# General Formula for Water Surface Profile over Side Weir in the Combined, Trapezoidal and Exponential, Channels

Abdulrahman Abdulrahman

***Abstract***— A side weir is a hydraulic structure set into the side of a channel. This structure is used for water level control in channels, to divert flow from a main channel into a side channel when the water level in the main channel exceeds a specific limit and as storm overflows from urban sewerage system. Computation of water surface over the side weirs is essential to determine the flow rate of the side weir. Analytical solutions for water surface profile along rectangular side weir are available only for the special cases of rectangular and trapezoidal channels considering constant specific energy. In this paper, a rectangular side weir located in a combined (trapezoidal with exponential) channel was considered. Expanding binominal series of integer and fraction powers and the using of reduction formula of cosine function integrals, a general analytical formula was obtained for water surface profile along a side weir in a combined (trapezoidal with exponential) channel. Since triangular, rectangular, trapezoidal and parabolic cross-sections are special cases of the combined cross section, the derived formula, is applicable to triangular, rectangular, trapezoidal cross-sections as analytical solution and semi-analytical solution to parabolic cross-section with maximum relative error smaller than 0.76%. The proposed solution should be a useful engineering tool for the evaluation and design of side weirs in open channel.

***Keywords***—Analytical solution, combined channel, exponential channel, side weirs, trapezoidal channel, water surface profile.

## I. Introduction

A SIDE weir is a hydraulic structure, installed at one side of channel to divert flow from a main channel into a side channel when the water level in the main channel exceeds a specific limit. Such structures are widely used for water level control in hydraulics, irrigation and drainage canal system and environmental engineering applications, for flood protection works.

Flow over side weir is a typical case of spatially varied flow, the behavior of flow over side weir has been the subject of many investigations. Most of the previous theoretical analysis and experimental research are limited to the flow over side weirs in rectangular main channels [among them: [1], [4], [6], [7], in circular main channels [8], [9], in triangular main channels [10], in trapezoidal main channels [11], in parabolic main channels [12], and in U- shaped main channels.

In spite of numerous investigations for the water surface profile along the side weir, the above literature review reveals that there are analytical and semi-analytical solutions only for specific channel shapes. In practice a channel shape which is a combination of trapezoidal and exponential channel allows the modeling of a natural channel and man-made channel shapes. Most of irrigation and drainage channels have rectangular, triangular, trapezoidal and parabolic cross sections, and they are special cases of a combined channel. Therefore, investigation of the water surface profile is important for a combined cross section for proper estimation of discharge over side weir.

In the present study, general formula for the water surface profile along the side weir in the combined (trapezoidal and exponential) main channel is derived by making use of energy relationships.

A. Abdulrahman is with Aleppo University, Aleppo Faculty of Civil Eng. Water Eng. Dept., Syria. He is now the Dean of Civil Eng. Faculty of AL-hasakeh, AL-furat University, Syria (e-mail: abdulrahman21959@gmail.com).

## II. Proposed Solution

### A. Geometric Properties of a Combined Cross Section

A cross sectional shape of combined channel is shown in Fig. 1. The flow width $T_y$ at any height $y$ above the channel bed, and the cross sectional area $A$ are considered to be given by:

$$T_y = b + cy^{n-1} \tag{1}$$

$$A = by + \frac{c}{n}.y^n \tag{2}$$

In which, $y$ -Bottom width and $c$ and $n$ are constants which define the channel shape.

### B. Flow Equation for Side Weirs

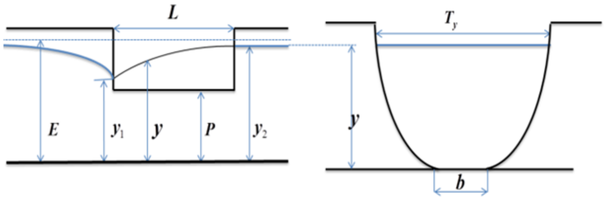


Fig. 1 Definition sketch for rectangular side weir in combined trapezoidal and exponential channel

