Properties and Approximation Distribution Reductions in Multigranulation Rough Set Model

Chen Wu, Youquan Xu, Dandan Li, Ronghua Yang, Lijuan Wang, Xibei Yang

Abstract—Some properties of approximation sets are studied in multi-granulation optimist model in rough set theory using maximal compatible classes. The relationships between or among lower and upper approximations in single and multiple granulation are compared and discussed. Through designing Boolean functions and discernibility matrices in incomplete information systems, the lower and upper approximation sets and reduction in multi-granulation environments can be found. By using examples, the correctness of computation approach is consolidated. The related conclusions obtained are suitable for further investigating in multiple granulation RSM.

Keywords—Incomplete information system, maximal compatible class, multi-granulation rough set model, reduction.

I. INTRODUCTION

ROUGH set theory is a useful tool for analyzing and studying information systems concerning imprecise, vague, undetermined knowledge. Since rough set model was proposed by Pawlak in 1982s, it has been widely applied in many scientific areas such as knowledge inference, decision making and pattern classification [1]-[3]. Because the phenomenon of missing attribute values for any object does not exist in complete information systems, an indiscernibility relation or equivalence relation is easy to be built and used to process complete information systems. The results obtained are also intuitive and ideal. But for an incomplete information system (IIS), such an equivalence relation is not conveniently constructed since missing values are not allowed to be compared with real existed values. Some people fill the missing value or null value by statistics or other methods, transforming it into complete system, called indirect method. Other people construct new relations between objects to deal with IIS. These relations may not be equivalence relations. These approaches are called direct methods; for example, tolerance relation [4]; non-symmetric similarity relation [5]; limited tolerant relation [6]. With the granular view of points, maximal consistent block technique for rule acquisition is put forward in [7]; some algorithms for lower and upper approximation sets with maximal compatible classes as primitive granules in IIS are designed in [8]. Other approaches can be read in [9]-[11].

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Nowadays, direct approaches to dispose IIS have been become a hot topic in RSM field because complete information system can be regarded as a special case of IIS [12]. Furthermore, experts now study granule computing and multi-granulation RSM from different viewpoints. Multi-granulation rough set model is suggested in [13]-[15]. The lower and upper assignment reductions in incomplete and inconsistent decision tables are introduced in [16]. Researches on knowledge reductions in inconsistent system using tolerance relation and tolerance classes are also done in [16]. Combining multi-granulation RSM with similarity relation in incomplete information table is also discussed by [17]. So studying on their mixture forms has become an interesting topic in the RSM field. In addition to discussing relationships between or among tolerance relation, non-symmetric similarity relation, limited relation and etc., constructing useful relations on the universe from incomplete information systems by exerting strict condition on them becomes current development tendencies. A chief aim is at breaking through the limitation of traditional RSM and extending study of tolerance relation instead of using generalized decision rules [4].

The present paper engages some work in defining lower and upper approximation sets (to some extent, it is a kind of optimistic multi-granulation RSM) in the case of one subset of attributes and those in the multi-granulation RSM (MGRSM) using maximal compatible classes as granules of tolerance relation to enhance the processing ability for IIS. The solving approaches for lower and upper approximation distribution reductions in MGRSM view are suggested. Properties of distribution approximations and relationships between single and multi-granulation RSM are investigated. The main goal of it is to obtain some useful and related results through analyzing the lower and upper approximation distribution law in MGRSM. A method to acquire decision rules from consistent IIS is presented. Through proofs, examples and experiments, the method is verified to be correct. So this knowledge acquisition approach is meaningful.

II. DEFINITIONS AND CONCEPTS

Maximal compatible classes as primitive granules and multi-granulation approach are our important concepts, so we first give some related definitions and explain some terminology, and then introduce models.

A quadruple $IIS = (U, A, V, f)$ is called an IIS (see the definitions in [4]) where all elements are defined.

$$SIM(B) = \{(x, y) \in U \times U : \forall a \in B, f(x, a) = f(y, a)$$

$\vee f(x,a) = * \vee f(y,a) = *\}$ is called a tolerance relation, where $B \subseteq A$. $O(B) = \{X \subseteq U : \max\{X^2 \subseteq SIM(B)\}\}$ is called complete cover with compatible classes, where max means conclusion $\subseteq$. $O(B)$ is also called a knowledge expression system over U. $O_B(x) = \max\{X : x \in X, X^2 \subseteq SIM(B)\}$ is called compatible class(es) containing x where $x \in U$.

$C \in O(B)$ is viewed as primitive granule presently. Arbitrarily, two elements in $C \in O(B)$ are mutually compatible, not like in tolerance class in [4], maybe not mutually compatible. $O(B) = \cup_{x \in U} O_B(x)$ is obvious.

Definition 1. Let $IIS = (U, A, V, f)$ be an IIS, $B \subseteq A$, $X \subseteq U$. Then

$$\overline{B}(X) = \cup_{C \in O(B), C \cap X \neq \varnothing} C, \underline{B}(X) = \cup_{C \in O(B), C \subseteq X} C$$

are called upper and lower approximations for X in knowledge expression system $O(B)$ respectively.

$\alpha_B(X) = |\underline{B}(X)| / |\overline{B}(X)|$ is called the approximation precision.

In [13]-[15], Qian et al. propose optimistic and pessimistic multi-granulation rough set models (MGRSM). Combined with the above compatible granules, we can introduce the above definition and concepts into MGRSMs using complete cover $O(B)$ and then get some related new results.

Definition 2. Let $IIS = (U, A, V, f)$ be an IIS, $B_1, B_2, ..., B_m \subseteq A$ be m attribute subsets, $M = \{1, 2, ..., m\}$, $X \subseteq U$. Then the optimistic multi-granulation lower approximation of X is referred to

$$\sum_{i=1}^m \underline{B_i}(X) = \{x \in U : \exists i \in M(x \in \underline{B_i}(X))\};$$

the upper is $\overline{\sum_{i=1}^m B_i}(X) = \sim \sum_{i=1}^m \underline{B_i}(\sim X)$.

$Bn_{\sum_{i=1}^m B_i}(X) = \overline{\sum_{i=1}^m B_i}(X) - \sum_{i=1}^m \underline{B_i}(X)$ is called the optimistic multi-granulation boundary region of X. $|\sum_{i=1}^m \underline{B_i}(X)| / |\overline{\sum_{i=1}^m B_i}(X)|$ is called approximation precision.

