded, there is a constant $M>0$ such that $|H_{i,\varepsilon}(\xi, Z_{i,\varepsilon})| \le M|Z_{i,\varepsilon}|$ for $\xi\in[0,T]$. Let $S\subset C[0,T]$ be the set of continuous function such that $f\in S$ if

$$|f(\tau)| \le M_w e^{M\tau} \quad \text{for } \tau\in[0,T],$$

where $M_w = \max_{i=1,\ldots,n}\{|\bar{w}_i|+\delta\}$, and define mapping $K: S\mapsto C[0,T]$ by

$$(Kf)(\tau) = z_i(T) + \int_0^\tau H_{i,\varepsilon}(\xi, f(\xi))\,d\xi.$$

Then for any $f\in S$ we have

$$\begin{aligned} |(Kf)(\tau)| &\le |z_i(T)| + M\int_0^\tau |f(\xi)|\,d\xi \\ &\le M_w + MM_w\int_0^\tau e^{M\xi}d\xi = M_w e^{M\tau} \end{aligned}$$

for any $\tau\in[0,T]$. This proves that $KS\subset S$. Thus $\{z_{i,\varepsilon}\}$ is uniformly bounded on $[0,T]$.

We next show that $z_{i,\varepsilon}(s)$ is equicontinuous. Let $\tau_1,\tau_2\in[0,T]$. Then

$$\begin{aligned} &|(Kf)(\tau_1) - (Kf)(\tau_2)| = \left|\textstyle\int_{\tau_1}^{\tau_2} H_{i,\varepsilon}(\xi, f(\xi))\,d\xi\right| \\ &\qquad \le \left|\textstyle\int_{\tau_1}^{\tau_2} M|f(\xi)|\,d\xi\right| \le M|w_i(T)|e^{MT}|\tau_1-\tau_2|. \end{aligned}$$

This proves that $KS$ is equi-continuous.

Thus, by the Ascoli-Arzera Theorem, there is a sequence $\varepsilon_k\to 0$ such that $z_{i,\varepsilon_k}$ converges to a continuous function $z_i$ on $[0,T]$. By the construction of $\phi^*_\varepsilon$, it follows that $\phi^*_{\varepsilon_k}(x, z_{i,\varepsilon_k}) \to \phi^*(x,z_i)$ pointwise in each $D_{il}$. Thus $\phi^*_{\varepsilon_k}(x, z_{i,\varepsilon_k}) \to \phi^*(x,z_i)$ a.e. in $\Omega\times S$. Let $w_{i,\varepsilon_k}$ and $w_i$ be the function such that $w_{i,\varepsilon_k}(x(s)) = z_{i,\varepsilon_k}(s)$ and $w_i(x(s)) = z_i(s)$, where $x(s)$ is the solution of the first equation of (15). Since the first equation of (15) has a unique solution for any initial value $x_0\in\Omega$, $w_{i,\varepsilon_k}(x)$ is defined on $\Omega$ and satisfy the partial differential equation

$$\lambda w_{i,\varepsilon_k} = g^*_{i,\varepsilon_k}(x, w_{i,\varepsilon_k}) + F(x)\partial_x w_{i,\varepsilon_k} + Q^*_{\varepsilon_k}(x, w_{i,\varepsilon_k}) w_{i,\varepsilon_k} \tag{17}$$

classically in $\Omega$. So for any $v\in L^2(\Omega;\mathbb{R}^m)$ by the dominated convergence theorem we haves

$$\begin{aligned} &\textstyle\int_\Omega \langle v, g^*_{i,\varepsilon_k}(x, w_{i,\varepsilon_k}) + F(x)\partial_x w_{i,\varepsilon_k} \\ &\qquad -\lambda w_{i,\varepsilon_k} + Q^*_{\varepsilon_k}(x, w_{i,\varepsilon_k}) w_{i,\varepsilon_k}\rangle\,dx = 0. \end{aligned} \tag{18}$$

Since $w_{i,\varepsilon_k}\to w_i$ in $C(\Omega)$ and $g^*_{i,\varepsilon_k}(x, w_{i,\varepsilon_k}(x))$ and $Q^*_{\varepsilon_k}(x, w_{i,\varepsilon_k}(x))$ converge to $g_i(x, \phi^*(x, w_i(x)))$ and $Q(x, \phi^*(x, w_i(x)))$ a.e. in $\Omega$, respectively, it follows that

$$\begin{aligned} &\lim_{k\to\infty}\left[\int_\Omega \langle v, g^*_{i,\varepsilon_k}(x, w_{i,\varepsilon_k}) - \lambda w_{i,\varepsilon_k}\rangle\,dx \right. \\ &\qquad \left. + \int_\Omega \langle v, Q^*_{\varepsilon_k}(x, w_{i,\varepsilon_k}) w_{i,\varepsilon_k}\rangle\,dx\right] \\ &= \int_\Omega \langle v, g^*_i(x,w_i) - \lambda w_i + Q^*(x,w_i)w_i\rangle\,dx \end{aligned}$$

for any $v\in L^2(\Omega;\mathbb{R}^m)$. It remains to show that

$$\lim_{k'\to\infty}\int_\Omega \langle v, F(x)\partial_x w_{i,\varepsilon_{k'}}\rangle\,dx = \int_\Omega \langle v, F(x)\partial_x w_i\rangle\,dx \tag{19}$$

for a subsequence $(w_{i,\varepsilon_{k'}})$ of $(w_{i,\varepsilon_k})$. For this purpose we show that $(w_{i,\varepsilon_k})$ is bounded in $H^1(\Omega\backslash N_\delta(\bar{x}))$. The boundedness of $w_{i,\varepsilon_k}$ in $L^2(\Omega\backslash N_\delta(\bar{x}))$ follows directly from the boundedness of $S$. To see that $\partial_x w_{i,\varepsilon_k}$ is bounded, we use (17), which leads to

$$F(x)\partial_x w_{i,\varepsilon_k} = \lambda w_{i,\varepsilon_k} - g^*_{i,\varepsilon_k}(x, w_{i,\varepsilon_k}) - Q^*_{\varepsilon_k}(x, w_{i,\varepsilon_k}) w_{i,\varepsilon_k}$$

to conclude that $F(x)\,\partial_x w_{i,\varepsilon_k}$ is bounded on $\Omega\backslash N_\delta(\bar{x})$ since $w_{i,\varepsilon_k}$, $g^*_{i,\varepsilon_k}$ and $Q^*_{\varepsilon_k}$ are bounded. Observe that since $F(x) \neq \mathbf{0}$ if $x \neq \bar{x}$ in $\Omega$, there is a constant $\rho > 0$ such that $|F(x)| > \rho$ in $\Omega\backslash N_\delta(\bar{x})$. Hence, the above equation implies that $|\partial_x w_{i,\varepsilon_k}|_{L^\infty}$ is bounded in $\Omega\backslash N_\delta(\bar{x})$. Since $\Omega$ is bounded, it follows that $\partial_x w_{i,\varepsilon}$ is bounded in $L^2(\Omega\backslash N_\delta(\bar{x}))$.

By the weak compactness of bounded sets in $H^1(\Omega\backslash N_\delta(\bar{x}))$, there is a subsequence $(w_{i,\varepsilon_{k'}})$ that is weakly convergent. Thus, since in $N_\delta(\bar{x})$ all $w_{i,\varepsilon_k} = w_i$ for all $k$, we have

$$\begin{aligned}
&\lim_{k'\to\infty}\int_\Omega \left\langle v, F(x)\,\partial_x w_{i,\varepsilon_{k'}}\right\rangle dx \\
=\;& \int_{N_\delta(\bar{x})} \langle v, F(x)\,\partial_x w_i\rangle\,dx \\
&\qquad + \lim_{k'\to\infty}\int_{\Omega\backslash N_\delta(\bar{x})} \left\langle v, F(x)\,\partial_x w_{i,\varepsilon_{k'}}\right\rangle dx \\
=\;& \int_{N_\delta(\bar{x})} \langle v, F(x)\,\partial_x w_i\rangle\,dx \\
&\qquad + \int_{\Omega\backslash N_\delta(\bar{x})} \langle v, F(x)\,\partial_x w_i\rangle\,dx \\
=\;& \int_\Omega \langle v, F(x)\,\partial_x w_i\rangle\,dx.
\end{aligned}$$

This proves (19).

As we have shown that

$$\begin{aligned}
&\lim_{k'\to\infty}\int_\Omega \Big\langle v, g^*_{i,\varepsilon_{k'}}\left(x, w_{i,\varepsilon_{k'}}\right) + F(x)\,\partial_x w_{i,\varepsilon_{k'}} \\
&\qquad -\lambda w_{i,k'} + Q^*_{\varepsilon_{k'}}\left(x, w_{i,\varepsilon_{k'}}\right) w_{i,\varepsilon_{k'}}\Big\rangle\,dx \\
=\;& \int_\Omega \langle v, g^*_i(x, w_i) + F(x)\,\partial_x w_i - \lambda w_i\rangle\,dx \\
& + \int_\Omega \langle v, Q^*(x, w_i)\,w_i\rangle\,dx
\end{aligned}$$

as $k \to \infty$, by (18), $\{w, \phi\}$ is a weak solution of (1)–(3). This completes the proof of the theorem. ∎

### B. Computation of Solution

Based on the idea of the proof of Theorem 1, we propose the following method of computing approximate solution of (1)-(3).