Referring to Fig. 1, at any section $x$, for small bottom slope and hydrostatic pressure distribution, the specific energy $E$ can be written as

$$E = y + \frac{\alpha Q^2}{2gA^2} \tag{3}$$

where $y$ is the flow depth at a distance $x$ from upstream of the weir, $\alpha$ is the velocity distribution coefficient, $Q$ is the discharge of the main channel at a distance $x$, $A$ is cross-sectional area and $g$ is gravitational acceleration. Differentiating both sides of (3) with respect to $x$ gives

$$\frac{dE}{dx} = \frac{dy}{dx}\left(1 - \frac{\alpha Q^2 T}{gA^3}\right) + \frac{dQ}{dx}\left(\frac{\alpha Q}{gA^2}\right) \tag{4}$$

in which $T$ is width of the channel at water surface and $x$ is horizontal distance from upstream of the weir. Neglecting the effect of variation of specific energy $(dE/dx = 0)$ along the side weir, the water surface slope $dy/dx$ can be expressed as

$$\frac{dy}{dx} = \frac{-dQ/dx\,(\alpha Q/gA^2)}{1-(\alpha Q^2 T/gA^3)} \tag{5}$$

For a short side weir, the hypothesis of constant specific energy, which is equivalent to assuming $S_f = S_0$ or $S_f = 0_0$ and $S_0 = 0\,(S_f)$ is friction loss and $S_0$ is bed slope of the channel), is acceptable [5].

Although the direction of the side weir is parallel to the flow, the equation of a weir is assumed for discharge along the side weir [2], [3],

$$q_s = -\frac{dQ}{dx} = \frac{2}{3} C_m (2g)^{1/2} (y-p)^{3/2} \tag{6}$$

where $q_s$ discharge per unit width of the weir is, $p$ is the weir height and $C_m$ is the discharge coefficient. A conventional weir equation for discharge per unit length is assumed and this assumption is approximately valid [10].

Assuming that the specific energy $E$ is constant along the length of the side weir, the discharge in the main channel $Q$ can be obtained by solving (3) for $Q$ as

$$Q = A\left(\frac{2g}{\alpha}(E-y)\right)^{1/2} \tag{7}$$

Using (7) and knowing the flow depth $y$, the discharge over the side weir $Q_s$ can be determined as $Q_s = Q_1 - Q$ in which $Q_1$ is the discharge at section 1 in the main channel. As noted earlier, determination of the water surface profile along the side weir is essential to estimate the discharge over the side weir. Substituting the variation of discharge over the side weir $(dQ/dx)$ from (6) and the discharge $Q$ of the main channel at a distance $x$ from (7)"into (5) yields:

$$\frac{dy}{dx} = \frac{4}{3}\alpha^{1/2} C_m \left(\frac{(y-p)^{3/2}(E-y)^{1/2}}{A - 2(E-y)T}\right) \tag{8}$$

Substituting $T_y$ and $A$ from (1) and (2) into (8) yields

$$dx = \frac{3}{4C_m \alpha^{1/2}}\left[\frac{3by + c(\frac{1}{n}+2)y^n - 2bE - 2cEy^{n-1}}{\sqrt{(E-y)(y-p)^3}}\right]dy \tag{9a}$$

Equation (9a) can be transformed into dimensionless form

$$\mu dx^* = \left(\frac{3b^* y^* + E^{n-2}c(\frac{1}{n}+2)y^{*n} - 2b^* - 2cE^{n-2}y^{*n-1}}{\sqrt{(1-y^*)(y^*-p^*)^3}}\right)dy^* \tag{9b}$$

where $x^* = x/E$, $b^* = b/E$, $y^* = y/E$, $p^* = p/E$, $\mu$ represents the discharge coefficient of the side weir flow and is equal to $\mu = 4\alpha^{1/2}C_m/3$. Integrating (9b) gives:

$$\mu x^* = (I_1 + I_2 + I_3 + I_4) \tag{9c}$$

in which:

$$I_1 = 3b^* \int \frac{y^* dy^*}{\sqrt{(1-y^*)(y^*-p^*)^3}} \tag{10a}$$

$$I_2 = E^{n-2}c(2+\frac{1}{n}) \int \frac{y^{*n} dy^*}{\sqrt{(1-y^*)(y^*-p^*)^3}} \tag{10b}$$

$$I_3 = -2b^* \int \frac{dy^*}{\sqrt{(1-y^*)(y^*-p^*)^3}} \tag{10c}$$

$$I_4 = -2cE^{n-2} \int \frac{y^{*n-1} dy^*}{\sqrt{(1-y^*)(y^*-p^*)^3}} \tag{10d}$$