III. MAIN FEATURES AND RELATIONSHIPS

An equivalent calculation method of lower and upper approximations in Definition 1 is given in the following theorem.

Theorem 1. Let $IIS = (U, A, V, f)$ be an IIS, $B \subseteq A$. Then

1. $\overline{B}(X) = \{x \in U : \exists C \in O(B)(x \in C \wedge (C \cap X \neq \varnothing))\}$;
2. $\underline{B}(X) = \{x \in U : \exists C \in O(B)(x \in C \wedge C \subseteq X)\}$.

Proof.

(1) $y \in \{x \in U : \exists C \in O(B)(x \in C \wedge C \cap X \neq \varnothing)\}$
$\Rightarrow y \in \cup_{C \in O(B), C \cap X \neq \varnothing} C$. Thus $\{x \in U : \exists C \in O(B)$
$\subseteq \cup_{C \in O(B), C \cap X \neq \varnothing} C (x \in C \wedge C \cap X \neq \varnothing)\}$
$y \in \cup_{C \in O(B), C \cap X \neq \varnothing} C \Rightarrow \exists C \in O(B)(C \cap X \neq \varnothing)$.
So $y \in C$. Therefore,

$$y \in \{x \in U : \exists C \in O(B)(x \in C \wedge C \cap X \neq \varnothing)\}$$

$\cup_{C \in O(B), C \cap X \neq \varnothing} C \subseteq \{x \in U : \exists C \in O(B)(x \in C \wedge C \cap X \neq \varnothing)\}$.

Hence

$$\cup_{C \in O(B), C \cap X \neq \varnothing} C = \{x \in U : \exists C \in O(B)(x \in C$$
$$\wedge C \cap X \neq \varnothing)\} = \overline{B}(X). \text{ So}$$

$$\overline{B}(X) = \{x \in U : \exists C \in O(B)(x \in C \wedge (C \cap X \neq \varnothing))\}.$$

(2) $y \in \underline{B}(X) = \{x \in U : \exists C \in O(B)(x \in C \wedge C \subseteq X)\}$
$\Rightarrow \exists C \in O(B)(y \in C \wedge C \subseteq X) \Rightarrow y \in \cup_{C \in O(B), C \subseteq X} C$.
Thus $\{x \in U : \exists C \in O(B)(x \in C \wedge C \subseteq X)\} \subseteq \cup_{C \in O(B), C \subseteq X} C$.
$\forall y \in \cup_{C \in O(B), C \subseteq X} C \Rightarrow \exists C \in O(B) (C \subseteq X)$ such that $y \in C$.
Thus $y \in \{x \in U : \exists C \in O(B)(x \in C \wedge C \subseteq X)\} = \underline{B}(X)$. That is, $\cup_{C \in V(B), C \subseteq X} C \subseteq \{x \in U : \exists C \in O(B)(x \in C \wedge C \subseteq X)\}$.
So $\cup_{C \in C(A), C \subseteq X} C = \{x \in U : \exists C \in O(B)(x \in C \wedge C \subseteq X)\} = \underline{B}(X)$.

This theorem means that $\underline{B}(X)$ can be expressed by

$$\underline{B}(X) = \{x \in U : \exists C \in O(B)(x \in C \wedge C \subseteq X)\}$$

The upper approximation of a subset X can be calculated by lower approximation as in the following theorem.

Theorem 2. Let $IIS = (U, A, V, f)$ be IIS, $B \subseteq A, X \subseteq U$. Then $\sim \underline{B}(\sim X) = \overline{B}(X)$.

Proof. $y \in \sim \underline{B}(\sim X) \Leftrightarrow y \in \cup_{C \in O(B), C \cap X \neq \varnothing} C = \overline{B}(X)$.

Lemma. $y \in \sim \underline{B}(\sim X) \Leftrightarrow y \in \cup_{C \in O(B), C \cap X \neq \varnothing} C = \overline{B}(X)$.

The upper approximation of $X \subseteq U$ also complies with the following theorem.

Theorem 3. $\overline{B}(X) = \{x \in U : \forall C \in O(B)(x \in C \rightarrow C \cap X \neq \varnothing)\}$.

Proof. $y \in \sim \underline{B}(\sim X) \Leftrightarrow \forall C \in O(B)(y \in C \rightarrow C \cap X \neq \varnothing)$. So $\overline{B}(X) = \{x \in U : \forall C \in O(B)(x \in C \rightarrow C \cap X \neq \varnothing)\}$.

We can obtain properties of lower and upper approximation for the union or intersection of a series of subsets as follows.

Theorem 4. Let $IIS = (U, A, V, f)$ be an IIS, $B \subseteq A, M = 1, 2, ..., k$; $X_i \subseteq U$ $(i \in M)$. Then

1. $\underline{B}(\cap_{i=1}^{k} X_i) \subseteq \cap_{i=1}^{k} \underline{B}(X_i)$;
2. $\cup_{i=1}^{k} \underline{B}(X_i) \subseteq \underline{B}(\cup_{i=1}^{k} X_i)$;
3. $\overline{B}(\cap_{i=1}^{k} X_i) \subseteq \cap_{i=1}^{k} \overline{B}(X_i)$;
4. $\overline{B}(\cup_{i=1}^{k} X_i) = \cup_{i=1}^{k} \overline{B}(X_i)$

Proof.

(1) $\forall y \in \underline{B}(\cap_{i=1}^{k} X_i) \Rightarrow y \in \cup_{C \in O(B), C \subseteq \cap_{i=1}^{k} X_i} C$

$\Rightarrow y \in \cap_{i=1}^{k} \underline{B}(X_i)$. Thus $\underline{B}(\cap_{i=1}^{k} X_i) \subseteq \cap_{i=1}^{k} \underline{B}(X_i)$.

(2) $\forall y \in \cup_{i=1}^{k} \underline{B}(X_i) \Rightarrow \exists i \in M, y \in \underline{B}(X_i)$

$\Rightarrow y \in \underline{B}(\cup_{i=1}^{k} X_i)$. Thus $\cup_{i=1}^{k} \underline{B}(X_i) \subseteq \underline{B}(\cup_{i=1}^{k} X_i)$.

(3) $\overline{B}(\cap_{i=1}^{k} X_i) = \cup_{C \in O(B), C \cap (\cap_{i=1}^{k} X_i) \neq \varnothing} C$, so

$\forall y \in \cup_{C \in O(B), C \cap (\cap_{i=1}^{k} X_i) \neq \varnothing} C \Rightarrow y \in \cap_{i=1}^{k} \overline{B}(X_i)$.

