

# The Spectral Power Amplification on the Regular Lattices

Kotbi Lakhdar, Hachi Mostefa

**Abstract**—We show that a simple transformation between the regular lattices (the square, the triangular, and the honeycomb) belonging to the same dimensionality can explain in a natural way the universality of the critical exponents found in phase transitions and critical phenomena. It suffices that the Hamiltonian and the lattice present similar writing forms. In addition, it appears that if a property can be calculated for a given lattice then it can be extrapolated simply to any other lattice belonging to the same dimensionality. In this study, we have restricted ourselves on the spectral power amplification (SPA), we note that the SPA does not have an effect on the critical exponents but does have an effect by the criticality temperature of the lattice; the generalisation to other lattice could be shown according to the containment principle.

**Keywords**—Ising model, phase transitions, critical temperature, critical exponent, spectral power amplification.

## I. INTRODUCTION

MANY web-like structures such as the Internet, world-wide web, power supply and communications networks, etc., which are of high importance for modern society, belong to a class of complex networks [1]-[5]. In many cases, such networks exhibit scale-free (SF) topology, i.e., their degree distribution (distribution of the number of edges, or connections, per node) obeys a power scaling law  $p(k) \propto k^{-\gamma}$ ,  $\gamma > 2$  [2]-[4]. Apart from the topological properties of complex arrays, various physical phenomena in systems with the structure of complex networks have recently become a subject of interest, including stochastic resonance [6] (SR, for review see [7], [8]) in systems on small world networks [9]-[12] (with partial rewiring of regular connections [1]) and on SF networks [13], [14], coherence resonance [15], [16] and deterministic amplification of weak signals in SF networks of bistable oscillators [17]. Also, SR in the Ising model with ferromagnetic coupling on small-world [18] and SF [19], [20] networks was studied, with a weak periodic magnetic field as the input signal, time-dependent order parameter as the output signal, and thermal fluctuations playing the role of noise. As in the case of SR in the Ising model on regular or globally coupled arrays [21]-[27], the response of the system to the oscillating magnetic field was maximum in the vicinity of the critical temperature for the ferromagnetic transition due to the divergence of the magnetic susceptibility. Besides, in the case of the Ising model on SF networks constructed as evolving networks with preferential attachment [2], [4] under certain assumptions, the response to the oscillating field showed

additional maximum in the ordered phase, below the critical temperature. As a result, the SPA, defined as the strength of the Fourier component of the output signal at the frequency of the input signal divided by the strength of the input signal, exhibited double maxima as a function of the temperature [20]. This is an example of stochastic multi-resonance [28]-[30].

This paper is organized as follows: In Section II, we shall define a certain “writing form” for a given lattice in order to develop on the Ising model. Then in Section III, we shall examine the criticality temperature and the universality of the critical exponent on the planar lattices. In Section IV, we show that the SPA on an arbitrarily lattice could be related only by the criticality temperature of another lattice belonging to the same dimensionality. Finally, the conclusion is reported in Section V.

## II. THE “WRITING” OF A LATTICE

Obviously, there are many rigorous mathematical representations of the lattices in terms of graphs, subgraphs, and edges. However, we shall propose a simple picture that gives a simple representation of the lattice. Consider a set of  $N$  sites (atoms) located at the vertices of a regular lattice denoted by  $\mathfrak{T}(z, d, N)$  where  $z$  is the coordination number and  $d$  the Euclidean dimensionality of the corresponding space. To assure the condition of regularity of the lattice, the size  $N$  would be infinite. Obviously, the lattice appears as a mixture of  $N$  sites and  $n_b$  bonds, with  $n_b = \frac{z}{2}N$ . In Fig. 1, we have drawn some common planar regular lattices.

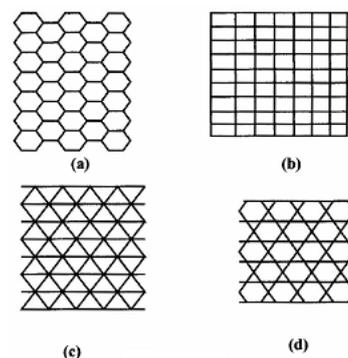


Fig. 1 Some bi-dimensional regular lattices: the triangular (a), the square (b), the honeycomb (c) and the Kagome (d)

Denoting site  $i$  by  $\langle\langle i \rangle\rangle$  and the bond  $(i, j)$  by  $[i :: j]$ , we can adopt the following symbolic representation referred to as

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a “writing form” for the lattice

$$\mathfrak{X}(z, d) = \sum_i \ll i \gg \oplus \sum_{(i,j)} [i :: j] \quad (1)$$

Here, the  $\sum_i$  and  $\sum_{(i,j)}$  indicate summations over the atoms and the bonds of the lattice, respectively.