## III. General Proposed Analytical Solution for Water Surface Profile

Let $\tan u = \sqrt{\frac{1-y^*}{y^*-p^*}} \Rightarrow dy^* = -2(1-p^*)\sin u.\cos u.du$

$$y^* = p^* + \frac{1-p^*}{1+\tan^2 u}.$$

Equations (10a)-(10d) then become

$$I_1 = -\frac{6b^*}{1-p^*}\int (1 + p^* \tan^2 u)du \Rightarrow$$
$$I_1 = -6b^*\left(p^* \tan u + u(1-p^*)\right) \tag{11a}$$

$$I_2 = -\frac{2E^{n-2}}{1-p^*} c(2+\frac{1}{n}) p^{*n} \int \left(1+p^* \tan^2 u\right)^n \cos^{2n-2} u.du \quad (11b)$$

$$I_3 = \frac{4b^*}{1-p^*} \int \frac{du}{\cos^2 u} \Rightarrow$$
$$I_3 = \frac{4b^*}{1-p^*} \tan u \quad (11c)$$

$$I_4 = \frac{4c}{1-p^*} E^{n-2} p^{*n-1} \int \left(1+p^* \tan^2 u\right)^{n-1} \cos^{2n-2} u.du \quad (11d)$$

Integrants $I_2, I_4$ can be evaluated by expansion in Binomial series as:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 + ........\infty$$

$$I_2 = -\frac{2E^{n-2}}{1-p^*} c(2+\frac{1}{n}) \int \begin{pmatrix} a_1 \cos^{2n-2} u + \\ a_2 \cos^{2n-4} u + \\ a_3 \cos^{2n-6} u + \\ a_4 \cos^{2n-8} u + .. \end{pmatrix} du \quad (12a)$$

$$I_4 = \frac{4c}{1-p^*} E^{n-2} \int \begin{pmatrix} b_1 \cos^{2n-4} u + \\ b_2 \cos^{2n-6} u + \\ b_3 \cos^{2n-8} u + \\ b_4 \cos^{2n-10} u + .... \end{pmatrix} du \quad (12b)$$

where:

$$a_1 = \left(1 - np^* + \frac{n(n-1)}{2} p^{*2} - \frac{n(n-1)(n-2)}{6} p^{*3}\right) \quad (13a)$$

$$a_2 = \left(np^* - n(n-1) p^{*2} + \frac{n(n-1)(n-2)}{2} p^{*3}\right) \quad (13b)$$

$$a_3 = \left(\frac{n(n-1)}{2} p^{*2} - \frac{n(n-1)(n-2)}{2} p^{*3}\right) \quad (13c)$$

$$a_4 = \left(\frac{n(n-1)(n-2)}{6} p^{*3}\right) \quad (13d)$$

$$b_1 = \left(1-(n-1)p^* + \frac{(n-1)(n-2)}{2} p^{*2} - \frac{(n-1)(n-2)(n-3)}{6} p^{*3}\right) \quad (14a)$$

$$b_2 = \left((n-1)p^* - (n-1)(n-2) p^{*2} + \frac{(n-1)(n-2)(n-3)}{2} p^{*3}\right) \quad (14b)$$

$$b_3 = \left(\frac{(n-1)(n-2)}{2} p^{*2} - \frac{(n-1)(n-2)(n-3)}{2} p^{*3}\right) \quad (14c)$$

$$b_4 = \left(\frac{(n-1)(n-2)(n-3)}{6} p^{*3}\right) \quad (14d)$$

## IV. Special Cases

The general solution developed in this study may be applied to channels of different cross sections shapes

### A. Solution for Trapezoidal Channel

For a side weir in a trapezoidal channel of side slope $m$ (1 vertical to $m$ horizontal), the constants of channel shape are $c = 2m$, $n = 2$. Substituting into (12a), (13), (12b) and (14):

$$a_1 = (1-p^*)^2,\ a_2 = 2p^*(1-p^*),\ a_3 = p^{*2},\ a_4 = 0$$

$$I_2 = -\frac{5m}{1-p^*} \int \left((1-p^*)^2 \cos^2 u + 2p^*(1-p^*) + \frac{p^{*2}}{\cos^2 u}\right) du \quad (15a)$$

$$b_1 = \left(1-p^*\right),\ b_2 = p^*,\ b_3 = 0,\ b_4 = 0$$

$$I_4 = \frac{8m}{1-p^*} \int \left((1-p^*) + \frac{p^*}{\cos^2 u}\right) du \quad (15b)$$

Integrating each of (15a) and (15b) is made by using the following reduction formulae

$$\int \cos^m u du = \frac{\sin^m u.\cos^{m-1} u}{m} + \frac{m-1}{m} \int \cos^{m-2} u.du$$