1) For any $i = 1, \ldots, n$, we first find an approximate solution near the equilibrium point $(\bar{x}, \bar{w}_i)$. Since $(\bar{x}, \bar{w}_i)$ is in the interior of a subdomain $D_{il}$ in which $\phi^*$ of (3) is constant, one can construct an approximate solution using the Taylor expansion. To do so, differentiate (1) to obtain equations for the derivatives of $w_{ji}$ at $\bar{x}$,

$$\begin{aligned}
&\lambda\,(w_{ji})_{x_\mu}(\bar{x}) \\
=\;& \sum_{k=1}^{d} (F_k)_{x_\mu}(\bar{x})\,(w_{ji})_{x_k}(\bar{x}) \\
&+ \sum_{k=1}^{d} (g_{ji})_{x_\mu}(\bar{x}, \phi^*(\bar{x})) \\
&+ \sum_{l=1}^{m} (q_{jl})_{x\mu}(\bar{x}, \phi^*(\bar{x}))\,\bar{w}_{li} \\
&+ \sum_{l=1}^{m} q_{jl}(\bar{x}, \phi^*(\bar{x}))\,(w_{li})_{x_\mu}(\bar{x}), \\
&\lambda\,(w_{ji})_{x_\mu x_\nu}(\bar{x}) \\
=\;& \sum_{k=1}^{d} (F_k)_{x_\mu x_\nu}(\bar{x})\,(w_{ji})_{x_k}(\bar{x}) \\
&+ \sum_{k=1}^{d} 2\,(F_k)_{x_\mu}(\bar{x})\,(w_{ji})_{x_k x_\nu}(\bar{x}) \\
&+ \sum_{k=1}^{d} (g_{ji})_{x_\mu x\nu}(\bar{x}, \phi^*(\bar{x})) \\
&+ \sum_{l=1}^{m} (g_{jl})_{x_\mu x_\nu}(\bar{x}, \phi^*(\bar{x}))\,\bar{w}_{li} \\
&+ 2\sum_{l=1}^{m} (g_{jl})_{x_\mu}(\bar{x}, \phi^*(\bar{x}))\,(w_{li})_{x_\nu}(\bar{x}) \\
&+ \sum_{l=1}^{m} g_{jl}(\bar{x}, \phi^*(\bar{x}))\,(w_{li})_{x_\mu x_\nu}(\bar{x}), \ldots
\end{aligned}$$

The derivatives can be solved from each equation because the eigenvalues of $D_x F(\bar{x})$ and $(q_{jl}(\bar{x}, \phi^*))$ are negative or complex with a negative real part. Thus, we can use a Taylor polynomial

$$\begin{aligned}
w_{ji}(x) \;\approx\;& w_{ji}(\bar{x}) + \nabla_x w_{ji}(\bar{x})\cdot(x - \bar{x}) \\
&+ \frac{1}{2}(x - \bar{x})^T D_x^2 w_{ji}(\bar{x})\,(x - \bar{x}) + \cdots
\end{aligned}$$

to approximate $w_{ji}(x)$ for $x$ near $\bar{x}$. The function $\phi(x, w)$ can then be approximated by solving (3).

2) We then use an iterative scheme to construct an approximate solution. The first step is to assumes a function $\phi^{(0)}(x)$ and substitute it for $\phi^*$ in (1) to obtain a numerical solution $w_i^{(1)}(x)$. We then solve (3) to obtain $\phi^{(1)}$ from $w_i^{(1)}$. In general, if $\phi^{(k)}$ has been obtained, we use it to substitute for $\phi^*$ in (1) and solve the equations for $w_i^{(k+1)}$, and then solve (3) to obtain $\phi^{(k+1)}$. In general, since there are finite many possible values of $\phi$, the two sequences $\left\{w_i^{(k)}\right\}$ and $\left\{\phi^{(k)}\right\}$ are likely to converge to a cyclic limits. Pressumably, the smaller the stepsize in the numerical approximation of the solution $w_i^{(k)}$ to the differential equations (1), the smaller the diviation of the solutions $\left\{w_i^{(k)}, \phi^{(k)}\right\}$ in the cyclic limits, and thus the better approximation.

3) After solving the function $\phi^*(x)$, the last step is to solve the dynamical system (6), (8) with $u = \phi^*(x)$ and with any given initial conditions $x(0)$ and $p(0)$. This is an initial-value problem of a system of ordinary differential equations, whose solution is easy to obtain.

This general approach is illustrated in Section III by a two-player democratization model.

## III. A Model of Democratization

In this section we apply the method developed in Section II to a differential game model of democratization as an illustration. Mathematical modeling of changing societies using game theories has been an active research area for decades. Various game theories have been used in the study of politico-economic phenomena. Major societal transformations such as institutional changes in non-democracy and democratization processes have been extensively investigated (cf. [1]-[5], [10]-[12], [14]-[18] and references therein). In particular, there is a large body of literature on the co-evolution of the economic and political development of the society ([2], [7], [10], [12], [13], [18], [23]). On the other hand, many models are formulated as discrete time dynamic games rather than continuous time differential games. Since in many cases there are continuously varying state variables involved in the transition of a society, it is often more convenient to formulate differential game models. We formulate a two-player democratization model below as an example.

### A. Model Description

In [12] a model of democratization in a society is proposed that consists of four social groups, the monarch, landowners, capitalists, and labors. The underlying concept of the modelling is that democratization is considered as a transition process of the political power from being highly concentrated in the hands of a small number of people to being widely shared by the general population. As the history exhibits, this process takes multiple stages. Different social groups enter into the political arena at different times. Typically a social group's quest for political power begins with the group becoming economically powerful, capable of challenging the ruler. Eventually confrontations break out resulting in either the challenging group being adopted into the ruling class peacefully, or the challenging group overtakes the ruling group in a revolt and becomes the new ruler. The model in [12] divides this process into two steps, (1) from monarchy to oligarchy, during which period capitalists, with or without help of landowners, gain political power from the monarch; and (2) from oligarchy to democracy, during which period labors and the general population gain political power. As each stage involves confrontations between only two parties, the model is a game of two players, the ruler and a challenging group. The original model is formulated as an infinite-horizon discrete-time repeated game. To incorporate the continuous growth of the state variable, which is the total physical capital in the state, we reformulate the model as a continuous-time differential game. Specifically, we focus on the first stage, from monarchy to oligarchy, with the monarch and capitalists as players, and use $i = m, c$ to denote the monarch and the capitalists, respectively.

The elements of the model include the payoff rates of the players, the strategies of the players, the continuously-varying state variables, the modes of the society and their transition rates.

The payoff rates of the players are their after-tax incomes. Each social group in the society has certain gross (before-tax) income $I_i$ which depends on the total amount of physical capital, $K$, and the total amount of human capital, $H$, available in the society at the time. Following [12] we assume that $H$ is a constant during the transition period from monarchy to oligarchy. Hence only $K = K(t)$ varies with time. Let $I_m$, $I_c$, $I_l$, and $I_w$ represent the before-tax income of the monarch, the capitalists, the landowners, and the labors, respectively. As shown by Proposition 1 in Appendix, the gross incomes depend on $K$ in the form

$$\begin{aligned} &I_m = C_m (L+K)^{-\alpha}, \quad I_l = C_l (L+K)^{-\alpha}, \\ &I_c = C_c K (L+K)^{-\alpha}, \quad I_w = C_w (L+K)^{1-\alpha}, \end{aligned} \tag{20}$$

where $C_m$, $C_l$, $C_c$, $C_w$ and $L$ are positive constants. The after-tax income of a social group is its before-tax income plus or minus an amount of tax revenue. Tax is collected at a fixed rate $r_T \in (0,1)$ from all individuals who are not in the ruling body. So an individual having the before-tax income $I$ pays tax $r_T I$. There is a tax collecting cost so that the ruler receives $\hat{r}_T I$ from the individual for some constant $\hat{r}_T < r_T$. The tax revenue is shared among members of the ruling group in proportion to their economic power. The after-tax income also depends on the mode of the society. The society can be either in a peaceful mode or in the aftermath of a revolt. The former is a time when there has not been a revolt recently. In this mode a non-ruler only pays the tax, so his after-tax income is $\Pi = (1 - r_T) I$. The latter is a time when the society just endured a revolt and needs to be recovered. During this period individuals on the defeated side pays reparation in proportion to his before-tax income. Thus we assume that there is a constant $\theta \in (0,1)$ such that $\Pi = \theta (1 - r_T) I$. In the case where both players are rulers, the recovery cost is paid by an extra tax collected from non-rulers.

Hence there are six modes of the society, depending on who is the ruler and whether the society is in peaceful mode or in the aftermath of a revolt. Let $\sigma_j$, $j = 1, \ldots, 6$ denote the states

$$\begin{aligned} \sigma_1 &= (m,p), \quad \sigma_2 = (m,a), \quad \sigma_3 = (c,p), \\ \sigma_4 &= (c,a), \quad \sigma_5 = (b,p), \quad \sigma_6 = (b,a) \end{aligned}$$

where the first component $m$, $c$, or $b$ indicates the ruler being the monarch, the capitalists, or both players, and the second component $p$ or $a$ indicates whether the society is in a peaceful mode or in the aftermath of a revolt. Let $g_{ji}$ be the after-tax income for $i = m, c$ in states $\sigma_j$, $j = 1, \ldots, 6$. Then the above rules lead to the after-tax income of both players,

$$g_{1m} = g_{2m} = I_m + \hat{r}_T \left(I_c + I_l + I_w\right),$$

$$g_{3m} = \left(1 - r_T\right) I_m, \; g_{4m} = \theta \left(1 - r_T\right) I_m,$$

$$\begin{aligned} g_{5m} &= I_m + \tfrac{I_m}{I_m+I_c} \hat{r}_T \left(I_l + I_w\right), \\ g_{6m} &= I_m + \tfrac{I_m}{I_m+I_c} \theta \hat{r}_T \left(I_l + I_w\right), \\ g_{1c} &= \left(1 - r_T\right) I_c, \quad g_{2c} = \theta \left(1 - r_T\right) I_c, \end{aligned} \tag{21}$$

$$\begin{aligned} g_{3c} &= g_{4c} = I_c + \hat{r}_T \left(I_m + I_l + I_w\right) \\ g_{5c} &= I_c + \tfrac{I_c}{I_m+I_c} \hat{r}_T \left(I_l + I_w\right), \\ g_{6c} &= I_c + \tfrac{I_c}{I_m+I_c} \theta \hat{r}_T \left(I_l + I_w\right). \end{aligned}$$

Each player's objective is to maximize its total discounted payoff

$$\begin{aligned} U_i\left(t\right) &= \mathbb{E}_i^t \int_t^\infty e^{-\lambda(\tau - t)} \Pi_i\left(\tau\right) d\tau \\ &= \int_t^\infty e^{-\lambda(\tau - t)} \left\langle p\left(\tau\right), g_i\left(K\left(\tau\right)\right)\right\rangle d\tau \end{aligned}$$

using its available actions, where $p\left(\tau\right) = \left(p_1\left(\tau\right), \ldots, p_6\left(\tau\right)\right)$ is the probability distribution of the modes of the state $\sigma_1, \ldots, \sigma_6$.