The linear chain ( $lc$ ) is a one-dimensional lattice with  $z = 2$ . It presents one principal direction so that:

$$\mathfrak{X}_{lc}(z, d) = \sum_k \ll k \gg \oplus \sum_k [k :: k + 1] \quad (2)$$

The square lattice ( $sq$ ) is a two-dimensional lattice with  $z = 4$ . It is characterized by two principal directions ( $\vec{k}, \vec{l}$ ). It is given by:

$$\mathfrak{X}_{sq}(z, d) = \sum_{(k,l)} \ll k, l \gg \oplus \sum_{(k,l)} [(k, l) :: (k, l + 1)] \oplus [(k, l) :: (k + 1, l)] \quad (3)$$

In the triangular lattice ( $z = 4$ ), there are three principal directions ( $\vec{k}, \vec{l}, \vec{m}$ ) but the direction  $\vec{m}$  can be expressed in terms of the two other directions so that:

$$\mathfrak{X}_t(z, d) = \sum_{(k,l)} \ll k, l \gg \oplus \sum_{(k,l)} [(k, l) :: (k, l + 1)] \oplus [(k, l) :: (k + 1, l)] \oplus \sum_{(k,l)} [(k, l) :: (k + 1, l + 1)] \quad (4)$$

#### A. The Model

For the sake of simplicity and without a loss of generality, we restrict ourselves to the branch of magnetism and particularly to the Ising model. Let  $S_i$  be the random variable of spin on atom  $i$ . We say that a function  $f$  is a free lattice function if it has the following form:

$$f(\{S_i\}, a, b, c) = a \sum_i S_i^2 + b \sum_i S_i + c \sum_{(i,j)} S_i S_j \quad (5)$$

The  $\sum_{(i,j)}$  indicates a sum over pairs of nearest neighbouring sites that form bonds of the lattices. Such a function can be developed “freely” on the lattice because it possesses a similar writing form like the lattice one: the quantities  $S_i^2$  and  $S_i$  are relevant to atom  $\ll i \gg$ , while  $S_i S_j$  reflects the bond  $[i :: j]$ . We say that the free function is in perfect harmony with the lattice.

The Hamiltonian of the system in the Ising model is given as:

$$H(\{S_i\}, \Delta, B, J) = -\frac{J}{\langle k \rangle} \sum_{(i,j)} S_i S_j \quad (6)$$

In order to observe SR the in-put periodic signal in the form of the external oscillating magnetic field,  $h(t) = h_0 \sin \omega_0 t$  is applied to all spins. The Hamiltonian for the model is:

$$H(\{S_i\}) = -\frac{J}{\langle k \rangle} \sum_{(i,j)} S_i S_j - h_0 \sin \omega_0 t \sum_{i,j} S_{i,j} \quad (7)$$

An expansion of the Hamiltonian (7) on the square lattice leads to:

$$H_{sq}(\{S_{i,j}\}) =$$

$$-\frac{J_{sq}}{\langle k \rangle} \sum_{(i,j)} S_{i,j} (S_{i,j+1} + S_{i+1,j}) - h_0 \sin \omega_0 t \sum_{i,j} S_{i,j} \quad (8)$$

The transition rate between two-spin configuration, which differs by a single flip of on spin, i.e., that in node  $(i, j)$  is given by the Glauber Dynamics:

$$w_{i,j}(s_{i,j}) = \frac{1}{2} (1 - s_{i,j} \tanh(\frac{I_{i,j}(t)}{T})) \quad (9)$$

with

$$I_{i,j}(t) = \frac{J_{sq}}{\langle k \rangle} (s_{i-1,j} + s_{i+1,j} + s_{i,j-1} + s_{i,j+1}) + (h_0 \sin \omega_0 t) \quad (10)$$

is a local field acting on the spin  $(i, j)$  (with degree  $k_i, k_j$ ) at time  $t$ ,  $T$  is the temperature. The output signal is the time-dependent order parameter  $S(t)$ ,

$$S(t) = \frac{1}{N \langle k \rangle} \sum_{i,j} k_i k_j S_{i,j} \quad (11)$$

In order to observe SR the SPA is evaluated from the output signal,

$$SPA = \frac{|p_1|^2}{k_0^2}, p_1 = \lim_{\tau \rightarrow \infty} \sum_{t=0}^{\tau-1} S(t) \exp(-i\omega_0 t) \quad (12)$$