Thus;

$$\int \cos^2 u du = \frac{1}{2}\left(\sin u.\cos u + u\right)$$

$$I_2 = -\frac{10m}{1-p^*} \begin{pmatrix} p^{*2} \tan u + \frac{(1-p^*)^2}{2}\left(\sin u.\cos u + u\right) + \\ 2p^*(1-p^*)u \end{pmatrix} \quad (16a)$$

$$I_4 = \frac{8m}{1-p^*} \left(p^* \tan u + (1-p^*)u\right) \quad (16b)$$

Substituting (11a), (16a), (11c) and (16b) into (9c) yields

$$\mu x^* = 3\left(m(1-5p^*) - 2b^*\right) \tan^{-1} \sqrt{\frac{1-y^*}{y^*-p^*}} +$$
$$\frac{2}{(1-p^*)}\left(mp^*(4-5p^*) + b^*(2-3p^*)\right)\sqrt{\frac{1-y^*}{y^*-p^*}}$$
$$-5m\sqrt{(1-y^*)(y^*-p^*)} + \text{constant} \quad (17)$$

Noting that: $u = \tan^{-1}\sqrt{\frac{1-y^*}{y^*-p^*}}$, $\cos u = \sqrt{\frac{y^*-p^*}{1-p^*}}$, $\sin u = \sqrt{\frac{1-y^*}{1-p^*}}$.

Equation (17) gives the length of the side weir for trapezoidal channel when the water depth varies from $y_1$ to $y_2$ along the side weir at the main channel axis in the flow direction". Equations (7) and (17) are used to determine the discharge over the side weir". These equations can be used for designing a side weir pass a certain discharge into a side channel.

The integration constant in (17) can be eliminated as

$$\mu L / E = \psi(y_2^*) - \psi(y_1^*) \tag{18}$$

In which $L$ is the length of the side weir and $\psi(y^*)$ is defined as:

$$\begin{aligned}\psi(y^*) = 3\left(m(1-5p^*) - 2b^*\right)\tan^{-1}\sqrt{\frac{1-y^*}{y^*-p^*}} + \\ \frac{2}{(1-p^*)}\left(mp^*(4-5p^*) + b^*(2-3p^*)\right)\sqrt{\frac{1-y^*}{y^*-p^*}} - \\ 5m\sqrt{(1-y^*)(y^*-p^*)}\end{aligned} \tag{19}$$

*B. Exponential Channels*

1) Rectangular Cross-Section

The shape constants are: $c = 0,\ n = 1,\ \Rightarrow I_2 = I_4 = 0$,

$$I_1 = -6b^*\left(\frac{p^*}{1-p^*}\tan u + u\right) = -6b^*\left(\begin{aligned}\frac{p^*}{1-p^*}\sqrt{\frac{1-y^*}{y^*-p^*}} + \\ \tan^{-1}\sqrt{\frac{1-y^*}{y^*-p^*}}\end{aligned}\right)$$

$$I_3 = \frac{4b^*}{1-p^*}\tan u = \frac{4b^*}{1-p^*}\sqrt{\frac{1-y^*}{y^*-p^*}}$$

Substituting into (9c), yields

$$\mu x^* = 2b^*\left(\frac{2-3p^*}{1-p^*}\sqrt{\frac{1-y^*}{y^*-p^*}} - 3\tan^{-1}\sqrt{\frac{1-y^*}{y^*-p^*}}\right) + \text{constant} \tag{20}$$

The integration constant in (20) can be eliminated as

$$\mu L / E = \psi(y_2^*) - \psi(y_1^*)$$

in which $L$ is the length of the side weir and $\psi(y^*)$ is defined as

$$\psi(y^*) = 2b^*\left(\frac{2-3p^*}{1-p^*}\sqrt{\frac{1-y^*}{y^*-p^*}} - 3\tan^{-1}\sqrt{\frac{1-y^*}{y^*-p^*}}\right) \tag{21}$$