So $\overline{B}(\cap_{i=1}^{k} X_i) \subseteq \cap_{i=1}^{k} \overline{B}(X_i)$.

(4) $y \in \cup_{i=1}^{k} \overline{B}(X_i) \Leftrightarrow y \in \overline{B}(\cup_{i=1}^{k} X_i)$.

Thus $\cup_{i=1}^{k} \overline{B}(X_i) \subseteq \overline{B}(\cup_{i=1}^{k} X_i)$.

$\forall y \in \overline{B}(\cup_{i=1}^{k} X_i) \Rightarrow y \in \cup_{i=1}^{k} \overline{B}(X_i)$.

Thus $\overline{B}(\cup_{i=1}^{k} X_i) \subseteq \cup_{i=1}^{k} \overline{B}(X_i)$.

So $\overline{B}(\cup_{i=1}^{k} X_i) = \overline{B}(\cup_{i=1}^{k} X_i)$.

An equivalent expression about the upper approximation of a subset in MGRSM is as follows:

Theorem 5. $\overline{\sum_{i=1}^{m} B_i}(X) = \{x \in U : \forall i \in M(\forall C \in O(B_i)$
$(x \in C \to C \cap X \neq \varnothing))\}$

Proof. $\overline{\sum_{i=1}^{m} B_i}(X) = \sim \underline{\sum_{i=1}^{m} B_i}(\sim X)$, so $y \in \overline{\sum_{i=1}^{m} B_i}(X)$

$\Leftrightarrow y \in \sim \underline{\sum_{i=1}^{m} B_i}(\sim X) \Leftrightarrow y \notin$

$\{x \in U : \exists i \in M(\exists C \in O(B_i)(x \in C \land (C \cap X = \varnothing)))\}$.

Since $\neg \exists i \in M(\exists C \in O(B_i)(y \in C \land (C \cap X = \varnothing)))$

$\Leftrightarrow \forall i \in M(\forall C \in O(B_i)(y \in C \to C \cap X \neq \varnothing))$.

Thus, $\sim \underline{\sum_{i=1}^{m} B_i}(\sim X) = \{x \in U : \forall i \in M(\forall C \in O(B_i)$

$(x \in C \to C \cap X \neq \varnothing))\}$.

Another equivalent expression about the upper approximation in MGRSM is illustrated as follows.

Theorem 6. $\overline{\sum_{i=1}^{m} B_i}(X) = \{x \in U : \forall i \in M(x \in \overline{B_i}(X))\}$

Proof. $\overline{\sum_{i=1}^{m} B_i}(X) = \sim \underline{\sum_{i=1}^{m} B_i}(\sim X)$, so we have $y \in$

$\overline{\sum_{i=1}^{m} B_i}(X) \Leftrightarrow y \in \sim \underline{\sum_{i=1}^{m} B_i}(\sim X)$

$\Leftrightarrow \{x \in U : \exists i \in M(x \in \underline{B_i}(\sim X))\}$.

$\neg \exists i \in M(x \in \underline{B_i}(\sim X)) \Leftrightarrow \forall i \in M(x \in \overline{B_i}(X))$.

Thus,

$y \in \overline{\sum_{i=1}^{m} B_i}(X) \Leftrightarrow y \in \{x \in U : \forall i \in M(x \in \overline{B_i}(X))\}$.

So

$$\overline{\sum_{i=1}^{m} B_i}(X) = \{x \in U : \forall i \in M(x \in \overline{B_i}(X))\}.$$

The lower approximation of a subset in MGRSM can be computed by the union of the lower approximations of it in single granulation models, while the upper approximation can be calculated by the intersection of the upper approximations.

Theorem 7. Let $IIS = (U, A, V, f)$ be an IIS, $B_i \subseteq A$ $(i = 1, 2, ..., m)$, $X \subseteq U$. Then

1. $\underline{\sum_{i=1}^{m} B_i}(X) = \cup_{i=1}^{m} \underline{B_i}(X)$;

2. $\overline{\sum_{i=1}^{m} B_i}(X) = \cap_{i=1}^{m} \overline{B_i}(X)$.

Proof.

(1) $\underline{\sum_{i=1}^{m} B_i}(X) = \{x \in U : \exists i \in M(\exists C \in O(B_i)$
$(x \in C \land (C \subseteq X)))\}$.

$y \in \underline{\sum_{i=1}^{m} B_i}(X) \Rightarrow y \in \cup_{i=1}^{m} \underline{B_i}(X)$.

$\forall y \in \cup_{i=1}^{m} \underline{B_i}(X) \Rightarrow y \in \underline{\sum_{i=1}^{m} B_i}(X)$.

So, $\underline{\sum_{i=1}^{m} B_i}(X) = \cup_{i=1}^{m} \underline{B_i}(X)$.

(2) $\overline{\sum_{i=1}^{m} B_i}(X) = \{x \in U : \forall i \in M(x \in \overline{B_i}(X))\}$. So,

$$\overline{\sum_{i=1}^{m} B_i}(X) = \cap_{i=1}^{m} \overline{B_i}(X).$$

Theorem 8. Let $IIS = (U, A, V, f)$ be an IIS and $B_i \subseteq A(i = 1, 2, ..., m)$, $X \subseteq U$. Then

(1) $\underline{\sum_{i=1}^{m} B_i}(X) \subseteq X \subseteq \overline{\sum_{i=1}^{m} B_i}(X)$;

(2) $\underline{\sum_{i=1}^{m} B_i}(\varnothing) = \overline{\sum_{i=1}^{m} B_i}(\varnothing) = \varnothing$,

$\underline{\sum_{i=1}^{m} B_i}(U) = \overline{\sum_{i=1}^{m} B_i}(U) = U$;

(3) $\underline{\sum_{i=1}^{m} B_i}(\underline{\sum_{i=1}^{m} B_i}(X)) = \underline{\sum_{i=1}^{m} B_i}(X)$,

$\overline{\sum_{i=1}^{m} B_i}(\overline{\sum_{i=1}^{m} B_i}(X)) = \overline{\sum_{i=1}^{m} B_i}(X)$

Proof. Here we only prove (3).