Similarly, the development of the Hamiltonian (7) on the triangular lattice gives:

$$H_{tr}(\{S_{i,j,k}\}) = -\frac{J_{tr}}{\langle k \rangle} \sum_{(i,j,k)} S_{i,j,k} (S_{i,j+1,k} + S_{i+1,j,k} + S_{i,j,k+1}) - h_0 \sin \omega_0 t \sum_{i,j,k} S_{i,j,k} \quad (13)$$

Or by,

$$H_{tr}(\{S_{i,j,k}\}) = -\frac{J_{tr}}{\langle k \rangle} \sum_{(i,j,k)} S_{i,j,k} (S_{i,j+1,k} + S_{i+1,j,k} + S_{i,j,k+1}) - h_0 \sin \omega_0 t \sum_{(i,j,k)} S_{i,j,k} \quad (14)$$

In the same case, the transition rate between two-spin configuration which differ by a single flip of on spin, i.e., that in node  $(i, j, k)$  is given by the Glauber Dynamics:

$$w_{i,j,k}(s_{i,j,k}) = \frac{1}{2} (1 - s_{i,j,k} \tanh(\frac{I_{i,j,k}(t)}{T})) \quad (15)$$

with

$$I_{i,j,k}(t) = \frac{J_{tr}}{\langle k \rangle} (s_{i-1,j,k} + s_{i+1,j,k} + s_{i,j-1,k} + s_{i,j+1,k} + s_{i,j,k-1} + s_{i,j,k+1}) + (h_0 \sin \omega_0 t) \quad (16)$$

is a local field acting on the spin  $(i, j, k)$  (with degree  $k_i, k_j, k_k$ ) at time  $t$ ,  $T$  is the temperature. The output signal is the time-dependent order parameter  $S(t)$ ,

$$S(t) = \frac{1}{N \langle k \rangle} \sum_{i,j,k} k_i k_j k_k S_{i,j,k} \quad (17)$$

In order to observe SR the SPA is evaluated from the output signal,

$$SPA = \frac{|p_1|^2}{k_0^2}, p_1 = \lim_{\tau \rightarrow \infty} \sum_{t=0}^{\tau-1} S(t) \exp(-i\omega_0 t) \quad (18)$$

### III. THE CRITICAL TEMPERATURE AND THE UNIVERSALITY OF THE CRITICAL EXPONENT

#### A. The Critical Temperature

The critical temperature varies linearly in the three: lattice, square, triangular and honeycomb as:

$$T_{sq}^c(S, \{J_{i,j}^{sq}\}, Z_{sq}, d) = \sum_{i,j} a_{i,j}^{sq} J_{i,j}^{sq} \quad (19)$$

$$T_{tr}^c(S, \{J_{i,j}^{tr}\}, Z_{tr}, d) = \sum_{i,j} a_{i,j}^{tr} J_{i,j}^{tr} \quad (20)$$

$$T_{h.d}^c(S, \{J_{i,j}^{h.d}\}, Z_{h.d}, d) = \sum_{i,j} a_{i,j}^{h.d} J_{i,j}^{h.d} \quad (21)$$

where the coefficients  $a_{i,j}^{sq}, a_{i,j}^{tr}, a_{i,j}^{h.d}$  are depending on the magnitude of the spin S and on the bonds of the lattice. So for a system is the same type of bonds,

$$a_{i,j}^{sq} = a_{i,j}^{tr} = a_{i,j}^{h.d} = a \text{ for all bonds of the lattice}$$

$$n_{sq}^b, n_{tr}^b, n_{h.d}^b. \text{ Since } n_{sq}^b = \frac{N}{2} Z_{sq}, n_{tr}^b = \frac{N}{2} Z_{tr}, n_{h.d}^b = \frac{N}{2} Z_{h.d}.$$