2) Triangular Cross-Section

For triangular channel of side slopes $m_1$ and $m_2$ the shape constants are $c = m_1 + m_2 = m$, $n = 2$, $b = 0 \Rightarrow I_1 = I_3 = 0$

$$a_1 = (1-p^*)^2,\ a_2 = 2p^*(1-p^*),\ a_3 = p^{*2},\ a_4 = 0$$

$$I_2 = -\frac{5m}{1-p^*}\int\left((1-p^*)^2\cos^2 u + 2p^*(1-p^*) + \frac{p^{*2}}{\cos^2 u}\right)du$$

$$I_2 = -\frac{5m}{1-p^*}\left(p^{*2}\tan u + \frac{(1-p^*)^2}{2}\left(\sin u.\cos u + u\right) + 2p^*(1-p^*)u\right)$$

$$b_1 = (1-p^*),\ b_2 = p^*,\ b_3 = 0,\ b_4 = 0$$

$$I_4 = \frac{4m}{1-p^*}\int\left((1-p^*) + \frac{p^*}{\cos^2 u}\right)du$$

$$I_4 = \frac{4m}{1-p^*}\left(p^*\tan u + (1-p^*)u\right)$$

Substituting into (9c), yields

$$\mu x^* = m\left(\begin{aligned}\frac{3}{2}(1-5p^*)\tan^{-1}\sqrt{\frac{1-y^*}{y^*-p^*}} + \\ \frac{p^*}{1-p^*}(4-5p^*)\sqrt{\frac{1-y^*}{y^*-p^*}} - \\ \frac{5}{2}\sqrt{(1-y^*)(y^*-p^*)}\end{aligned}\right) + \text{constant} \tag{22}$$

The integration constant in (22) can be eliminated as

$$\mu L / E = \psi(y_2^*) - \psi(y_1^*)$$

In which $L$ is the length of the side weir and $\psi(y^*)$ is defined as

$$\psi(y^*) = m\left(\begin{aligned}\frac{3}{2}(1-5p^*)\tan^{-1}\sqrt{\frac{1-y^*}{y^*-p^*}} + \\ \frac{p^*}{1-p^*}(4-5p^*)\sqrt{\frac{1-y^*}{y^*-p^*}} - \frac{5}{2} \\ \sqrt{(1-y^*)(y^*-p^*)}\end{aligned}\right) \tag{23}$$

$$\mu x^* = z\begin{pmatrix} 3(1-5p^*)\tan^{-1}\sqrt{\dfrac{1-y^*}{y^*-p^*}} + \\ \dfrac{2p^*}{1-p^*}(4-5p^*)\sqrt{\dfrac{1-y^*}{y^*-p^*}} - \\ 5\sqrt{(1-y^*)(y^*-p^*)} \end{pmatrix} + \text{const.} \tag{24}$$

and $\psi(y^*)$ is defined as

$$\psi(y^*) = z\begin{pmatrix} 3(1-5p^*)\tan^{-1}\sqrt{\dfrac{1-y^*}{y^*-p^*}} + \\ \dfrac{2p^*}{1-p^*}(4-5p^*)\sqrt{\dfrac{1-y^*}{y^*-p^*}} - \\ 5\sqrt{(1-y^*)(y^*-p^*)} \end{pmatrix} \tag{25}$$

3) Parabolic Cross-Section

A channel section with function:

$$Y = kX^2 \tag{26}$$

where $Y$ is ordinate, $X$ is abscissa and $k$ is a parameter for which the function takes different shapes (narrow or flat). For purposes of this study, it is convenient to express the parabolic function in $X$ and $Y$ coordinate system, where the $X$ -axis is horizontally located at the channel bed $Y$ - axis is placed at the center of the channel bed. Since the section is symmetrical, the half top width of the water surface, $T/2$, is obtained by substituting the flow depth $y$ for $Y$ in (26).

The top width and section area which correspond to the flow depth $y$ are required for water surface computations. Top width of the water surface, $T$, can be expressed as

$$T = \frac{2}{\sqrt{k}} y^{1/2} \tag{27}$$

The flow area of the channel, $A$, for the flow depth $y$, can be computed as

$$A = 2\int_0^{T/2} (y - kX^2)\,dX \tag{28}$$

So, the shape constants for parabolic section are: $b = 0$, $n = 3/2$, $c = 2/\sqrt{k} \Rightarrow I_1 = I_3 = 0$.

Since $n = 3/2$, integrants $I_2$ and $I_4$ can't be evaluated exactly and we should seek the domain of its convergence. Checking the convergence of binominal series $(1+x)^n$ indicated that this series converges for $|x| < 1$. This means that

$$\left|\frac{2}{(1+p^*)}\right| < \frac{y^*}{p^*}.$$

Hence the solution for parabolic section is applicable for the interval $\dfrac{p^*}{y^*} < \dfrac{1+p^*}{2}$.