Because $\underline{\sum_{i=1}^{m} B_i}(X) \subseteq X$ for any $X \subseteq U$,

$\underline{\sum_{i=1}^{m} B_i}(\underline{\sum_{i=1}^{m} B_i}(X)) \subseteq \underline{\sum_{i=1}^{m} B_i}(X)$. So, we just need to prove $\underline{\sum_{i=1}^{m} B_i}(X) \subseteq \underline{\sum_{i=1}^{m} B_i}(\underline{\sum_{i=1}^{m} B_i}(X))$.

For $\forall x \notin \underline{\sum_{i=1}^{m} B_i}(\underline{\sum_{i=1}^{m} B_i}(X))$, we have

$$\forall i \forall C \in O(B_i)(x \in C \to C \not\subseteq \underline{\sum_{i=1}^{m} B_i}(X)) \Rightarrow C \not\subseteq X.$$

Thus,

$$x \notin \underline{\sum_{i=1}^{m} B_i}(X).$$

$$\underline{\sum_{i=1}^{m} B_i}(X) \subseteq \underline{\sum_{i=1}^{m} B_i}(\underline{\sum_{i=1}^{m} B_i}(X)).$$

Conclusively,

$$\underline{\sum_{i=1}^{m} B_i}(\underline{\sum_{i=1}^{m} B_i}(X)) = \underline{\sum_{i=1}^{m} B_i}(X).$$

Next, since $X \subseteq \overline{\sum_{i=1}^{m} B_i}(X)$, thus $\overline{\sum_{i=1}^{m} B_i}(X) \subseteq \overline{\sum_{i=1}^{m} B_i}(\overline{\sum_{i=1}^{m} B_i}(X))$. For $\forall x \in \overline{\sum_{i=1}^{m} B_i}(\overline{\sum_{i=1}^{m} B_i}(X))$, it follows $x \in \sim \underline{\sum_{i=1}^{m} B_i}(\sim \overline{\sum_{i=1}^{m} B_i}(X)) \Rightarrow x \in \overline{\sum_{i=1}^{m} B_i}(X)$. Therefore, $\overline{\sum_{i=1}^{m} B_i}(\overline{\sum_{i=1}^{m} B_i}(X)) = \overline{\sum_{i=1}^{m} B_i}(X)$.

TABLE I
AN IIS FOR CARS

U	P	M	S	X	D
1	high	low	full	low	good
2	low	*	full	low	good
3	*	*	compact	low	poor
4	high	*	full	high	good
5	*	*	full	high	excellent
6	low	high	full	*	good

The lower and upper approximations of a set of subsets in MGRSM have relationships between their lower and upper approximations in single granulation model as follows.

Theorem 9. Let $IIS = (U, A, V, f)$ be an IIS, $B_i \subseteq A (i = 1, 2, ..., m), X_j \subseteq U (j = 1, 2, ..., k)$. Then

(1) $\underline{\sum_{i=1}^{m} B_i}(\cap_{j=1}^{k} X_j) = \cup_{i=1}^{m} \underline{B_i}(\cap_{j=1}^{k} X_j) \subseteq \cup_{i=1}^{m}(\cap_{j=1}^{k} \underline{B_i}(X_j))$;

(2) $\underline{\sum_{i=1}^{m} B_i}(\cup_{j=1}^{k} X_j) = \cup_{i=1}^{m} \underline{B_i}(\cup_{j=1}^{k} X_j) \supseteq \cup_{i=1}^{m}(\cup_{j=1}^{k} \underline{B_i}(X_j))$

(3) $\overline{\sum_{i=1}^{m} B_i}(\cap_{j=1}^{k} X_j) = \cap_{i=1}^{m} \overline{B_i}(\cap_{j=1}^{k} X_j) \subseteq \cap_{i=1}^{m}(\cap_{j=1}^{k} \overline{B_i}(X_j))$;

(4) $\overline{\sum_{i=1}^{m} B_i}(\cup_{j=1}^{k} X_j) = \cap_{i=1}^{m} \overline{B_i}(\cup_{j=1}^{k} X_j) = \cap_{i=1}^{m}(\cup_{j=1}^{k} \overline{B_i}(X_j))$.

This theorem can be proven according to theorem 4 and theorem 7. Here the proof is omitted for saving space.

Example 1. Table I shows an IIS, in which Price, Mileage, Size, Max-Speed are all condition attributes, $d = D$ is a decision attribute. Use P, M, S, X to denote Price, Mileage, Size, Max-Speed respectively in Table I [4].

Let $B = \{P, M, S, X\}$. $O(B) = \{\{1\}, \{2,6\}, \{3\}, \{4,5\}, \{5,6\}\}$. $O_B(1) = \{\{1\}\}$, $O_B(2) = \{\{2,6\}\}$, $O_B(3) = \{\{3\}\}$, $O_B(4) = \{\{4,5\}\}$, $O_B(5) = \{\{4,5\}, \{5,6\}\}$, $O_B(6) = \{\{2,6\}, \{5,6\}\}$. $X = d_{good} = \{1,2,4, 6\}$. We obtain: $\underline{B}(X) = \{1,2,6\}$, $\overline{B}(X) = \{1,2,4,5,6\}$.

$$\alpha_B(X) = |\underline{B}(X)| / |\overline{B}(X)| = 3/5 = 0.5.$$

Example 2. In Table I, let $B_1 = \{P, M\}$, $B_2 = \{S, X\}$, $B_3 = \{M, X\}$. Then $O(B_1) = \{\{1,3,4,5\}, \{2,3,5,6\}\}$, $O(B_2) = \{\{1,2,6\}, \{3\}, \{4,5, 6\}\}$, $O(B_3) = \{\{1,2,3\}, \{2,3,6\}, \{4,5,6\}\}$. $\underline{\sum_{i=1}^{m} B_i}(X) = \cup \underline{B_i}(X) = \{1,2,6\}$, $\overline{\sum_{i=1}^{m} B_i}(X) = \cup_{i=1}^{m} \overline{B_i}(X) = \{1,2,3,4, 5,6\}$.

The optimistic approximation precision in MGRSM is

$$|\underline{\sum_{i=1}^{m} B_i}(X)| / |\overline{\sum_{i=1}^{m} B_i}(X)| = 3/6 = 0.5.$$

IV. Approximation Distribution Reduction

Now we discuss approximation distribution reduction in MGRSM based on our granules--maximal compatible classes and suggest upper and lower approximation distribution reductions and then solve simplest certain and possible decision rules in incomplete decision table.