The available actions for the non-ruling player at any time are to challenge and not to challenge the ruler, and the available actions for the ruler are to repress and to compromise with the challenger. The actions cause the society to change from one mode to another. If the non-ruling player does not challenge the ruler and if the society is in a peaceful mode, the mode will not be changed. If the society is in the aftermath of a revolt, it transfers to a peaceful mode at a fixed transition rate. We choose a time scale so that this transition rate is 1. If the non-ruling group challenges the ruler and the ruler compromises with the challenger, the challenger's status will be transferred to a ruler at the transition rate 1. In this case if previously the society is in the aftermath of a revolt, it will be transferred to a peaceful mode. If the ruler represses the challenger, then either the ruler remains in power or the power changes hands, according to the relative coercive capacities of the players, and the society transfers to the mode of aftermath of a revolt at rate 1 if it was previously in a peaceful mode. The coercive capacity of a player is the strength of the player in a confrontation against the other player. It depends on the player's resources and skill of using the resources. Let $\pi_m\left(t\right)$ and $\pi_c\left(t\right)$ be the coercive capacities of the monarch and capitalists, respectively. Following [12], we assume that

$$\pi_i\left(\tau\right) = e_i I_i\left(\tau\right), \quad i = m, c \tag{22}$$

where $e_i$ is the player's organizing effectiveness. We choose a scale so that $e_m = 1$. In addition, the ruler enjoys an "incumbency advantegy" represented by a factor $\chi > 1$. So if a confrontation occurs, the transition rate for the challenging group to overtake the ruler is

$$q = \frac{\pi_c}{\pi_c + \pi_r}$$

where $\pi_c$ and $\pi_r$ are the coercive capacities of the challenging group and the ruler, respectively.

We write the transition matrix $\left(q_{j\mu}\right)$ as follows. Let $\left(\phi_{\mu m}, \phi_{\mu c}\right) \in \left\{\left(0,0\right), \left(0,1\right), \left(1,0\right), \left(1,1\right)\right\}$ represent the actions of the monarch and capitalists, where 0 means no challenge for the non-ruler and compromise for the ruler, and 1 means revolt for the non-ruler and repress for the ruler. Not all elements of the set are available strategies in any mode of the state. For example in $\sigma_5$ and $\sigma_6$, no action is available for any player. If the current state is $\sigma_1$ or $\sigma_2$, then the available pure strategies are $(0,0)$, $(0,1)$ and $(1,1)$. The mixed strategies are

$$\left(1 - \phi_{jc}\right)\left(0,0\right) + \left(1 - \phi_{jm}\right)\phi_{jc}\left(0,1\right) + \phi_{jm}\phi_{jc}\left(1,1\right)$$

with $\phi_{jm}, \phi_{jc} \in [0,1]$ for $j = 1, 2$. Based on the rule of game described above, we have

$$\begin{aligned} q_{11} &= -\phi_{1c}, \quad q_{12} = \phi_{1m}\phi_{1c}\frac{\chi\pi_m}{\chi\pi_m + \pi_c}, \\ q_{14} &= \phi_{1m}\phi_{1c}\frac{\pi_c}{\chi\pi_m + \pi_c}, \quad q_{15} = \left(1 - \phi_{1m}\right)\phi_{1c}, \\ q_{21} &= 1 - \phi_{2c}, \quad q_{22} = -1 + \phi_{2m}\phi_{2c}\frac{\chi\pi_m}{\chi\pi_m + \pi_c}, \\ q_{24} &= \phi_{2m}\phi_{2c}\frac{\pi_c}{\chi\pi_m + \pi_c}, \quad q_{26} = \left(1 - \phi_{2m}\right)\phi_{2c}. \end{aligned}$$

Similarly, if the current state is $\sigma_3$ or $\sigma_4$, then the total mixed strategies are

$$\left(1 - \phi_{jm}\right)\left(0,0\right) + \left(1 - \phi_{jc}\right)\phi_{jm}\left(1,0\right) + \phi_{jm}\phi_{jc}\left(1,1\right)$$

with $\phi_{jm}, \phi_{jc} \in [0,1]$ for $j = 3, 4$. One can show that the transition rates are

$$\begin{aligned} q_{32} &= \phi_{3m}\phi_{3c}\frac{\pi_m}{\pi_m + \chi\pi_c}, \quad q_{33} = -\phi_{3m}, \\ q_{34} &= \phi_{3m}\phi_{3c}\frac{\chi\pi_c}{\pi_m + \chi\pi_c}, \quad q_{35} = \left(1 - \phi_{3c}\right)\phi_{3m}, \\ q_{42} &= \phi_{4m}\phi_{4c}\frac{\pi_m}{\pi_m + \chi\pi_c}, \quad q_{43} = 1 - \phi_{4m}, \\ q_{44} &= -1 + \phi_{4m}\phi_{4c}\frac{\chi\pi_c}{\pi_m + \chi\pi_c}, \quad q_{46} = \left(1 - \phi_{4c}\right)\phi_{4m}. \end{aligned}$$

By (22), we denote

$$\begin{aligned} \frac{\chi\pi_m}{\chi\pi_m + \pi_c} &= \frac{\chi I_m}{\chi I_m + e_c I_c} \equiv \eta\left(K\right), \\ \frac{\pi_c}{\chi\pi_m + \pi_c} &= \frac{e_c I_c}{\chi I_m + e_c I_c} = 1 - \eta\left(K\right), \\ \frac{\pi_m}{\pi_m + \chi\pi_c} &= \frac{I_m}{I_m + \chi e_c I_c} \equiv 1 - \delta\left(K\right), \\ \frac{\chi\pi_c}{\pi_m + \chi\pi_c} &= \frac{\chi e_c I_c}{I_m + \chi e_c I_c} \equiv \delta\left(K\right). \end{aligned}$$

Then the transition matrix, $Q\left(\phi_m, \phi_c\right) \equiv \left(q_{kl}\right)_{6\times 6}$, with the mixed strategy, $\left(\phi_m, \phi_c\right) \in [0,1]^8$, has the entries

$$\begin{gathered} q_{12} = \phi_{1m}\phi_{1c}\eta, \; q_{14} = \phi_{1m}\phi_{1c}\left(1 - \eta\right), \\ q_{15} = \left(1 - \phi_{1m}\right)\phi_{1c}, \quad q_{21} = 1 - \phi_{2c}, \\ q_{24} = \phi_{2m}\phi_{2c}\left(1 - \eta\right), \; q_{16} = \left(1 - \phi_{2m}\right)\phi_{2c}, \\ q_{32} = \phi_{3m}\phi_{3c}\left(1 - \delta\right), \; q_{34} = \phi_{3m}\phi_{3c}\delta, \\ q_{35} = \phi_{3m}\left(1 - \phi_{3c}\right), \quad q_{42} = \phi_{4m}\phi_{4c}\left(1 - \delta\right), \\ q_{43} = 1 - \phi_{4m}, \; q_{46} = \phi_{4m}\left(1 - \phi_{4c}\right), \\ q_{65} = 1, \quad q_{kk} = -\textstyle\sum_{l \neq k} q_{kl}, \quad k = 1, \ldots, 6, \end{gathered} \tag{23}$$

and all other entries are zero. It follows that the probability distribution $p(t)$ is governed by (8), which has the component form

$$\begin{aligned}
dp_1/dt &= -\phi_{1c}p_1 + (1-\phi_{2c})\,p_2,\\
dp_2/dt &= \phi_{1m}\phi_{1c}\eta p_1 + (-1+\phi_{2m}\phi_{2c}\eta)\,p_2\\
&\quad +\phi_{3m}\phi_{3c}(1-\delta)\,p_3 + \phi_{4m}\phi_{4c}(1-\delta)\,p_4,\\
dp_3/dt &= -\phi_{3m}p_3 + (1-\phi_{4m})\,p_4,\\
dp_4/dt &= \phi_{1m}\phi_{1c}(1-\eta)\,p_1 + \phi_{2m}\phi_{2c}(1-\eta)\,p_2\\
&\quad +\phi_{3m}\phi_{3c}\delta p_3 + (-1+\phi_{4m}\phi_{4c}\delta)\,p_4,\\
dp_5/dt &= (1-\phi_{1m})\,\phi_{1c}p_1 + \phi_{3m}(1-\phi_{3c})\,p_3 + p_6,\\
dp_6/dt &= (1-\phi_{2m})\,\phi_{2c}p_2 + \phi_{4m}(1-\phi_{4c})\,p_4 - p_6.
\end{aligned} \tag{24}$$