In the tight-binding picture,  $J_{i,j}^{sq} = J_{sq}, J_{i,j}^{tr} = J_{tr}, J_{i,j}^{h.d} = J_{h.d}$  for the nearest neighbouring sites and zero elsewhere. Then the critical temperature in the three lattices is written as:

$$T_{sq}^c(S, \{J_{i,j}\}, Z_{sq}, d) = \frac{N.a}{2} Z_{sq} J_{sq} \quad (22)$$

$$T_{tr}^c(S, \{J_{i,j}\}, Z_{tr}, d) = \frac{N.a}{2} Z_{tr} J_{tr} \quad (23)$$

$$T_{h.d}^c(S, \{J_{i,j}\}, Z_{h.d}, d) = \frac{N.a}{2} Z_{h.d} J_{h.d} \quad (24)$$

This result could be extended to any lattice  $\mathbf{L}(\mathbf{z}, \mathbf{d}, \mathbf{N})$  as:

$$T_z^c(S, \{J_{i,j}\}, z, d) = \frac{N.a}{2} zJ \quad (25)$$

#### B. Universality of the Critical Exponent

The Universality in a critical phenomenon means that some remarkable identities remain unchanged when changing the geometries of a system. We will look in this paragraph at universality in its deep meaning from universality in different lattice in the same Euclidean space. Consider a physical quantity F has a singular behaviour near the critical temperature. We write this singularity to the triangular lattice  $l_{tr}(z_{tr}, d, N)$  and to the square lattice  $l_{sq}(z_{sq}, d, N)$ . We can write:

$$F_{tr}(S, J_{tr}, (z_{tr}, d, N)) = A_{tr}(T_{tr}^c - T)^{\sigma_{tr}}, T \rightarrow T_{tr}^c \quad (26)$$

$$F_{sq}(S, J_{sq}, (z_{sq}, d, N)) = A_{sq}(T_{sq}^c - T)^{\sigma_{sq}}, T \rightarrow T_{sq}^c \quad (27)$$

Assume now that there is a value  $J_{tr}$  such that the quantities  $F_{tr}(S, J_{tr}, (z_{tr}, d, N))$ , and  $F_{sq}(S, J_{sq}, (z_{sq}, d, N))$  are mathematically identical, allowing us to write:

$$A_{tr}(T_{tr}^c - T)^{\sigma_{tr}} = A_{sq}(T_{sq}^c - T)^{\sigma_{sq}} \quad (28)$$

Indeed, by applying the functional "log" in (28), we get:

$$\ln A_{tr} + \sigma_{tr} \ln(T_{tr}^c - T) = \ln A_{sq} + \sigma_{sq} \ln(T_{sq}^c - T) \quad (29)$$

As the quantity  $\ln(T^c - T)$  is a monotonic function of the variable T, then the only solutions to (29) are given by:

$$A_{tr}(S, \{J_{tr}\}, Z_{tr}, d) = A_{sq}(S, \{J_{tr}\}, Z_{tr}, d) \quad (30)$$

$$\sigma_{tr}(S, \{J_{tr}\}, Z_{tr}, d) = \sigma_{sq}(S, \{J_{tr}\}, Z_{tr}, d) \quad (31)$$

$$T_{tr}^c(S, \{J_{tr}\}, Z_{tr}, d) = T_{sq}^c(S, \{J_{sq}\}, Z_{sq}, d) \quad (32)$$

Similarly to the triangular lattice and honeycomb is obtained as :

$$A_{tr}(S, \{J_{tr}\}, Z_{tr}, d) = A_{sq}(S, \{J_{tr}\}, Z_{tr}, d) \quad (33)$$

$$\sigma_{tr}(S, \{J_{tr}\}, Z_{tr}, d) = \sigma_{sq}(S, \{J_{tr}\}, Z_{tr}, d) \quad (34)$$

$$T_{tr}^c(S, \{J_{tr}\}, Z_{tr}, d) = T_{sq}^c(S, \{J_{sq}\}, Z_{sq}, d) \quad (35)$$

This result could be extended to any network  $\mathbf{L}(\mathbf{z}, \mathbf{d}, \mathbf{N})$  as:

$$A_{tr}(S, \{J_{tr}\}, Z_{tr}, d) = A(S, \{J\}, z, d) \quad (36)$$

$$\sigma_{tr}(S, \{J_{tr}\}, Z_{tr}, d) = \sigma(S, \{J\}, z, d) \quad (37)$$

$$T_{tr}^c(S, \{J_{tr}\}, Z_{tr}, d) = T(S, \{J\}, z, d) \quad (38)$$

The second solution provides that the critical exponent  $\sigma(S, \{J\}, z, d)$  is independent of the lattice form and energies correlations, i.e., the critical exponent is invariant under transformations of the lattice.