Definition 4. Let $IIS = (U, A \cup \{d\}, V, f)$ be an incomplete decision table, $B = \{b_1, b_2, ..., b_m\} \subseteq A$. Then

$$\underline{B}(\{d\}) = \{\underline{\sum_{i=1}^{m} b_i}(X_1), \underline{\sum_{i=1}^{m} b_i}(X_2), ..., \underline{\sum_{i=1}^{m} b_i}(X_l)\},$$

$$\overline{B}(\{d\}) = \{\overline{\sum_{i=1}^{m} b_i}(X_1), \overline{\sum_{i=1}^{m} b_i}(X_2), ..., \overline{\sum_{i=1}^{m} b_i}(X_l)\}$$

are called forming the lower and upper approximations of decision class family referring to B in MGRSM respectively. If $B = \{c_1, c_2, ..., c_n\}$, then:

$$\underline{B}(\{d\}) = \{\underline{\sum_{i=1}^{n} c_i}(X_1), \underline{\sum_{i=1}^{n} c_i}(X_2), ..., \underline{\sum_{i=1}^{n} c_i}(X_l)\},$$

$$\overline{B}(\{d\}) = \{\overline{\sum_{i=1}^{n} c_i}(X_1), \overline{\sum_{i=1}^{n} c_i}(X_2), ..., \overline{\sum_{i=1}^{n} c_i}(X_l)\}.$$

The lower and upper approximation distribution reductions in MGRSM are defined in the following definitions, respectively.

Definition 5. Let $IIS = (U, A \cup \{d\}, V, f)$ be an incomplete decision table, $B = \{b_1, b_2, ..., b_m\} \subseteq A$, then B is called a lower approximation distribution reduction in MGRSM if, and only if $\underline{B}(\{d\}) = \underline{A}(\{d\})$, $\forall C \subset B$, $\underline{C}(\{d\}) \neq \underline{A}(\{d\})$. B is called a upper approximation distribution reduction if, and only if $\overline{B}(\{d\}) = \overline{A}(\{d\})$, $\forall C \subset B$, $\overline{C}(\{d\}) \neq \overline{A}(\{d\})$.

Definition 6. Let $IIS = (U, A \cup \{d\}, V, f)$ be an incomplete decision table, $B = \{b_1, b_2, ..., b_m\} \subseteq A$, $x \in U$. Then

$$\overline{B}(x) = \{X \in U / IND(\{d\}) : x \in \overline{\sum_{i=1}^{m} b_i(X)}\}, \text{ and}$$

$$\underline{B}(x) = \{X \in U / IND(\{d\}) : x \in \underline{\sum_{i=1}^{m} b_i(X)}\}.$$

Theorem 10. Let $IIS = (U, A \cup \{d\}, V, f)$ be an incomplete decision table, $B \subseteq A$. Then
(1) $\underline{B}(\{d\}) = \underline{A}(\{d\})$ iff $\forall x \in U(\underline{B}(x) = \underline{A}(x))$;
(2) $\overline{B}(\{d\}) = \overline{A}(\{d\})$ iff $\forall x \in U(\overline{B}(x) = \overline{A}(x))$.

Proof.
(1) "$\Rightarrow$": Since $\underline{B}(\{d\}) = \underline{A}(\{d\})$, $\forall X_k \in U / IND(\{d\})$,

$\underline{\sum_{j=1}^{m} b_i(X_k)} = \underline{\sum_{i=1}^{n} c_i(X_k)}$, that is, $\forall x \in U$,

$x \in \underline{\sum_{j=1}^{m} b_i(X_k)} \Leftrightarrow x \in \underline{\sum_{i=1}^{n} c_i(X_k)}$ and

$x \notin \underline{\sum_{j=1}^{m} b_i(X_k)} \Leftrightarrow x \notin \underline{\sum_{i=1}^{n} c_i(X_k)}$. Thus

$\underline{B}(x) = \underline{A}(x)$ for $\forall x \in U$. "$\Leftarrow$": Since $\underline{B}(x) = \underline{A}(x)$ for

$\forall x \in U$, $\underline{\sum_{j=1}^{m} b_i(X_k)} = \underline{\sum_{i=1}^{n} c_i(X_k)}$ for

$\forall X_k \in U / IND(\{d\})$, thus $\underline{B}(\{d\}) = \underline{A}(\{d\})$.

The proof of (2) is similar to that of (1).

Theorem 10 is useful for finding the lower and upper distribution reduction of x. But it is not so easy. Now we give an easier method.

Theorem 11. Let $IIS = (U, A \cup \{d\}, V, f)$ be an incomplete decision table, $B \subseteq A$. Then
(1) $\forall x \in U(\underline{B}(x) = \underline{A}(x)) \Leftrightarrow \exists a \in B(\underline{A}(x) \not\subset \underline{A}(y)$
$\rightarrow \exists C \in O(\{a\}), x \in C, \exists c \in A, C' \in O(\{c\}), y \in C, C \not\subseteq C')$
(2) $\forall x \in U(\overline{B}(x) = \overline{A}(x)) \Leftrightarrow \exists a \in B(\overline{A}(y) \not\subseteq \overline{A}(x) \rightarrow$

$\exists C \in O(\{a\}), x \in C, \exists c \in A, C' \in O(\{c\}), y \in C', C' \not\subseteq C$),

$\forall X_k \in \overline{B}(x) \forall a \in A \forall C \in O(\{a\})(x \in C \rightarrow C \cap X_k \neq \emptyset)$
Proof.
(1) "$\Rightarrow$": From $\underline{A}(x) \not\subseteq \underline{A}(y)$, there exists $X_k \in U / IND(\{d\})$ such that $X_k \in \underline{A}(x)$ and $X_k \notin \underline{A}(y)$. Because $\underline{B}(x) = \underline{A}(x)$ for $\forall x \in U$, then

$X_k \in \underline{B}(x) = \{X \in U / IND(\{d\}) : x \in \underline{\sum_{i=1}^{m} b_i(X)}\}$ and

$X_k \notin \underline{B}(y) = \{X \in U / IND(\{d\}) : y \in \underline{\sum_{i=1}^{m} b_i(X)}\}$. Therefore

$\exists a \in B$, $x \in \underline{a}(X_k)$ and $\forall c \in B$, $y \notin \underline{c}(X_k) \Rightarrow \exists a \in B$,

$\exists C \in O(\{a\}), x \in C$, $\exists c \in B$, $\exists C' \in O(\{c\}), y \in C', C \not\subseteq C'$.
"$\Leftarrow$": Since $B = \{b_1, b_2, \cdots, b_m\} \subseteq A = \{a_1, a_2, \cdots, a_n\}$,