There is only one dynamical state variable during the first stage of democratization, which is the total physical capital $K(t)$. The capital grows with the investment made by all individuals in the society. It is shown in [12] assuming a log-linear utility function that an individual having an income $I$ would make an investment by the amount

$$b(K) = \beta\,[I(K) - Z]_+ \tag{25}$$

for the future, where $\beta$ and $Z$ are positive constants. (See Appendix for details.) Let $N_m$, $N_c$, $N_l$, and $N_w$ be the populations of the capitalists, landowners and the labors. We may assume that an individual has the average income $I = I_m/N_m$, $I_c/N_c$, $I_l/N_l$ or $I_w/N_w$ depending on whether the individual is the monarch, a capitalist, a landowner, or a labor, respectively. Hence the investments made by the social groups are

$$b_m = \beta\left[\frac{I_m}{N_m} - Z\right]_+ = \beta\,[I_m - N_m Z]_+\,,$$

and similarly

$$\begin{aligned}
b_l &= \beta\,[I_l - N_l Z]_+\,, \quad b_c = \beta\,[I_c - N_c Z]_+\,,\\
b_w &= \beta\,[I_w - N_w Z]_+\,,
\end{aligned}$$

where $[a]_+ = \max\{a,0\}$ for any $a \in \mathbb{R}$. If we further assume that $N_m$ is negligible, then $b_m = \beta I_m$. As there is no other form of investment, we propose that all the investments goes to the physical capital. Thus $K(t)$ is governed by the initial value problem

$$dK/d\tau = -aK + B(K)\,, \quad K(t) = x, \tag{26}$$

where

$$B = \beta\left\{I_m + [I_l - N_l Z]_+ + [I_c - N_c Z]_+ + [I_w - N_w Z]_+\right\} \tag{27}$$

is the total investment and $a > 0$ is the capital depreciation rate. This concludes the description of the elements of the model.

### B. Existence of Solution

Note that the function on the right-hand side of (26) is independent of the mode $\sigma$, and quantities $g_{ji}$ and $q_{\mu\nu}$ are independent of time $t$. Thus (12) is valid and can be used to find the strategies. We use Theorem 1 to prove the existence of solution $\phi^*(t)$, $K(\tau)$ and $p(\tau)$ given the initial conditions $K(t) = x$ and $p(t) = y$ if the equilibrium $\bar{x}$ of the equation

$$dx/ds = -ax + B(x) \tag{28}$$

is either sufficiently large or sufficiently small. We first observe that by (20) and (27), $B(x)$ is continuous in $x \in \mathbb{R}^+$, $B(0) \geq \beta I_m(0) = \beta C_m L^{-\alpha} > 0$, and $B(x) \leq M(L+x)^{1-\alpha}$ in $\mathbb{R}$ for some constant $M$. Thus $B(x) < ax$ if $x$ is sufficiently large. Hence by the Intermediate Value Theorem, (28) has at least one positive equilibrium. In addition, an equilibrium is asymptotically stable if the derivative $f'(x) = -a + B'(x)$ is negative at the equilibrium.

We next show that the maximization problem (13) has a solution for any $x \in (\underline{x}, \bar{x}]$ and $w_i \in \mathbb{R}^m$, $i = m, c$. Note that since $g_m$ and $g_c$ are independent of $\phi$, (12) has the vector form

$$\begin{aligned}
\lambda w_m(x) &= g_m(x) + (-ax + B(x))\,w_m'(x)\\
&\quad +Q(\phi_m^*, \phi_c^*)\,w_m(x)\,,\\
\lambda w_c(x) &= g_c(x) + (-ax + B(x))\,w_c'(x)\\
&\quad +Q(\phi_m^*, \phi_c^*)\,w_c(x)\,.
\end{aligned} \tag{29}$$

Thus (13) takes the form

$$\underset{\phi_i \in X_i(\phi_{-i}^*)}{\text{maximize}}\; Q_j\left(x, \phi_i, \phi_{-i}^*\right) w_i, \quad i = m, c.$$

Recall that by the rule of game, at any moment $t$ the non-ruler first chooses its strategy as to whether or not to revolt, anticipating that the ruler will choose whether to represses the revolt or to compromises with the challenger according to its best interest.

For $j = 1, 2$ the monarch is the ruler. So the capitalists first choose their (mixed) strategy $\phi_{jc}^*$ and the monarch responds with a (possibly mixed) strategy $\phi_{jm}^*$ so that

$$\phi_{jm}^* = \underset{\phi_{jm} \in [0,1]}{\arg\max}\, Q_j\left(\phi_{jm}, \phi_{jc}^*\right) w_m(x)\,.$$

Note that $Q_j$ is bilinear in $\phi_{jm}$ and $\phi_{jc}$. Thus

$$\begin{aligned}
Q_j\left(\phi_{jm}, \phi_{jc}^*\right) w_m &= \phi_{jm} Q_j\left(1, \phi_{jc}^*\right) w_m\\
&\quad + \left(1 - \phi_{jm}\right) Q_j\left(0, \phi_{jc}^*\right) w_m.
\end{aligned}$$

In case $\phi_{jc}^* = 0$, there is no choice for the monarch except $\phi_{jm}^* = 0$. If $\phi_{jc}^* = 1$ then either $\phi_{jm}^* = 0$ if

$$Q_j(0,1)\,w_m \geq Q_j(1,1)\,w_m \tag{30}$$

or $\phi_{jm}^* = 1$ if the reversed inequality holds. With this knowledge the capitalists would choose $\phi_{jc}^* = 0$ if (30) holds and

$$Q_j(0,0)\,w_c \geq Q_j(0,1)\,w_c; \tag{31}$$

or if the reversed inequality of (30) holds and

$$Q_j(0,0)\,w_c \geq Q_j(1,1)\,w_c.$$

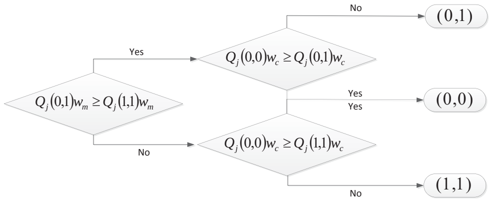


Fig. 1 Decision making when the monarch is in power ($j = 1, 2$)

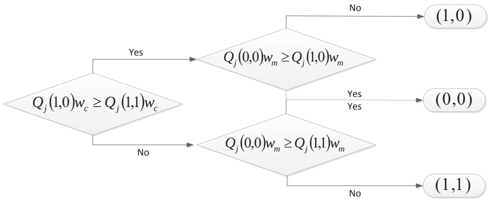


Fig. 2 Decision making when the capitalists are in power ($j = 3, 4$)

Hence, $\left(\phi_{jm}^*, \phi_{jc}^*\right) = (0, 1)$ if

$$Q_j(0,1)\, w_m \ge Q_j(1,1)\, w_m,\ Q_j(0,0)\, w_c < Q_j(0,1)\, w_c;$$

$\left(\phi_{jm}^*, \phi_{jc}^*\right) = (1, 1)$ if

$$Q_j(0,1)\, w_m < Q_j(1,1)\, w_m,\ Q_j(0,0)\, w_c < Q_j(1,1)\, w_c;$$

and $\left(\phi_{jm}^*, \phi_{jc}^*\right) = (0, 0)$ in the remaining cases (See Fig. 1).

Specifically, $\left(\phi_{jm}^*, \phi_{jc}^*\right) = (0, 1)$ if

$$w_{5m} \ge \eta w_{2m} + (1-\eta)\, w_{4m}, \quad w_{1c} < w_{5c}; \tag{32}$$

$(\phi_{1m}^*, \phi_{1c}^*) = (1, 1)$ if

$$w_{5m} < \eta w_{2m} + (1-\eta)\, w_{4m}, \quad w_{1c} < \eta w_{2c} + (1-\eta)\, w_{4c};$$

and $(\phi_{1m}^*, \phi_{1c}^*) = (0, 0)$ in the remaining cases. Similarly, $(\phi_{2m}^*, \phi_{2c}^*) = (0, 1)$ if

$$w_{6m} \ge \eta w_{2m} + (1-\eta)\, w_{4m}, \quad w_{1c} < w_{6c}; \tag{33}$$

$(\phi_{2m}^*, \phi_{2c}^*) = (1, 1)$ if

$$w_{6m} < \eta w_{2m} + (1-\eta)\, w_{4m}, \quad w_{1c} < \eta w_{2c} + (1-\eta)\, w_{4c};$$

and $(\phi_{2m}^*, \phi_{2c}^*) = (0, 0)$ in the remaining cases.

For $j = 3, 4$ the capitalists are the ruler. A similar reasoning shows that $\left(\phi_{jm}^*, \phi_{jc}^*\right) = (1, 0)$ if

$$Q_j(1,0)\, w_c \ge Q_j(1,1)\, w_c,\ Q_j(0,0)\, w_m < Q_j(1,0)\, w_m;$$

$\left(\phi_{jm}^*, \phi_{jc}^*\right) = (1, 1)$ if

$$Q_j(1,0)\, w_c < Q_j(1,1)\, w_c,\ Q_j(0,0)\, w_m < Q_j(1,1)\, w_m;$$

and $\left(\phi_{jm}^*, \phi_{jc}^*\right) = (0, 0)$ in the remaining cases (See Fig. 2).