## IV. SPECTRAL POWER AMPLIFICATION

### A. Mean Field Approximation

The master equation for the probability that at time t the system is in the spin configuration  $(s_{i,j})$  is given as:

$$\frac{d\langle s_{i,j} \rangle}{dt} = -\langle s_{i,j} \rangle + \langle \tanh\left(\frac{J_{i,j}(t)}{T}\right) \rangle \quad (39)$$

Divide the nodes of the square lattice according to their degrees  $k_i, k_j$  and assume that the average values of spin located in the nodes belonging to the classic with degree  $k_i, k_j$  are equal to  $s_{i,j}$ .

Multiply both sides of (39) by  $k_i, k_j$ , perform the sum over all nodes of the square lattice and replace with a sum over the classes of nodes.

$$\frac{d \langle s \rangle_{sq}}{dt} = -\langle s \rangle_{sq} + \sum_{k=m}^{k_{max}} \sum_{k'=m}^{k'_{max}} \frac{p_k p_{k'} k k'}{\langle k \rangle \langle k' \rangle} \tanh\left(\frac{j_{sq} k k' \langle s \rangle_{sq}}{\langle k \rangle \langle k' \rangle T}\right) + \frac{h_0}{T} \sin \omega_0 t \quad (40)$$

Similarly to the triangular lattice we obtained:

$$\frac{d \langle s \rangle_{tr}}{dt} = -\langle s \rangle_{tr} + \sum_{k=m}^{k_{max}} \sum_{k'=m}^{k'_{max}} \sum_{k''=m}^{k''_{max}} \frac{p_k p_{k'} p_{k''} k k' k''}{\langle k \rangle \langle k' \rangle \langle k'' \rangle} \tanh\left(\frac{j_{tr} k k' k'' \langle s \rangle_{tr}}{\langle k \rangle \langle k' \rangle \langle k'' \rangle T}\right) + \frac{h_0}{T} \sin \omega_0 t \quad (41)$$

**B. Stationary Values of the Order Parameter and Magnetization**

In the absence of the magnetic field, the system evolves towards a stable equilibrium with the corresponding value of the order parameter  $\langle s \rangle_0$  which can be obtained as a stable fixed point of (39), (40) with  $h_0 = 0$ , respectively.

$$\langle s \rangle_{sq_0} = \int_m^{k_{max}} \int_m^{k'_{max}} \frac{ABk^{-\gamma+1} k'^{-\gamma+1}}{\langle k \rangle \langle k' \rangle} \tanh\left(\frac{j_{sq} k k' \langle s \rangle_{sq_0}}{\langle k \rangle \langle k' \rangle T}\right) dk dk' \quad (42)$$

$$\langle s \rangle_{tr_0} = \int_m^{k_{max}} \int_m^{k'_{max}} \int_m^{k''_{max}} \frac{ABCk^{-\gamma+1} k'^{-\gamma+1} k''^{-\gamma+1}}{\langle k \rangle \langle k' \rangle \langle k'' \rangle} \tanh\left(\frac{j_{tr} k k' k'' \langle s \rangle_{tr_0}}{\langle k \rangle \langle k' \rangle \langle k'' \rangle T}\right) dk dk' dk'' \quad (43)$$

The corresponding stationary value of the magnetization  $\langle M \rangle_0$  is then,

$$\langle M \rangle_{sq_0} = \sum_{k=m}^{k_{max}} \sum_{k'=m}^{k'_{max}} p_k p_{k'} \tanh\left(\frac{j_{sq} k k' \langle s \rangle_{sq_0}}{\langle k \rangle \langle k' \rangle T}\right) = \int_m^{k_{max}} \int_m^{k'_{max}} ABk^{-\gamma+1} k'^{-\gamma+1} \tanh\left(\frac{j_{sq} k k' \langle s \rangle_{sq_0}}{\langle k \rangle \langle k' \rangle T}\right) dk dk' \quad (44)$$