$\forall X_k \in U / IND(\{d\})$, we have $\underline{\sum_{j=1}^{m} b_j(X_k)} = \{x \in U : \exists j \in \{1, 2, ..., m\}, C \in O(\{b_j\}), x \in C, C \subseteq X_k\}$,

$\underline{\sum_{i=1}^{n} c_i(X_k)} = \{x \in U : \exists j \in \{1, 2, ..., n\}, C \in O(\{c_j\}), x \in C, C \subseteq X_k\}$. Because $B = \{b_1, b_2, \cdots, b_m\} \subseteq \{a_1, a_2, ..., a_n\} = A$, $\underline{\sum_{j=1}^{m} b_j(X_k)} \subseteq \underline{\sum_{i=1}^{n} c_i(X_k)}$. Therefore, we have $\underline{B}(x) \subseteq \underline{A}(x)$ for $\forall x \subseteq U$. So in the following we only have to prove that $\underline{B}(x) \supseteq \underline{A}(x)$ for $\forall x \subseteq U$.

Due to the given condition, we have

$\exists a \in B, \exists C \in O(\{a\}), x \in C, \quad \exists c \in B, \exists C' \in O(\{c\}), y \in C', C \not\subseteq C'$
$\Rightarrow \exists a \in B, \exists C \in O(\{a\}), x \in C, \exists c \in A, \exists C' \in O(\{c\}), y \in C', C \not\subseteq C'$.

$\neg(\exists a \in B, \exists C \in O(\{a\}), x \in C, \exists c \in A, \exists C' \in O(\{c\}), y \in C', C \not\subseteq C') \Leftrightarrow \forall a \in B, \forall C \in O(\{a\}), x \in C,$
$\forall c \in A, \forall C' \in O(\{c\}), y \in C, C \subseteq C' \Rightarrow \underline{A}(x) \subseteq \underline{A}(y)$.

For $\forall X_k \in \underline{A}(x) = \{X \in U / IND(\{d\}) : x \in \underline{\sum_{i=1}^{n} c_i(X)}\}$
$\Rightarrow X_k \in \underline{A}(y) \Rightarrow \exists c' \in A, \exists C'' \in O(\{c'\}), y \in C'', C'' \subseteq X_k$.
$\because \forall a \in B, \forall C \in O(\{a\}), x \in C, \forall c \in A, \forall C' \in O(\{c\}), y \in C', C \subseteq C'$
$\therefore \forall a \in B, \forall C \in O(\{a\}), x \in C, \exists c' \in A, \exists C'' \in O(\{c'\}), y \in C'', C \subseteq C'' \subseteq X_k \Rightarrow x \in \underline{B}(X_k) \Rightarrow X_k \in \underline{B}(x)$.

(2) "$\Rightarrow$": Suppose $\overline{A}(y) \not\subseteq \overline{A}(x)$. Then there must exist an $X_k \in U / IND(\{d\})$ such that $X_k \in \overline{A}(y)$ and $X_k \notin \overline{A}(x)$. Because $\overline{B}(x) = \overline{A}(x)$ for $\forall x \in U$, thus $X_k \in \overline{B}(y)$ and $X_k \notin \overline{B}(x)$.

$X_k \in \overline{B}(y) = \{X \in U / IND(\{d\}) : y \in \overline{B}(X)\}$
$= \{X \in U / IND(\{d\}) : y \in \cap_{a \in A} \overline{a}(X)\}$
$\Rightarrow \forall a \in B \forall C' \in O(\{a\})(y \in C' \rightarrow C' \cap X_k \neq \emptyset)$
$X_k \notin \overline{B}(x) = \{X \in U / IND(\{d\}) : x \in \overline{B}(X)\}$
$\Rightarrow \exists a \in B \exists C \in O(\{a\})(x \in C \wedge C \cap X_k = \emptyset)$.

So, summing up the above two results, we have

$\exists a \in B \exists C \in O(\{a\}) \exists C' \in O(\{a\})(x \in C \wedge y \in C' \wedge C' \not\subseteq C)$.

For $\forall X_k \in \overline{B}(x) = \overline{A}(x)$

$\Rightarrow \forall a \in A \forall C \in O(\{a\})(x \in C \rightarrow C \cap X_k \neq \emptyset)$

" $\Leftarrow$ ": For $B \subseteq A, \overline{B}(x) = \{X_k \in U / IND(\{d\}) : x \in \overline{B}(X_k)\}$

$= \{X_k \in U / IND(\{d\}) : x \in \cap_{a \in A} \overline{a}(X_k)\}$,

$\overline{A}(x) = \{X_k \in U / IND(\{d\}) : x \in \overline{A}(X_k)\}$

$= \{X_k \in U / IND(\{d\}) : x \in \cap_{a \in A} \overline{a}(X_k)\}$, $\cap_{a \in A} \overline{a}(X_k)$

$\subseteq \cap_{a \in B} \overline{a}(X_k)$, therefore, $\overline{A}(x) \subseteq \overline{B}(x)$ for $\forall x \subseteq U$.

In the following we are going to prove $\overline{B}(x) \subseteq \overline{A}(x)$ for $\forall x \subseteq U$ under given condition. From given condition, we have:

$\neg \exists a \in B \exists C \in O(\{a\}) \exists C' \in O(\{a\})(x \in C \wedge y \in C' \wedge C' \not\subseteq C)$

$\Rightarrow \overline{A}(y) \subseteq \overline{A}(x)$, i.e. $\forall a \in B \forall C \in O(\{a\}) \forall C' \in O(\{a\})$

$(x \in C \wedge y \in C' \rightarrow C' \subseteq C) \Rightarrow \overline{A}(y) \subseteq \overline{A}(x)$

$O(\{a\})$ is a complete cover, so $\forall a \in B$ $\forall C \in O(\{a\})$ $\forall C' \in O(\{a\})(x \in C \wedge y \in C' \rightarrow C' \subseteq C)$ equals to $\forall a \in B \forall C \in O(\{a\}) \forall C' \in O(\{a\})(x \in C \wedge y \in C' \rightarrow C' = C)$. Thus, $\overline{B}(x) = \overline{B}(y)$.

If $X_k \in \overline{B}(x)$, then $X_k \notin \overline{A}(x)$.

$X_k \in \overline{B}(x) \Rightarrow \forall a \in B \forall C \in O(\{a\})(x \in C \rightarrow C \cap X_k \neq \varnothing)$.