In terms of components of $w_m$ and $w_c$, $(\phi_{3m}^*, \phi_{3c}^*) = (1, 0)$ if

$$w_{5c} \ge (1-\delta)\, w_{2c} + \delta w_{4c}, \quad w_{3m} < w_{5m};$$

$(\phi_{3m}^*, \phi_{3c}^*) = (1, 1)$ if

$$w_{5c} < (1-\delta)\, w_{2c} + \delta w_{4c}, \quad w_{3m} < (1-\delta)\, w_{2m} + \delta w_{4m};$$

and $(\phi_{3m}^*, \phi_{3c}^*) = (0, 0)$ in the remaining cases. Similarly, $(\phi_{4m}^*, \phi_{4c}^*) = (1, 0)$ if

$$w_{6c} \ge (1-\delta)\, w_{2c} + \delta w_{4c}, \quad w_{3m} < w_{6m};$$

$(\phi_{4m}^*, \phi_{4c}^*) = (1, 1)$ if

$$w_{6c} < (1-\delta)\, w_{2c} + \delta w_{4c}, \quad w_{3m} < (1-\delta)\, w_{2m} + \delta w_{4m};$$

and $(\phi_{4m}^*, \phi_{4c}^*) = (0, 0)$ in the remaining cases.

Hence, the rule of game completely determines pure strategies $\phi_m^*(x, w_i)$ and $\phi_c^*(x, w_i)$ for any $x$ and $w_i$.

Let $\bar{x}$ be an equilibrium of (28). We next show that the equations for the steady states, (14), has a solution if $\bar{x}$ is sufficiently large or sufficiently small. Note that equations in (14) have the form

$$\begin{aligned} \lambda w_m(\bar{x}) &= g_m(\bar{x}) + Q(\phi_m^*, \phi_c^*)\, w_m(\bar{x}), \\ \lambda w_c(\bar{x}) &= g_c(\bar{x}) + Q(\phi_m^*, \phi_c^*)\, w_c(\bar{x}). \end{aligned} \tag{34}$$

*Lemma 2:* If $\bar{x}$ is sufficiently large then there is a steady state solution $(\bar{w}_m, \bar{w}_c)$ that satisfies

$$\begin{aligned} \bar{w}_{1i} &= \frac{\lambda \bar{g}_{1i} + \bar{g}_{5i}}{\lambda(\lambda+1)}, \quad \bar{w}_{2i} = \frac{\bar{g}_{2i}}{\lambda+1} + \frac{\bar{g}_{5i} + \lambda \bar{g}_{6i}}{\lambda(\lambda+1)^2}, \quad \bar{w}_{3i} = \frac{\bar{g}_{3i}}{\lambda}, \\ \bar{w}_{4i} &= \frac{\bar{g}_{3i} + \lambda \bar{g}_{4i}}{\lambda(\lambda+1)}, \quad \bar{w}_{5i} = \frac{\bar{g}_{5i}}{\lambda}, \quad \bar{w}_{6i} = \frac{\bar{g}_{5i} + \lambda \bar{g}_{6i}}{\lambda(\lambda+1)} \end{aligned}$$

for $i = m, c$ corresponding to $\left(\phi_{jm}^*, \phi_{jc}^*\right) = (0, 1)$ for $j = 1, 2$ and $\left(\phi_{jm}^*, \phi_{jc}^*\right) = (0, 0)$ for $j = 3, 4$, where $\bar{g}_{ji} = g_{ji}(\bar{x})$. Similarly, if $\bar{x}$ is sufficiently small then there is a steady state solution with $\left(\phi_{jm}^*, \phi_{jc}^*\right) = (0, 0)$ for $j = 1, 2$ and $\left(\phi_{jm}^*, \phi_{jc}^*\right) = (1, 0)$ for $j = 3, 4$.

Proof*:* For $\left(\phi_{jm}^*, \phi_{jc}^*\right) = (0, 1)$ for $j = 1, 2$ and $\left(\phi_{jm}^*, \phi_{jc}^*\right) = (0, 0)$ for $j = 3, 4$ we compute

$$Q = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}. \tag{35}$$

Thus

$$Q w_i = (-w_{1i} + w_{5i}, -w_{2i} + w_{6i}, 0, w_{3i} - w_{4i}, 0, w_{5i} - w_{6i})$$

for $i = m, c$. Equations in (34) have the form

$$\begin{aligned} \lambda w_{1i} &= g_{1i} - w_{1i} + w_{5i},\ \lambda w_{2i} = g_{2i} - w_{2i} + w_{6i}, \\ \lambda w_{3i} &= g_{3i},\ \lambda w_{4i} = g_{4i} + w_{3i} - w_{4i}, \\ \lambda w_{5i} &= g_{5i},\ \lambda w_{6i} = g_{6i} + w_{5i} - w_{6i}. \end{aligned}$$

We verify that (32), (33) hold if $\bar{x}$ is sufficiently large. This will imply that $\left(\phi_{jm}^*, \phi_{jc}^*\right) = (0, 1)$ for $j = 1, 2$. The first inequality in (32) has the form

$$\frac{g_{5m}}{\lambda} \ge \eta \left( \frac{g_{2m}}{\lambda+1} + \frac{g_{5m} + \lambda g_{6m}}{\lambda(\lambda+1)^2} \right) + (1-\eta) \frac{g_{3m} + \lambda g_{4m}}{\lambda(\lambda+1)}. \tag{36}$$

If $\bar{x} \gg 1$ then $\eta \ll 1$. By (21)

$$g_{4m} < g_{3m} < g_{6m} < g_{5m}. \tag{37}$$

Thus the right-hand side of (36) is close to

$$\frac{g_{3m}+\lambda g_{4m}}{\lambda(\lambda+1)}<\frac{g_{5m}}{\lambda}.$$

The second inequality in (32) has the form

$$\frac{\lambda g_{1c}+g_{5c}}{\lambda(\lambda+1)}<\frac{g_{5c}}{\lambda}$$

which is equivalent to $g_{1c}<g_{5c}$. This is true by (21). The first inequality in (33) has the form

$$\begin{aligned}\frac{g_{5m}+\lambda g_{6m}}{\lambda(\lambda+1)} &\geq \eta\left(\frac{g_{2m}}{\lambda+1}+\frac{g_{5m}+\lambda g_{6m}}{\lambda(\lambda+1)^2}\right)\\ &\quad+(1-\eta)\frac{g_{3m}+\lambda g_{4m}}{\lambda(\lambda+1)}.\end{aligned}$$

For $\eta \ll 1$ the right-hand side is close to $(g_{3m}+\lambda g_{4m})/\lambda(\lambda+1)$. It is less than the left-hand side due to (37). The second inequality in (33) has the form

$$\frac{\lambda g_{1c}+g_{5c}}{\lambda(\lambda+1)}<\frac{g_{5c}+\lambda g_{6c}}{\lambda(\lambda+1)}.$$

In view of (21) which implies $g_{1c}<g_{6c}$, the above inequality holds. We next show that

$$\begin{aligned} w_{5c} &< (1-\delta)w_{2c}+\delta w_{4c}, \quad w_{3m}<w_{5m}, &(38)\\ w_{3m} &\geq (1-\delta)w_{2m}+\delta w_{4m} &(39)\end{aligned}$$

if $\bar{x}\gg 1$. By the definition of $\delta$, $\bar{x}\gg 1$ implies $\delta$ is close to 1. Thus by (21) the right-hand side of the first inequality is close to

$$w_{4c}=\frac{g_{3c}+\lambda g_{4c}}{\lambda(\lambda+1)}=\frac{g_{3c}}{\lambda}>\frac{g_{5c}}{\lambda}=w_{5c}.$$

By (37)

$$w_{3m}=\frac{g_{3m}}{\lambda}<\frac{g_{5m}}{\lambda}=w_{5m},$$

the second inequality of (38) holds. For $\delta$ close to 1 the right-hand side of the third inequality in (38) is close to

$$w_{4m}=\frac{g_{3m}+\lambda g_{4m}}{\lambda(\lambda+1)}.$$

By (37)

$$\frac{g_{3m}+\lambda g_{4m}}{\lambda(\lambda+1)}<\frac{g_{3m}}{\lambda}=w_{3m}.$$

This completes the proof of (38). Hence $(\phi^*_{3m},\phi^*_{3c})=(0,0)$. Finally, we can show that

$$w_{6c}<(1-\delta)w_{2c}+\delta w_{4c}, \quad w_{3m}<w_{6m}, \qquad (40)$$

if $\bar{x}\gg 1$. By the definition of $\delta$, $\delta$ is close to 1. Thus by (21) the right-hand side of the first inequality is close to

$$w_{4c}=\frac{g_{3c}+\lambda g_{4c}}{\lambda(\lambda+1)}>\frac{g_{5c}+\lambda g_{6c}}{\lambda(\lambda+1)}=w_{6c}.$$

By (37)

$$w_{3m}=\frac{g_{3m}}{\lambda}<\frac{g_{5m}+\lambda g_{6m}}{\lambda(\lambda+1)}=w_{6m},$$

the second inequality of (40) holds. This completes the proof of (40). This implies that $(\phi^*_{4m},\phi^*_{4c})=(0,0)$. ■

With these preparation we have

*Theorem 3:* Let $\bar{x}$ be an equilibrium of (28) which is asymptotically stable and $B'(\bar{x})<a$. Suppose $\bar{x}$ is sufficiently large such that the conclusion of Lemma 2 holds, and $\phi^*(x,w_i)=\phi^*(\bar{x},\bar{w}_i)$ in a neighborhood of $(\bar{x},\bar{w}_i)$. Let $\underline{x}$ be either 0 or the largest equilibrium of (28) less than $\bar{x}$. Then Problem (29) has a solution $(w_m(x),w_c(x))$ at any $x\in(\underline{x},\bar{x}]$.