Similarly, the triangular lattice is then,

$$\langle M \rangle_{tr_0} = \sum_{k=m}^{k_{max}} \sum_{k'=m}^{k'_{max}} p_k p_{k'} \tanh\left(\frac{j_{tr} k k' \langle s \rangle_{tr_0}}{\langle k \rangle \langle k' \rangle T}\right) = \int_m^{k_{max}} \int_m^{k'_{max}} \int_m^{k''_{max}} ABCk^{-\gamma+1} k'^{-\gamma+1} k''^{-\gamma+1} \tanh\left(\frac{j_{tr} k k' k'' \langle s \rangle_{tr_0}}{\langle k \rangle \langle k' \rangle \langle k'' \rangle T}\right) dk dk' dk'' \quad (45)$$

Equations (44), (45) have one stable fixed point  $\langle s \rangle_{sq_0} = 0, \langle s \rangle_{tr_0} = 0$  for  $T > T_{sq}^c, T > T_{tr}^c$  corresponding to the paramagnetic phase, and two stable symmetric fixed points  $\bar{\langle s \rangle}_{sq_0}, \bar{\langle s \rangle}_{tr_0}$  with  $\langle s \rangle_{sq_0} > 0, \langle s \rangle_{tr_0} > 0$  for  $T < T_{sq}^c, T < T_{tr}^c$  corresponding to the ferromagnetic phase. The temperature is

$$T_{sq}^c = J_{sq} \frac{\langle k^2 \rangle \langle k'^2 \rangle}{\langle k \rangle^2 \langle k' \rangle^2} = \frac{N \cdot a}{2} Z_{sq} J_{sq} \quad (46)$$

$$T_{tr}^c = J_{tr} \frac{\langle k^2 \rangle \langle k'^2 \rangle \langle k''^2 \rangle}{\langle k \rangle^2 \langle k' \rangle^2 \langle k'' \rangle^2} = \frac{N \cdot a}{2} Z_{tr} J_{tr} \quad (47)$$

where  $\langle k^2 \rangle, \langle k'^2 \rangle, \langle k''^2 \rangle$  is the second moment of the distribution  $p(k), p(k'), p(k'')$  depends on the scaling exponent  $\gamma$  and on the number of bonds  $n_{sq}^b, n_{tr}^b$ . For  $\gamma > 3$ , the system undergoes a ferromagnetic phase transition at the critical temperature (taking into account that the critical exponent is invariant under transformations of the lattice), we have:

$$T_{sq}^c = J_{sq} \frac{(\gamma-2)}{(\gamma-1)(\gamma-3)} \quad (48)$$

$$T_{tr}^c = J_{tr} \frac{(\gamma-2)}{(\gamma-1)(\gamma-3)} \quad (49)$$

for  $2 < \gamma \leq 3$  the critical temperature diverges in the thermodynamic limit, however, for finite  $n_{sq}^b, n_{tr}^b$  there is a crossover temperature:  $T_{sq}^c \propto \ln(n_{sq}^b), T_{tr}^c \propto \ln(n_{tr}^b)$  for  $\gamma = 3$  and  $T_{sq}^c \propto (n_{sq}^b)^{\frac{(3-\gamma)}{(\gamma-1)}}, T_{tr}^c \propto (n_{tr}^b)^{\frac{(3-\gamma)}{(\gamma-1)}}$  for  $\gamma < 3$  separating the ordered and disordered phases.

**C. Linear Response Theory**

The response of the model to the weak oscillating magnetic field  $h_0 \rightarrow 0$  for given T can be studied in the MF approximation in the framework of the LRT. It is assumed that the MF order parameter  $S(t)$  oscillates around the stable stationary state, i.e.,  $\langle s(t) \rangle = s_0 + \varepsilon(t)$  where  $\varepsilon(t) \rightarrow 0$ . Inserting this in to (40), (41), expanding the tanh function in the Taylor series up to linear terms with respect to  $Jk\varepsilon(t) \langle k \rangle^{-1} + T^{-1} h_0 \sin \omega_0 t$ , we have:

$$\frac{d\varepsilon}{dt} = -\frac{\varepsilon}{\tau_{MF}} + \frac{Q}{T} h_0 \sin \omega_0 t \quad (50)$$