Taking the first four terms and putting them into the integrants $I_2$ and $I_4$, one obtains

$$a_1 = \left(1 - \frac{3}{2}p^* + \frac{3}{8}p^{*2} + \frac{1}{16}p^{*3}\right),$$

$$a_2 = \left(\frac{3}{2}p^* - \frac{3}{4}p^{*2} - \frac{3}{16}p^{*3}\right),\ a_3 = \left(\frac{3}{8}p^{*2} + \frac{3}{16}p^{*3}\right),$$

$$a_4 = \left(-\frac{1}{16}p^{*3}\right)$$

$$I_2 = \frac{-32}{3\sqrt{kE}(1-p^*)}\int\begin{pmatrix} (1-\frac{3}{2}P^* + \frac{3}{8}P^{*2} + \frac{p^{*3}}{16})\cos u + \\ (\frac{3}{2}P^* - \frac{3}{4}P^{*2} - \frac{3}{16}P^{*3})\dfrac{1}{\cos u} + \\ (\frac{3}{8}p^{*2} + \frac{3}{16}p^{*3})\dfrac{1}{\cos^3 u} - \\ (\frac{p^{*3}}{16})\dfrac{1}{\cos^5 u} \end{pmatrix} du$$

$$I_2 = \frac{-32}{3\sqrt{kE}(1-p^*)}\begin{pmatrix} (1-\frac{3}{2}P^* + \frac{3}{8}P^{*2} + \frac{p^{*3}}{16})\sin u + \\ (\frac{3}{2}P^* - \frac{3}{4}P^{*2} - \frac{3}{16}P^{*3})\ln\dfrac{1+\sin u}{\cos u} + \\ \frac{1}{2}(\frac{3}{8}p^{*2} + \frac{3}{16}p^{*3})(\ln\dfrac{1+\sin u}{\cos u} + \dfrac{\sin u}{\cos^2 u}) - \\ (\frac{p^{*3}}{128})\left(\dfrac{\sin u}{\cos^2 u}(2\tan^2 u + 5) + 3\ln(\dfrac{1+\sin u}{\cos u})\right) \end{pmatrix}$$

or;

$$I_2 = \frac{-32}{3\sqrt{kE}(1-p^*)}\begin{pmatrix} (1-\frac{3}{2}P^* + \frac{3}{8}P^{*2} + \frac{p^{*3}}{16})\sqrt{\dfrac{1-y^*}{1-p^*}} + \\ (\frac{3}{2}P^* - \frac{9}{16}P^{*2} - \frac{15}{128}P^{*3})\ln\sqrt{\dfrac{(1-y^*)(1-p^*)}{y^*-p^*}} + \\ \begin{pmatrix} \frac{3}{16}p^{*2} + \\ \frac{1}{128}p^{*3}(7 - 2\dfrac{1-y^*}{y^*-p^*}) \end{pmatrix}\dfrac{(\sqrt{1-p^*} + \sqrt{1-y^*})}{y^*-p^*} \end{pmatrix} \tag{29a}$$

and

$$b_1 = \left(1 - \frac{p^*}{2} - \frac{p^{*2}}{8} - \frac{p^{*3}}{16}\right),\ b_2 = \left(\frac{p^*}{2} + \frac{p^{*2}}{4} + \frac{3p^{*3}}{16}\right),$$

$$b_3 = -\left(\frac{p^{*2}}{8} + \frac{3p^{*3}}{16}\right),\ b_4 = \left(\frac{P^{*3}}{16}\right)$$

$$I_4=\frac{8}{\sqrt{kE}(1-p^*)}\left(\begin{array}{l}(1-\frac{1}{2}P^*-\frac{1}{8}P^{*2}-\frac{1}{16}p^{*3})\int\frac{du}{\cos u}+\\(\frac{1}{2}P^*-\frac{1}{4}P^{*2}+\frac{3}{16}P^{*3})\int\frac{du}{\cos^3 u}\\-\left(\begin{array}{l}\frac{1}{8}p^{*2}+\\\frac{3}{16}p^{*3}\end{array}\right)\int\frac{du}{\cos^5 u}+\frac{1}{16}p^{*3}\int\frac{du}{\cos^7 u}\end{array}\right)$$