*Proof.:* Hypothesis (H)–(1) and (2) are obviously satisfied. Note that the $\phi^*(x,w_i)$ is uniquely determined by the rules described in Figs. 1 and 2, and is piecewise constant. The boundaries of the subdomains in which $\phi^*$ is constant are given by one of the equations

$$\begin{aligned} Q_j(0,1)w_m(x) &= Q_j(1,1)w_m(x),\\ Q_j(0,0)w_c(x) &= Q_j(0,1)w_c(x),\\ Q_j(0,0)w_c(x) &= Q_j(1,1)w_c(x) \quad \text{for } j=1,2,\end{aligned}$$

and

$$\begin{aligned} Q_j(1,0)w_c(x) &= Q_j(1,1)w_c(x),\\ Q_j(0,0)w_m(x) &= Q_j(1,1)w_m(x),\\ Q_j(0,0)w_m(x) &= Q_j(1,0)w_m(x) \quad \text{for } j=3,4.\end{aligned}$$

In component form, the equations are

$$\begin{aligned} w_{5m}-\eta w_{2m}-(1-\eta)w_{4m} &= 0, \quad w_{1c}-w_{5c}=0,\\ w_{1c}-\eta w_{2c}-(1-\eta)w_{4c} &= 0,\\ w_{6m}-\eta w_{2m}-(1-\eta)w_{4m} &= 0, \quad w_{1c}-w_{6c}=0,\\ w_{5c}-(1-\delta)w_{2c}-\delta w_{4c} &= 0,\\ w_{3m}-(1-\delta)w_{2m}-\delta w_{4m} &= 0, \quad w_{3m}-w_{5m}=0,\\ w_{6c}-(1-\delta)w_{6c}-\delta w_{4c} &= 0, \quad w_{3m}-w_{5m}=0.\end{aligned}$$

The functions $\gamma_{l,\mu}$ that define the boundaries $\partial D_l$ are the left-hand sides of the above equations. The conclusion then follows from Theorem 1. ■

We present an example to show how the solution can be computed.

### C. A Numerical Example

Let us choose the parameters

$$\begin{aligned} \alpha &= \beta=1/2, \quad AH^\alpha=3/2, \quad L_m=0.4, \quad L_l=0.6,\\ N_c &= 0.1, \quad N_l=0.4, \quad N_w=0.5, \quad Z=15/\sqrt{8}\end{aligned}$$

and

$$a=\frac{39}{1100}\sqrt{11}-\frac{3}{160}\sqrt{2}$$

Then by (61)

$$C_m=3/10, \quad C_l=9/20, \quad C_c=C_w=3/4, \quad L=L_m+L_l=1,$$

and by (20)

$$\begin{aligned} I_m(x) &= \frac{3}{10\sqrt{1+x}}, \quad I_l(x)=\frac{9}{20\sqrt{1+x}},\\ I_c(x) &= \frac{3x}{4\sqrt{1+x}}, \quad I_w(x)=\frac{3}{4}\sqrt{1+x}.\end{aligned} \qquad (41)$$

Hence, by (27),

$$\begin{aligned} B(x) &= \frac{1}{2}\left\{\frac{3}{10\sqrt{1+x}}+\left[\frac{3x}{4\sqrt{1+x}}-\frac{3}{4\sqrt{2}}\right]_+\right. \\ &\left.+\left[\frac{3}{4\sqrt{1+x}}-\frac{3}{\sqrt{2}}\right]_+ +\left[\frac{3}{4}\sqrt{1+x}-\frac{15}{8}\sqrt{2}\right]_+\right\}. \end{aligned}$$

It can be seen that $-a\bar{x}+B(\bar{x})=0$ at $\bar{x}=10$. Since

$$\frac{3}{4\sqrt{1+x}}<\frac{3}{\sqrt{2}}, \quad \frac{3}{4}\sqrt{1+x}\leq\frac{3\sqrt{11}}{4}<\frac{15\sqrt{2}}{8}$$

for $x\in[0,10]$, it follows that

$$\begin{aligned} B(x) &= \frac{3}{20}\left\{\frac{1}{\sqrt{1+x}}+5\left[\frac{x}{\sqrt{1+x}}-\frac{1}{\sqrt{2}}\right]_+\right\} \\ &= \begin{cases} \frac{3}{20\sqrt{1+x}} & \text{if } x\leq 1, \\ \frac{3}{20}\left[\frac{1+5x}{\sqrt{1+x}}-\frac{1}{\sqrt{2}}\right] & \text{if } 1<x\leq 10. \end{cases} \end{aligned}$$

It can be seen that the only positive solution of the equation

$$-ax+B(x)=0$$

is $x=10$.

We next choose the values

$$\begin{aligned} r_T &= 0.2, \quad \hat{r}_T=0.15, \quad \theta=0.5, \\ \lambda &= 0.6, \quad \chi=1.5, \quad e_c=0.7 \end{aligned}$$

and computer $g_{ji}(\bar{x})$ by (21).

*Terminal values.:* Using (41) with $x=\bar{x}=10$, we find

$$\begin{aligned} I_m(\bar{x}) &= \frac{3}{10\sqrt{11}}, \quad I_l(\bar{x})=\frac{9}{20\sqrt{11}}, \\ I_c(\bar{x}) &= \frac{15}{2\sqrt{11}}, \quad I_w(\bar{x})=\frac{3}{4}\sqrt{11}. \end{aligned}$$

Hence, by (21)

$$\begin{aligned} g_m(\bar{x}) &= (0.823, 0.823, 0.072, 0.036, 0.106, 0.098), \\ g_c(\bar{x}) &= (1.809, 0.905, 2.668, 2.668, 2.640, 2.451). \end{aligned}$$

By Lemma 2,

$$\begin{aligned} w_m(\bar{x}) &= (0.624, 0.621, 0.121, 0.098, 0.176, 0.171), \\ w_c(\bar{x}) &= (3.880, 3.241, 4.447, 4.447, 4.399, 4.281). \end{aligned}$$

It remains to verify that $\left(\phi^*_{jm},\phi^*_{jc}\right)(\bar{x},\bar{w}_i)=(0,1)$ for $j=1,2$ and $\left(\phi^*_{jm},\phi^*_{jc}\right)(\bar{x},\bar{w}_i)=(0,0)$ for $j=3,4$. For $j=1,2$, we verify the relations

$$Q_j(0,1)\bar{w}_m\geq Q_j(1,1)\bar{w}_m, \; Q_j(0,0)\bar{w}_c<Q_j(0,1)\bar{w}_c \tag{42}$$

(See Fig. 1). By (23),

$$\begin{aligned} Q_1(0,1) &= (-1,0,0,0,1,0), \\ Q_1(1,1) &= (-1,\eta,0,(1-\eta),0,0), \\ Q_1(0,0) &= (0,0,0,0,0,0), \end{aligned}$$

where

$$\eta=\frac{\chi I_m(\bar{x})}{\chi I_m(\bar{x})+e_c I_c(\bar{x})}\approx 0.07895.$$

The first inequality in (42) has the forms

$$-\bar{w}_{1m}+\bar{w}_{5m}\geq-\bar{w}_{1m}+\eta\bar{w}_{2m}+(1-\eta)\bar{w}_{4m}. \tag{43}$$

The inequality is true using the values of $\bar{w}_{jm}$. The second inequality in (42) has the form

$$0<-\bar{w}_{1c}+\bar{w}_{5c}. \tag{44}$$

It is also true in view of the values of $\bar{w}_{jc}$. Similarly, for $j=2$, by (23)

$$\begin{aligned} Q_2(0,1) &= (0,-1,0,0,0,1), \\ Q_2(1,1) &= (0,-1+\eta,0,1-\eta,0,0), \\ Q_2(0,0) &= (1,-1,0,0,0,0). \end{aligned}$$

The first inequality in (42) has the form

$$-\bar{w}_{2m}+\bar{w}_{6m}\geq(-1+\eta)\bar{w}_{2m}+(1-\eta)\bar{w}_{4m}. \tag{45}$$

It is true. The second inequality in (42) has the form

$$\bar{w}_{1c}-\bar{w}_{2c}<-\bar{w}_{2c}+\bar{w}_{6c}. \tag{46}$$

It again is true. This proves that $\left(\phi^*_{jm},\phi^*_{jc}\right)(\bar{x},\bar{w}_i)=(0,1)$ for $j=1,2$.

For $j=3,4$ we verify the relations

$$Q_j(1,0)\bar{w}_c<Q_j(1,1)\bar{w}_c, \; Q_j(0,0)\bar{w}_m\geq Q_j(1,1)\bar{w}_m. \tag{47}$$

(See Fig. 2.) For $j=3$

$$\begin{aligned} Q_3(1,0) &= (0,0,-1,0,1,0), \\ Q_3(1,1) &= (0,1-\delta,-1,\delta,0,0), \\ Q_3(0,0) &= (0,0,0,0,0). \end{aligned}$$

The first inequality has the form

$$-\bar{w}_{3c}+\bar{w}_{5c}<(1-\delta)\bar{w}_{2c}-\bar{w}_{3c}+\delta\bar{w}_{4c} \tag{48}$$

where

$$\delta=\frac{\chi e_c I_c(\bar{x})}{I_m(\bar{x})+\chi e_c I_c(\bar{x})}\approx 0.96330.$$

The inequality is true. The second inequality in (47) has the form

$$0\geq(1-\delta)\bar{w}_{2m}-\bar{w}_{3m}+\delta\bar{w}_{4m} \tag{49}$$

which is true. For $j=4$, we have

$$\begin{aligned} Q_4(1,0) &= (0,0,0,-1,0,1), \\ Q_4(1,1) &= (0,1-\delta,0,-1+\delta,0,0), \\ Q_4(0,0) &= (0,0,1,-1,0,0). \end{aligned}$$

The first inequality is

$$-\bar{w}_{4c}+\bar{w}_{6c}<(1-\delta)\bar{w}_{2c}+(-1+\delta)w_{4c}. \tag{50}$$

It is true. The second inequality is

$$\bar{w}_{3m}-\bar{w}_{4m}\geq(1-\delta)w_{2m}+(-1+\delta)w_{4m}, \tag{51}$$

which is also true. This verifies the conditions of Theorem 3.