$X_k \notin \overline{A}(x) \Rightarrow X_k \notin \{X \in U / IND(\{d\}) : x \in \overline{A}(X)\}$

$\Rightarrow \exists a \in A \exists C \in O(\{a\})(x \in C \wedge C \cap X_k = \varnothing)$.

From $\forall a \in B \forall C \in O(\{a\}) \forall C' \in O(\{a\})$ $(x \in C \wedge y \in C' \rightarrow C' = C)$, we obtain $\overline{B}(x) = \overline{B}(y)$. So, $\forall a \in B \forall C \in O(\{a\})(x \in C \rightarrow C \cap X_k \neq \varnothing)$ implies that we do not have $\exists a \in B \exists C \in O(\{a\})$ $(x \in C \wedge C \cap X_k = \varnothing)$, but only have $\exists a \in A - B$ $\exists C' \in O(\{a\})(y \in C' \wedge C' \cap X_k = \varnothing)$.

From $\forall X_k \in \overline{B}(x) \forall a \in A \forall C \in O(\{a\})(x \in C \rightarrow C \cap X_k \neq \varnothing)$, we get a contradiction. This means $X_k \notin \overline{A}(x)$. Therefore, $X_k \in \overline{A}(x)$. So $\overline{B}(x) \subseteq \overline{A}(x)$. Conclusively, we obtain $\overline{B}(x) = \overline{A}(x)$.

V. FINDING APPROXIMATION REDUCTIONS

So as to solve the lower or upper distribution approximation reductions from MGRSM with granules maximal compatible classes, we build discernibility matrices first and then choose discernible attributes.

Definition 7. Let $IIS = (U, A, V, f)$ be an incomplete decision table. We define

$R_{MLW} = \{(x, y) \in U^2 : \underline{A}(x) \not\subseteq \underline{A}(y)\}$,

$D_{MLW}(x, y) = \begin{cases} \{a \in A : \exists C, C' \in O(\{a\})(x \in C \\ \quad \wedge y \in C' \wedge C \not\subseteq C') \wedge (x, y) \in R_{MLW}\} \\ \varnothing, \qquad\qquad\qquad \text{otherwise} \end{cases}$

$M_{MLW} = \{D_{MLW}(x, y) : x, y \in U\}$

$R_{MUP} = \{(x, y) \in U^2 : \overline{A}(y) \not\subseteq \overline{A}(x)\}$,

$D_{MUP}(x, y) = \begin{cases} \{a \in A : \exists C, C' \in O(\{a\})(x \in C \\ \quad \wedge y \in C' \wedge C' \not\subseteq C), \forall X_k \in \overline{A}(x) \\ \forall a \in A \forall C \in V(\{a\})(x \in C \\ \quad \rightarrow C \cap X_k \neq \varnothing), (x, y) \in R_{MUP}\} \\ \varnothing, \qquad\qquad\qquad \text{otherwise} \end{cases}$

$M_{MUP} = \{D_{MUP}(x, y) : x, y \in U\}$

$D_{MLW}(x, y)$ is called the lower discernible attribute subset and $D_{MUP}(x, y)$ is the upper one; M_{MLW} is called the lower approximation distribution discernible matrix and M_{MUP} is the upper one.

Theorem 12. Let $IIS = (U, A \cup \{d\}, V, f)$ be an IIS, $B \subseteq A$. Then

(i) $\underline{B}(\{d\}) = \underline{A}(\{d\})$ $\Leftrightarrow$ if $D_{MLW}(x, y) \neq \varnothing$, then $B \cap D_{MLW}(x, y) \neq \varnothing$;

(ii) $\overline{B}(\{d\}) = \overline{A}(\{d\})$ $\Leftrightarrow$ if $D_{MUP}(x, y) \neq \varnothing$, then $B \cap D_{MUP}(x, y) \neq \varnothing$.

Proof.

(i) Since $\underline{B}(\{d\}) = \underline{A}(\{d\})$, we have $\forall x \in U$, $\underline{B}(x) = \underline{A}(x)$. Since $(x, y) \in R_{MLW}$, we have $\underline{A}(x) \not\subseteq \underline{A}(y)$ and $\exists a \in B \exists C \in O(\{a\})$,

$x \in C, \exists c \in B, C' \in O(\{c\}), y \in C', C \not\subseteq C'$. Therefore $a \in D_{MLW}(x, y) \neq \varnothing$ means that $B \cap D_{MLW}(x, y) \neq \varnothing$.

If $\forall D_{MLW}(x, y) \neq \varnothing$, then we have $\underline{A}(x) \not\subseteq \underline{A}(y)$. It follows that $\exists a \in B$, $\exists C$, $C' \in O(\{a\})(x \in C \wedge y \in C' \wedge C \not\subseteq C')$. Thus, $\forall x \in U$, $\underline{B}(x) = \underline{A}(x)$. So $\underline{B}(\{d\}) = \underline{A}(\{d\})$.

The proof of (ii) is the same as that of (i).

Definition 8. Let $S = (U, AT, V, f)$ be an incomplete decision table. We respectively define

$$\Delta_{MLW} = \bigwedge_{(x, y) \in R_{MLW}} \vee D_{MLW}(x, y);$$

$$\Delta_{MUP} = \bigwedge_{(x, y) \in R_{MUP}} \vee D_{MUP}(x, y).$$

Δ_{MLW} is called the lower approximation distribution

discernibility function, Δ_{MUP} is the upper one in MGRSM, where $\wedge$ is the conjunction, and $\vee$ is the disjunction.

Theorem 13. Let $IIS = (U, A \cup \{d\}, V, f)$ be an incomplete decision table, $B \subseteq A$. Then

(1) if, and only if $\wedge B$ is a necessity term of Δ_{MLW}, B is a lower approximation distribution reduction of the incomplete decision table;

(2) if, and only if $\wedge B$ is a necessity term of Δ_{MUP}, B is an upper approximation distribution reduction of the incomplete decision table.

The proof of it is similar to that of Theorem 15 in [14].

Theorem 14. Let IIS be a consistent incomplete decision table. $t = \wedge(a_i, v_{a_i})$, $s = (d, j)$, $j \in V_{\{d\}}$. Then

(1) If B is a lower approximation distribution reduction, then for any $x \in U$, r_x: $t \to s$ is a certain decision rule;

(2) If B is an upper approximation distribution reduction, then for any $x \in U$, r_x: $t \to s$ is a possible decision rule.