$$I_4=\frac{8}{\sqrt{kE}(1-p^*)}\left(\begin{array}{l}(1-\frac{1}{2}P^*-\frac{1}{8}P^{*2}-\frac{1}{16}p^{*3})\ln\left(\frac{1+\sin u}{\cos u}\right)+\\\frac{1}{2}(\frac{1}{2}P^*-\frac{1}{4}P^{*2}+\frac{3}{16}P^{*3})\left(\begin{array}{l}\ln\left(\frac{1+\sin u}{\cos u}\right)+\\\frac{\sin u}{\cos^2 u}\end{array}\right)\\-\frac{1}{8}\left(\frac{1}{8}p^{*2}+\frac{3}{16}p^{*3}\right)\left(\begin{array}{l}\frac{\sin u}{\cos^2 u}(2\tan^2 u+5)+\\3\ln(\frac{1+\sin u)}{\cos u}\end{array}\right)+\\\frac{1}{16}p^{*3}\left(\begin{array}{l}\frac{\sin u}{\cos^2 u}\left(\frac{25}{48}+\frac{5}{24}\tan^2 u+\frac{1}{6\cos^4 u}\right)\\+\frac{5}{16}\ln\left(\frac{1+\sin u}{\cos u}\right)\end{array}\right)\end{array}\right) \quad (29b)$$

or;

$$I_4=\frac{8}{\sqrt{kE}(1-p^*)}\left(\begin{array}{l}\left(\begin{array}{l}1-\frac{1}{4}P^*-\frac{3}{64}P^{*2}-\\\frac{5}{256}p^{*3}\end{array}\right)\ln\\\left(\sqrt{\frac{(1-y^*)(1-p^*)}{y^*-p^*}}\right)+\\\left(\begin{array}{l}\frac{1}{4}p^*+\frac{1}{64}P^{*2}\left(\begin{array}{l}3-\\2\frac{1-y^*}{y^*-p^*}\end{array}\right)+\\\frac{1}{768}P^{*3}\\\left(7-26\frac{1-y^*}{y^*-p^*}+\frac{8(1-p^*)^2}{(y^*-p^*)^2}\right)\end{array}\right)\\\left(\frac{\sqrt{(1-y^*)(1-p^*)}}{y^*-p^*}\right)\end{array}\right) \quad (30)$$

Noting that:

$$\int\frac{du}{\cos u}=\left(\ln(\frac{1+\sin u}{\cos u})\right)$$

$$\int\frac{du}{\cos^3 u}=\frac{1}{2}\left(\frac{\sin u}{\cos^2 u}+\ln(\frac{1+\sin u}{\cos u})\right)$$

$$\int\frac{du}{\cos^5 u}=\frac{1}{8}\left(\frac{\sin u}{\cos^2 u}(2\tan^2 u+5)+3\ln(\frac{1+\sin u}{\cos u})\right)$$

$$\int\frac{du}{\cos^7 u}=\left(\frac{\sin u}{\cos^2 u}\left(\frac{5}{24}\tan^2 u+\frac{25}{48}+\frac{1}{6\cos^4 u}\right)+\frac{5}{16}\ln(\frac{1+\sin u}{\cos u})\right)$$

Substituting (29) and (30) into (9c) yields

$$\mu_{par}x^*=\varepsilon\left(\begin{array}{l}\left(1-\frac{9}{4}P^*+\frac{45}{64}P^{*2}+\frac{35}{256}p^{*3}\right)\ln\\\left(\sqrt{\frac{(1-y^*)(1-p^*)}{y^*-p^*}}\right)+\\\frac{1}{12}\left(24p^*-6p^{*2}-p^{*3}-16\right)\\\sqrt{\frac{1-y^*}{1-p^*}}\\\left(\frac{1}{4}p^*-\frac{1}{64}P^{*2}\left(13+2\frac{1-y^*}{y^*-p^*}\right)+\right.\\P^{*3}\left(\begin{array}{l}-\frac{49}{768}-\frac{5(1-y^*)}{384(y^*-p^*)}+\\\frac{(1-p^*)^2}{96(y^*-p^*)^2}\end{array}\right)\left.\right)\\\left(\frac{\sqrt{(1-y^*)(1-p^*)}}{y^*-p^*}\right)\end{array}\right)+\text{constant} \quad (31)$$

where $\varepsilon=\frac{8}{\sqrt{kE}(1-P^*)}$, and $\mu_{Par.}=4(\alpha kE)^{1/2}C_m/3$.

In the range of $\frac{2}{1+p^*}<\frac{y^*}{p^*}$, the maximum percentage errors of this approximation are less than 0.76%.