We now compute the solution of (29) with the terminal condition

$$\begin{aligned} w_{jm}(10) &= \bar{w}_{jm}, \quad w_{jc}(10) = \bar{w}_{jc}, \\ \left(\phi^*_{jm}, \phi^*_{jc}\right)(10) &= (0,1) \quad \text{for } j = 1,2, \\ \left(\phi^*_{jm}, \phi^*_{jc}\right)(10) &= (0,0) \quad \text{for } j = 3,4. \end{aligned}$$

The system (29) has the component form

$$\begin{aligned} \lambda w_{1i} &= g_{1i} + F w'_{1i} - w_{1i} + w_{5i}, \\ \lambda w_{2i} &= g_{2i} + F w'_{2i} - w_{2i} + w_{6i}, \\ \lambda w_{3i} &= g_{3i} + F w'_{3i}, \\ \lambda w_{4i} &= g_{4i} + F w'_{4i} + w_{3i} - w_{4i}, \\ \lambda w_{5i} &= g_{5i} + F w'_{5i}, \\ \lambda w_{6i} &= g_{6i} + F w'_{6i} + w_{5i} - w_{6i}, \end{aligned}$$

for $i = m, c$, where $g_{ji}$ are given by (21) and $F(x) = -ax + B(x)$.

*Initial step–out of singularity.:* The equations are singular at the $\bar{x} = 10$. We use a Taylor expansion

$$w_i(x) \approx w_i(\bar{x}) + (x - \bar{x}) w'_i(\bar{x}) + \frac{1}{2}(x - \bar{x})^2 w''_i(\bar{x}) + \dots \tag{52}$$

for $i = m, c$ to find an approximate solution near $\bar{x}$. Differentiating the equations in (29) with respect to $x$, and evaluate the expressions at $\bar{x}$ we find

$$\begin{aligned} \lambda w'_i(\bar{x}) &= g'_i(\bar{x}) + F'(\bar{x}) w'_i(\bar{x}) + Q w'_i(\bar{x}), \\ \lambda w''_i(\bar{x}) &= g''_i(\bar{x}) + 2F'(\bar{x}) w''_i(\bar{x}) \\ &\quad + Q w''_i(\bar{x}) + F''(\bar{x}) w'_i(\bar{x}), \dots \end{aligned}$$

for $i = m, c$. Using terms up to $(x - \bar{x})^2$ we obtain approximate solution given by (52).

Computations show that inequalities

$$\begin{aligned} Q_j(0,1) w_m(x) &\geq Q_j(1,1) w_m(x), \\ Q_j(0,0) w_c(x) &< Q_j(0,1) w_c(x), \quad j = 1,2, \\ Q_j(1,0) w_c(x) &< Q_j(1,1) w_c(x), \\ Q_j(0,0) w_m(x) &\geq Q_j(1,1) w_m(x), \quad j = 3,4 \end{aligned}$$

all satisfied for $x \in [9.5, 10]$. In component form, these equations are

$$\begin{aligned} w_{5m}, w_{6m} &\geq \eta w_{2m} + (1-\eta) w_{4m}, \quad w_{1c} < w_{5c}, w_{6c}, \\ w_{5c}, w_{6c} &< (1-\delta) w_{2c} + \delta w_{4c}, \\ w_{3m} &\geq (1-\delta) w_{2m} + \delta w_{4m}. \end{aligned}$$

Thus we use this approximate solution for $[9.5, 10]$.

*The differential-algebraic equations.:* We continue the computation of the solution for $x \in [0, 9.5)$ using the terminal data as the value of the approximate solution at $x = 9.5$,

$$\begin{aligned} w_m(9.5) &= (0.618, 0.614, 0.123, 0.100, 0.180, 0.175), \\ w_c(9.5) &= (3.781, 3.158, 4.336, 4.336, 4.287, 4.171). \end{aligned}$$

Note that on this interval the system (29) is not singular.

The computation is carried out using an iteration scheme. In the initial step Problem (29) is solved with functions $\phi^*_m(x)$ and $\phi^*_c(x)$ subsituted by the functions

$$\phi^{(0)}_m(x) = (0,0,0,0), \quad \phi^{(0)}_c(x) = (1,1,0,0)$$

for $0 \leq x \leq 9.5$, respectively. The solution is denoted as $\left(w^{(1)}_m(x), w^{(1)}_c(x)\right)$. Using this solution we find functions $\phi^{(1)}_m(x)$ and $\phi^{(1)}_c(x)$ that satisfy

$$\begin{aligned} \phi^{(1)}_{jm}(x) &= \underset{\phi_{jm} \in [0,1]}{\arg\max} Q_j\left(\phi_{jm}, \phi^{(1)}_{jc}(x)\right) w^{(1)}_m(x), \\ \phi^{(1)}_{jc}(x) &= \underset{\phi_{jc} \in [0,1]}{\arg\max} Q_j\left(\phi^{(1)}_{jm}(x), \phi_{jc}\right) w^{(1)}_c(x), \end{aligned} \tag{53}$$

as described by Figs. 1 and 2. In general, if $\phi^{(k)}_m(x)$ and $\phi^{(k)}_c(x)$ have been obtained, we solve Problem (29) with $\phi^*_m(x)$ and $\phi^*_c(x)$ substituted by $\phi^{(k)}_m(x)$ and $\phi^{(k)}_c(x)$, respectively, and denote the solution as $\left(w^{(k+1)}_m(x), w^{(k+1)}_c(x)\right)$. We then find $\left(\phi^{(k+1)}_m, \phi^{(k+1)}_c\right)$ using (53) with the superscript "(1)" changed to "$(k+1)$." The process can repeated to generate two sequences of functions $\left(\phi^{(k)}_m(x)\right)$, $\left(\phi^{(k)}_c(x)\right)$ on $[0, 9.5]$. If the sequences converge, the limits are the maximizing strategies $\phi^*_m(x)$ and $\phi^*_c(x)$.

The iteration scheme is implemented as follows. The interval $[0, 9.5]$ is partitioned by $n$ points

$$0 < x_1 < x_2 < \cdots < x_n < 9.5.$$

A numerical differential equation solver is used to compute the solution $\left(w^{(k+1)}_m, w^{(k+1)}_c\right)$ at points $x_1, \dots, x_n$ given $\left(\phi^{(k)}_m, \phi^{(k)}_c\right)$ at these points. Then $\left(\phi^{(k+1)}_m, \phi^{(k+1)}_c\right)$ is computed at $x_1, \dots, x_n$. This process continues. Since given the points $x_1$,..., $x_n$ the there are only finitely many possible values of $\left(\phi^{(k)}_m, \phi^{(k)}_c\right)$, the process will lead to a finite sequence of solutions $\left\{\left(\phi^{(k)}_m, \phi^{(k)}_c\right), \left(w^{(k)}_m, w^{(k)}_c\right)\right\}$, $k = k_1, k_1+1, \dots, k_2$ such that $\left(w^{(k+1)}_m, w^{(k+1)}_c\right)$ satisfies Problem (29) with $(\phi^*_m, \phi^*_c)$ replaced by $\left(\phi^{(k)}_m, \phi^{(k)}_c\right)$ for $k = k_1, \dots, k_2 - 1$, and $\left(w^{(k_1)}_m, w^{(k_1)}_c\right)$ satisfies (29) with $(\phi^*_m, \phi^*_c)$ replaced by $\left(\phi^{(k_2)}_m, \phi^{(k_2)}_c\right)$. Among these solutions we choose $k^*$ such that

$$\begin{aligned} &\left\| \left(w^{(k^*)}_m, w^{(k^*)}_c\right) - \left(w^{(k^*+1)}_m, w^{(k^*+1)}_c\right) \right\|_{L^2[0,9.5]} \\ = &\min_{k=k_1,\dots,k_2} \left\| \left(w^{(k)}_m, w^{(k)}_c\right) - \left(w^{(k+1)}_m, w^{(k+1)}_c\right) \right\|_{L^2[0,9.5]} \end{aligned}$$

where $k_2 + 1$ is identified as $k_1$. It is expected that the smaller the stepsize $\max_i |x_{i+1} - x_i|$, the smaller the error

$$\varepsilon_{k^*} = \left\| \left(w^{(k^*)}_m, w^{(k^*)}_c\right) - \left(w^{(k^*+1)}_m, w^{(k^*+1)}_c\right) \right\|_{L^2[0,9.5]}.$$

Computation using equal stepsize $x_{i+1} - x_i = 0.01$ is carried out using matlab solver ode15s. The cyclic solutions $\left(w^{(k)}_m, w^{(k)}_c\right)$ are obtained for $k = 129, \dots, 133$. The approximation solution $\left(w^{(129)}_m, w^{(129)}_c\right)$ is chosen with the error $\varepsilon_{129} \approx 0.119$. Then the initial value problem (6) and (8) are solved using the initial conditions

$$x(0) = 0, \quad p(0) = (1,0,0,0,0,0).$$

These initial data represents the situation that at the beginning of social transformation, there is no capital in the

TABLE I
CHANGES OF STRATEGIES OF PLAYERS OVER TIME

| time periods | $\phi^*_{im}, \phi^*_{2m}, \phi^*_{3m}, \phi^*_{4m}$ | $\phi^*_{1c}, \phi^*_{2c}, \phi^*_{3c}, \phi^*_{4c}$ |
|---|---|---|
| $(0, 0.8)$ | $(0, 0, 1, 1)$ | $(0, 0, 0, 0)$ |
| $(0.8, 20.5)$ | $(1, 1, 1, 1)$ | $(1, 1, 0, 0)$ |
| $(20.5, 70.3)$ | $(0, 0, 1, 0)$ | $(1, 1, 0, 0)$ |
| $t > 70.3$ | $(0, 0, 0, 0)$ | $(1, 1, 0, 0)$ |

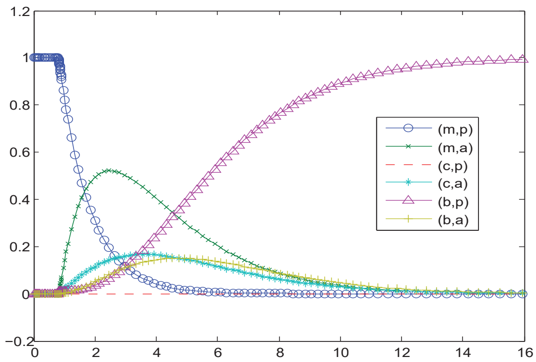

Fig. 3 Changes of probabilities of modes with time

society and the monarch is the ruler. Table 1 shows the approximations of $(\phi^*_m, \phi^*_c)$:

The probabilities of the modes are graphed in Fig. 3.