TABLE II
$D_{MLW}(x, y)$ WITH $(x, y) \notin SIM(A)$

	1	2	3	4	5	6
1			S	X	X	
2			X		X	
3	S	S		SX	SX	S
6			S			

VI. AN EXAMPLE REDUCTION IN MGRSM

In order to illustrate how to solve a reduction in MGRSM in incomplete decision table with our suggested method, here we give an example.

Example 3. Use the incomplete decision table shown in Table I. We can easily obtain $U / IND(\{d\} = \{X_1, X_2, X_3\}$, where $X_1 = \{1, 2, 4, 6\}$, $X_2 = \{3\}$, $X_3 = \{5\}$. $\underline{A}(X_1) = \{1, 2, 6\}$, $\underline{A}(X_2) = \{3\}$, $\underline{A}(\{d\}) = \{\underline{A}(X_1), \underline{A}(X_2), \underline{A}(X_3)\} = \{\underline{A}(\{1, 2, 4, 6\}), \underline{A}(\{3\}), \underline{A}(\{5\})\} = \{\{1, 2, 6\}, \{3\}, \varnothing\}$. $O(A) = \{\{1\}, \{2, 6\}, \{3\}, \{4, 5\}, \{5, 6\}\}$. $O_A(1) = \{\{1\}\}$, $O_A(2) = \{\{2, 6\}\}$, $O_A(3) = \{\{3\}\}$, $O_A(4) = \{\{4, 5\}\}$, $O_A(5) = \{\{5, 6\}\}$, $O_A(6) = \{\{2, 6\}, \{5, 6\}\}$.

$$\underline{A}(1) = \{X \in U / IND(\{d\}) : 1 \in \underline{A}(X)\} = \{X_1\},$$
$$\underline{A}(2) = \{X_1\}, \underline{A}(3) = \{X_2\}, \underline{A}(4) = \varnothing \underline{A}(5) = \varnothing \ \underline{A}(6) = \{X_1\}.$$
$$U / IND(\{P\} = \{\{1, 3, 4, 5\}, \{2, 3, 4, 6\}\},$$
$$U / IND(\{M\} = \{\{1, 2, 3, 4, 5\}, \{2, 3, 4, 5, 6\}\},$$
$$U / IND(\{S\} = \{\{1, 3, 4, 5, 6\}, \{3\}\},$$
$$U / IND(\{X\} = \{\{1, 2, 3, 6\}, \{4, 5\}\}.$$
$$R_{MLW} = \{(x, y) : \underline{A}(x) \not\subseteq \underline{A}(y)\}$$
$$= \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (3, 1),$$
$$(3, 2), (3, 4), (3, 5), (3, 6), (6, 3), (6, 4), (6, 5)\}.$$

$D_{MLW}(x, y)$ $((x, y) \notin SIM(A))$ with different pairs of x and y are shown in Table II. A blank means an empty.

TABLE III
$D_{MUP}(x, y)$ WITH $(x, y) \notin SIM(A)$

	1	2	3	4	5	6
1			S		X	
2			X		X	
3	S	S		SX	SX	S
4			SX			
5			SX			
6			S			

$$\Delta_{MLW} = S \wedge X \wedge X \wedge (P \vee X) \wedge X \wedge S \wedge S$$
$$\wedge (S \vee X) \wedge (S \vee X) \wedge S \wedge S = S \wedge X.$$

So, $\{S, X\}$ is the lower approximation distribution reduct of IIS. $\underline{S \wedge X}(\{d\}) = \{\{1, 2, 6\}, \{3\}, \varnothing\} = \underline{A}(\{d\})$. This result is identical to that in [4].

$$\overline{A}(X_1) = \overline{A}(\{1, 2, 4, 6\}) = \sim \underline{A}(\sim \{1, 2, 4, 6\}) = \sim \underline{A}(\{3, 5\})$$
$$= \sim \{3\} = \{1, 2, 4, 5, 6\}, \ \overline{A}(X_2) = \{3\}, \ \overline{A}(X_3) = \{4, 5\}. \ \overline{A}(\{d\})$$
$$= \{\overline{A}(X_1), \overline{A}(X_2), \overline{A}(X_3)\} = \{\{1, 2, 4, 5, 6\}, \{3\}, \{4, 5\}\}.$$
$$\overline{A}(1) = \{X \in U / IND(\{d\}) : 1 \in \overline{A}(X)\} = \{X_1\},$$
$$\overline{A}(2) = \{X_1\}, \overline{A}(3) = \{X_2\}, \ \overline{A}(4) = \{X_1\}, \ \overline{A}(5) = \{X_3\},$$
$$\overline{A}(6) = \{X_1\}.$$
$$R_{MUP} = \{(x, y) : \overline{A}(y) \not\subseteq \overline{A}(x)\} = \{(1, 3), (1, 5), (2, 3),$$
$$(2, 5), (3, 1), (3, 2), (3, 4), (3, 5), (3, 6), (4, 3), (4, 5), (5, 1),$$
$$(5, 2), (5, 3), (5, 4), (5, 6), (6, 3), (6, 5)\}.$$

$D_{MUP}(x, y)$ with different pair $(x, y) \notin SIM(A)$ are shown in Table III. $\Delta_{MUP} = S \wedge X \wedge (S \vee X) = S \wedge X$. So, $\{S, X\}$ is the upper approximation distribution reduct of IIS.

$$\overline{S \wedge X}(\{d\}) = \{\overline{S \wedge X}(X_1), \overline{S \wedge X}(X_2),$$
$$\overline{S \wedge X}(X_3)\} = \{\{1, 2, 4, 5, 6\}, \{3\}, \{4, 5\}\} = \overline{S \wedge X}(\{d\}).$$

This result is also identical to the one in [4].

VII. CONCLUSIONS

Using the knowledge representing system $O(B)$ (B is any subset of attribute set), a maximal compatible class set, as a primitive granule, the present paper defines the lower approximation distribution reduction and the upper one. It extends definitions of them from single granulation RSM to MGRSM. It discusses properties of the lower and upper approximations and reductions in single and multiple granulation models, and the relationships between both models. Disjunction and conjunction operations of subsets are also

explored. Under certain conditions, the lower and upper approximation in multi-granulation model is related to the lower and upper approximation distribution reductions in single models (see Theorems 10 and 11). It designs discernibility matrices and Boolean functions to find upper and lower approximation distribution reductions. It verifies the correctness of them through examples. This granular approach supplies us a view to study MGRSM in processing IIS.

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