The integration constant in (31) can be eliminated as

$$\mu L/E=\psi(y_2^*)-\psi(y_1^*)$$

in which $L$ is the length of the side weir and $\psi(y^*)$ is defined as

$$\psi(y^*)=\varepsilon\left(\begin{array}{l}\left(1-\frac{9}{4}P^*+\frac{45}{64}P^{*2}+\frac{35}{256}p^{*3}\right)\ln\\\left(\sqrt{\frac{(1-y^*)(1-p^*)}{y^*-p^*}}\right)+\frac{1}{12}\left(24p^*-6p^{*2}-p^{*3}-16\right)\\\sqrt{\frac{1-y^*}{1-p^*}}\left(\begin{array}{l}\frac{1}{4}p^*-\frac{1}{64}P^{*2}\left(13+2\frac{1-y^*}{y^*-p^*}\right)+\\P^{*3}\left(\begin{array}{l}-\frac{49}{768}-\frac{5(1-y^*)}{384(y^*-p^*)}+\\\frac{(1-p^*)^2}{96(y^*-p^*)^2}\end{array}\right)\end{array}\right)\\\left(\frac{\sqrt{(1-y^*)(1-p^*)}}{y^*-p^*}\right)\end{array}\right) \quad (32)$$

## V. Verification

To verify the proposed analytical solution, a comparison between analytical solution for water surface profile over side weir in rectangular, triangular and trapezoidal cross-sections proposed by Vatankhah [9], [10] and the analytical solution proposed by author is performed. The results show identical solutions were obtained by both authors. Also another comparison was made for parabolic section solved by Vatankhah using incomplete elliptic integrals and forth order Runge –Kutta method [11] and the solution proposed by the author, the results of comparison approximately are the same. Noting that the proposed solution is limited for the range $\frac{p^*}{y^*} < \frac{1+p^*}{2}$ and it is a direct (semi-analytical) solution.

## VI. Conclusion

General, analytical formula for calculating water surface profiles along a rectangular side weir in a combined channel has been derived. The proposed solution is an exact analytical one for rectangular, triangular, trapezoidal sections and semi analytical solution for parabolic section that involves expanding binominal series of power 3/2 and 1/2. The maximum percentage error of neglecting terms of powers more than 3 doesn't exceed 0.76% for $\frac{p^*}{y^*} < \frac{1+p^*}{2}$. The proposed solution is based on the constant specific energy, constant weir coefficient and constant velocity distribution coefficient along the side weir and allow for computing wholly subcritical or wholly supercritical water surface in rectangular, triangular, trapezoidal channels, and partially for ( $\frac{\boldsymbol{y}^*}{\boldsymbol{p}^*} > \frac{2}{1+\boldsymbol{p}^*}$ ) for parabolic channel. The efficient analytical and semi-analytical tool presented in this study will hopefully be useful in evaluating and designing side weirs in combined open channel.

## References

[1] Allen J. W. The discharge of water over side weirs in circular pipes. In: Proceedings of the institution of civil engineers, vol. 6. London, England; 1957: p. 270-87.
[2] Collinge VK. The discharge capacity of side weirs. Proceedings of the institution of civil engineers, vol. 6. London, England; 1957: p. 288-304
[3] Ranga Raju, K. G., Prasad, B., and Kupta, S. K., Side weir in rectangular channel. J. Hydraul. Div., 1977; 105(5), 547-554.
[4] Delo E. A., Saul A. J. Charts for the hydraulic design of high side-weirs in storm sewage of overflows. Proceeding of the Institution of Civil Engineers 1988;87 (2):175-193.
[5] Uymaz A., Smith R. H. Design procedure for flow over side weirs. Journal of Irrigation and Drainage Engineering 1991;117(1):79-90.
[6] Uymaz A, Muslu Y. Flow over side weirs in circular channels. Journal of Hydraulic Engineering 1985;111(1):144-160.
[7] Vatankhah A. R. New solution method for water surface profile along a side weir in circular channel. Journal of Irrigation and Drainage Eng. (ASCE) 2012; 138(9-c) 948-954.
[8] Uymaz A. Side weir in triangular channel. Journal of Irrigation and Drainage Eng. (ASCE) 1992; 118(6) 965-970.
[9] Vatankhah A. R. Analytical solution method for water surface profile along a side weir in triangular channel. Flow Measurement and Instrumentation 2012; 23(1) 76-79.
[10] Vatankhah A. R. Water surface profile over side weir in trapezoidal channel. Proceeding of the Institution of Civil Engineers (ICE) Water Management 2012; 165(5) 247-252.
[11] Vatankhah AR. Water surface profile along a side weir in parabolic channel. Flow Measurement and Instrumentation 2013; 32(1) 90-95.
[12] Vatankhah AR. Water Surface profile along a Side weir in a U- Shaped Channel (Analytical Findings). Journal of Hydrologic Eng. (ASCE) 2013; 18(5) 595-602.