As can be seen from Table 1 and Fig. 3, the monarch initially dominates the political power and the state is in the mode $(m, p)$ for a short period of time $0 < t < 0.8$. During this period the capitalists do not challenge the monarch since their strength is weak. As capital increases the capitalists become stronger. So, in the next period of time, $0.8 < t < 20.5$ the capitalists challenge the monarch and the monarch responds the challenge by repression if the monarch is in power, or the monarch challenges the capitalists and the latter repress the revolt if the capitalists are in power. During this period the state is found in 5 modes of $(m, p)$, $(m, a)$, $(c, a)$, $(b, p)$ and $(b, a)$, with the probability of $(m, p)$ decreasing, and the probability of the other the probability of $(b, p)$ increasing. As the capital continues to increasing, for $20.5 < t < 70.3$, the capitalists greatly over power the monarch. So whenever the monarch is in power the capitalists will revolt and the monarch will compromises with the challengers, and whenever the capitalists are in power, the monarch only revolts when the state is not in the aftermath of a revolt. During this period of time the probability of $(b, p)$ continues to rise, and the probabilities of the other three first rise then fall. Eventually, for $t > 70.3$, the capitalists always challenge the ruler and the monarch never represses if the monarch is in power, and the monarch nevery challenges the capitalists if the latter are in power. During this period the state $(b, p)$ prevails, and all other modes fade away.

## IV. CONCLUSION

The democratization model studied above can be extended to include multiple players. In this case, with the possibility of more than one social groups challenging the ruler simultaneously, the solution of the maximization problem (13) may involve Nash equilibria.

For the general case we have established the existence of solution and developed numerical scheme for the case where the continuously-varying state variables evolve independently of discretely-varying state variables and independently of strategies of the player. This is done because the Hamilton-Jacobi-Bellman equation is semilinear. The more general case where there is no such independence is much more difficult, because the Hamilton-Jacobi-Bellman equation is highly nonlinear.

## APPENDIX

We show in this appendix that $I_m$, $I_c$, $I_l$ and $I_w$ are functions of $K$ as in (20). Following [12] we assume that the land, together with physical capital and human capital, produce one single good that can be used for consumption and investment. The monarch and landowners own lands. Capitalists own the physical capital, and the workers own the human capital. Each social group earns income from the means of production that it owns. The land yield of an individual landowner is

$$Y_{l,i} = A\left(L_{l,i} + K_{l,i}\right)^{1-\alpha} H^{\alpha}_{l,i}$$

where $\alpha \in (0, 1)$ is a constant, $A$ is the knowledge stock, $L_{l,i}$, $K_{l,i}$ and $H_{l,i}$ are the land, the physical capital, and the human capital that the landowner $i$ utilizes. The earning of a capitalist is from renting the physical capital that he possesses, and that of a worker is from wage he received by offering his human capital. Let $r_K$ and $r_H$ be the rental rates of physical capital and the wage of a unit human capital, respectively. Then the before-tax incomes of capitalists and workers are

$$I_{c,i} = r_K K_{c,i}, \quad I_{w,i} = r_H H_{w,i} \tag{54}$$

where $K_{c,i}$ is the capital that capitalist $i$ possesses, and $H_{w,i}$ is the human capital that worker $i$ possesses. The before-tax income of the landowners, including the monarch, is their land yield minus cost of renting physical capital and wages for hiring workers. Thus the monarch and a landowner have the before-tax income

$$\begin{aligned} I_m &= A\left(L_m + K_m\right)^{1-\alpha} H^{\alpha}_m - r_K K_m - r_H H_m, \\ I_{l,i} &= A\left(L_i + K_{l,i}\right)^{1-\alpha} H^{\alpha}_i - r_K K_{l,i} - r_H H_{l,i}, \end{aligned} \tag{55}$$

respectively, where $L_m$, $K_m$ and $H_m$ are the land, the physical capital, and the human capital that the monarch utilizes.

The next proposition shows that the rates $r_K$ and $r_H$ are determined endogeneously by the market assuming the market clearing condition.

*Proposition 1:* Suppose $r_K$ and $r_H$ are market clearing prices of the physical capital and human capital, respectively. Then

$$\begin{aligned} r_K &= (1-\alpha) A \left(\frac{H}{L+K}\right)^{\alpha}, \\ r_H(t) &= \alpha A \left(\frac{L+K}{H}\right)^{1-\alpha} \end{aligned} \tag{56}$$

where $L = L_m + \sum_{i\in\text{landowners}} L_i$ is the total land in the state. Furthermore, $I_m$, $I_{l,i}$ defined in (55) have the form

$$\begin{aligned} I_m &= (1-\alpha)\, A \left(\tfrac{H}{L+K}\right)^{\alpha} L_m, \\ I_{l,i} &= (1-\alpha)\, A \left(\tfrac{H}{L+K}\right)^{\alpha} L_{l,i}, \\ I_{c,i} &= (1-\alpha)\, A \left(\tfrac{H}{L+K}\right)^{\alpha} K_{c,i}, \\ I_{w,i} &= \alpha A \left(\tfrac{L+K}{H}\right)^{1-\alpha} H_{w,i}. \end{aligned} \tag{57}$$

Proof.*:* By (55), the optimal demands for physical and human capitals are determined by

$$\begin{aligned} r_K &= (1-\alpha)\, A\, (L_m + K_m)^{-\alpha}\, H_m^{\alpha}, \\ r_H &= \alpha A\, (L_m + K_m)^{1-\alpha}\, H_m^{\alpha-1}. \end{aligned} \tag{58}$$

Similar identities hold if the subscript "$m$" is replaced by "$i$." Hence

$$L_m + K_m = H_m \left[\frac{(1-\alpha)\, A}{r_K}\right]^{1/\alpha} = H_m \left[\frac{\alpha A}{r_H}\right]^{1/(\alpha-1)}. \tag{59}$$

Similar identities hold for $L_i$, $K_i$ and $H_i$. Using the identities

$$\begin{aligned} L_m + \sum L_i &= L, \quad K_m + \sum K_i = K, \\ H_m + \sum H_i &= H \end{aligned}$$

we find

$$L + K = H \left[\frac{(1-\alpha)\, A}{r_K}\right]^{1/\alpha} = H \left[\frac{\alpha A}{r_H}\right]^{1/(\alpha-1)}. \tag{60}$$

These identities lead to (56).

To prove (57), we substitute (58) into (55) to obtain

$$\begin{aligned} I_m &= A\, (L_m + K_m)^{1-\alpha}\, H_m^{\alpha} \\ &\quad - (1-\alpha)\, A\, (L_m + K_m)^{-\alpha}\, H_m^{\alpha} K_m \\ &\quad - \alpha A\, (L_m + K_m)^{1-\alpha}\, H_m^{\alpha} \\ &= (1-\alpha)\, A\, (L_m + K_m)^{-\alpha}\, H_m^{\alpha} L_m. \end{aligned}$$

Since by (59) and (60)

$$\frac{L_m + K_m}{H_m} = \frac{L+K}{H}$$

the first identity in (57) follows. The second identity is proved similarly. The other two identities follow directly from 56. ■

Since

$$\begin{aligned} I_l &= N_l I_{l,i}, \quad I_c = N_c I_{c,i}, \quad I_w = N_w I_{w,i}, \\ L_l &= N_l L_{l,i}, \quad K = N_c K_{c,i}, \quad H = N_w H_{w,i}, \end{aligned}$$

(20) follows from (57) with constants $C_m$, $C_l$, $C_c$ and $C_w$ defined by

$$\begin{aligned} C_m &= (1-\alpha)\, A H^{\alpha} L_m, \quad C_l = (1-\alpha)\, A H^{\alpha} L_l, \\ C_c &= (1-\alpha)\, A H^{\alpha}, \quad C_w = \alpha A H^{\alpha}. \end{aligned} \tag{61}$$

We next give a justification of (25). Assuming as in [12] each individual in the society are identical in preference, which is represented by the utility function

$$v_i = (1-\beta) \ln c_i + \beta \ln (z + b_i)$$

where $c_i$ is the rate of consumption of individual $i$ and $b_i$ is the rate of the individual's bequest for offspring, $\beta \in (0,1)$ indicates the relative weight of bequest in utility, and $z > 0$ is a constant. The budget constraint $0 \le c_i + b_i \le I_i$ applies where $I_i$ is the individual's instantaneous rate of income. It is easy to see that the utility $v_i$ is maximized at

$$c_i^* = I_i - \beta\, [I_i - Z]_+, \quad b_i^* = \beta\, [I_i - Z]_+$$

where $Z = (1-\beta)\, z/\beta$. This proves (25).

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