In the square lattice we have:

$$-1 + \frac{J}{T \langle k' \rangle^2 \langle k \rangle^2} \sum_{k=m}^{k_{max}} \sum_{k'=m}^{k'_{max}} p_k p_{k'} k^2 k'^2 \cosh^{-2}\left(\frac{j_{sq} k k' \langle s \rangle_{sq_0}}{\langle k \rangle \langle k' \rangle T}\right) = \frac{-1}{\tau_{MF}^{sq}} = \quad (51)$$

$$Q_{sq} = \frac{1}{\langle k' \rangle \langle k \rangle} \sum_{k=m}^{k_{max}} \sum_{k'=m}^{k'_{max}} \sum_{k''=m}^{k''_{max}} p_k p_{k'} k k' k'' \cosh^{-2}\left(\frac{j_{sq} k k' \langle s \rangle_{sq_0}}{\langle k \rangle \langle k' \rangle T}\right) \quad (52)$$

Similarly to the triangular lattice we obtained:

$$\frac{-1}{\tau_{MF}^{tr}} = 1 + \frac{J}{T \langle k'' \rangle^2 \langle k' \rangle^2 \langle k \rangle^2} \sum_{k=m}^{k_{max}} \sum_{k'=m}^{k'_{max}} \sum_{k''=m}^{k''_{max}} p_k p_{k'} p_{k''} k^2 k'^2 k''^2 \cosh^{-2}\left(\frac{j_{tr} k k' k'' \langle s \rangle_{tr_0}}{\langle k \rangle \langle k' \rangle \langle k'' \rangle T}\right) \quad (53)$$

$$Q_{tr} = \frac{1}{\langle k'' \rangle \langle k' \rangle \langle k \rangle} \sum_{k=m}^{k_{max}} \sum_{k'=m}^{k'_{max}} \sum_{k''=m}^{k''_{max}} p_k p_{k'} p_{k''} k k' k'' \cosh^{-2}\left(\frac{j_{tr} k k' k'' \langle s \rangle_{tr_0}}{\langle k \rangle \langle k' \rangle \langle k'' \rangle T}\right) \quad (54)$$

where  $\tau_{MF}$  is the MF relaxation time (to evaluate  $\tau_{MF}$  and Q for  $T \leq T_c$  the integration by parts was performed, and (42)-(45) were taken into account). The asymptotic solution for

(50) is:

$$\varepsilon(t) = \varepsilon_0 \sin(w_0 - \theta) \quad (55)$$

$$\varepsilon_0 = \frac{h_0 Q}{T} \left( \frac{1}{\tau_{MF}^2} + w_0^2 \right) \quad (56)$$

$$\theta = \arctan(w_0 \tau_{MF}) \quad (57)$$

Thus, the SPA is

$$SPA = \frac{\varepsilon_0^2}{4h_0^2} \quad (58)$$

Hence, the SPA is proportional to  $\left| \frac{\partial s}{\partial h}(w_0) \right|^2$ , where  $\frac{\partial s}{\partial h}(w)$  is a dynamical susceptibility of the order parameter  $S(t)$ . In the paramagnetic phase with  $T > T^c$  there is  $\langle s \rangle_0 = 0$ ,  $Q = 1$  and

$$SPA^{sq} = \frac{1}{4T^2} \left[ \left( 1 - \frac{T_{sq}^c}{T} \right)^2 - w_0^2 \right]^2 \quad (59)$$

Similarly to the triangular lattice and honeycomb we have:

$$SPA^{tr} = \frac{1}{4T^2} \left[ \left( 1 - \frac{T_{tr}^c}{T} \right)^2 - w_0^2 \right]^2 \quad (60)$$

$$SPA^{hd} = \frac{1}{4T^2} \left[ \left( 1 - \frac{T_{hd}^c}{T} \right)^2 - w_0^2 \right]^2 \quad (61)$$

This result could be extended to any lattice  $\mathbf{L}(\mathbf{z}, \mathbf{d}, \mathbf{N})$  as:

$$SPA^z = \frac{1}{4T^2} \left[ \left( 1 - \frac{T_z^c}{T} \right)^2 - w_0^2 \right]^2 \quad (62)$$

## V. CONCLUSION

In this study we have restricted ourselves to the case of the two regular planar lattices: the triangular and the square. We have shown that the critical exponent is invariant under transformations of the lattices. On the other hand, we have observed that the SPA in this regular planar lattice belonging to the same dimensionality  $\mathbf{d}$  can effect only by the critical temperature and have no effect on the critical exponents. The generalization of this study to other planar lattices and to other dimensions would be discussed in a future paper.

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