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Numerical Solution of Manning's Equation in Rectangular Channels

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Abstract**—**When the Manning equation is used, a unique value of normal depth in the uniform flow exists for a given channel geometry, discharge, roughness, and slope. Depending on the value of normal depth relative to the critical depth, the flow type (supercritical or subcritical) for a given characteristic of channel conditions is determined whether or not flow is uniform. There is no general solution of Manning's equation for determining the flow depth for a given flow rate, because the area of cross section and the hydraulic radius produce a complicated function of depth. The familiar solution of normal depth for a rectangular channel involves 1) a trial-and-error solution; 2) constructing a non-dimensional graph; 3) preparing tables involving non-dimensional parameters. Author in this paper has derived semi-analytical solution to Manning's equation for determining the flow depth given the flow rate in rectangular open channel. The solution was derived by expressing Manning's equation in non-dimensional form, then expanding this form using Maclaurin's series. In order to simplify the solution, terms containing power up to 4 have been considered. The resulted equation is a quartic equation with a standard form, where its solution was obtained by resolving this into two quadratic factors. The proposed solution for Manning's equation is valid over a large range of parameters, and its maximum error is within -1.586%.

Keywords**—**Channel design, civil engineering, hydraulic engineering, open channel flow, Manning's equation, normal depth, uniform flow.

I. Introduction

ONE of the most important variables required in designing irrigation channels and analyzing gradually varied flow is the normal depth. Many relationships were derived to define normal depth in open channels such Manning's equation, Chezy's formula, and others [1]-[3]. Unfortunately, all of the available methods are time-consuming trial-and-error methods for determination of normal depth. Author in this paper has derived a semi-analytical solution for determination the normal flow depth of rectangular channel using Manning formula. The computation of normal depth of uniform flow using Manning’s equation may be obtained by rearranging the equation as [5]:

$$AR^{2/3} = \frac{A^{5/3}}{P^{2/3}} = \frac{nQ}{\sqrt{s_0}} \tag{1}$$

A= cross section area, *R*= hydraulic radius, *P*= wetted perimeter, *n*= roughness coefficient; *Q*= discharge; and s_0 = bed slope.

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In the case of a rectangular channel, the equation in non-dimensional form becomes:

$$\frac{nQ}{\sqrt{s_0}b^{8/3}} = \frac{(y/b)^{5/3}}{(2y/b+1)^{2/3}} \tag{2}$$

in which *b*=channel width; y=normal depth. Designating:

$$y' = \frac{y}{b};\frac{nQ}{\sqrt{s_0}b^{8/3}} = Z_{\mathrm{Re}c} \tag{3}$$

II. Proposed Solution

Combining (2) and (3) gives

$$Z^3{}_{\mathrm{Re}c} = y'^5/(2y'+1)^2 \tag{4}$$

Taking the logarithm of both sides of (4) yields:

$$\ln Z^3{}_{\mathrm{Re}c} = 5\ln y' - 2\ln(2y'+1) \tag{5}$$

Designating

$$y' = \frac{1}{2}e^x \tag{6}$$

Substituting into (5)

$$\ln Z^3{}_{\mathrm{Re}c} = 5x - 5\ln - 2\ln(1+e^x) \tag{7}$$

Expanding the function $\ln(1+e^x)$ using Maclaurin’s series, we obtain:

$$\ln(1+e^x) = \ln 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots\cdots \tag{8}$$

In order to simplify the solution, terms of expanded series containing power more than 4 will be omitted. The effect of omitting these terms will be shown later.

Substituting the function $\ln(1+e^x)$ from (8) into (7) yields

$$\ln Z^3{}_{\mathrm{Re}c} = 5x - 7\ln 2 - x - \frac{x^2}{4} + \frac{x^4}{96} \tag{9}$$

Rearranging (9)

$$x^4 - 24x^2 + 384x - 672\ln 2.Z^3{}_{\mathrm{Re}c} = 0 \tag{10}$$

Equation (10) is a quartic equation with a standard form

$$x^4 + \alpha_1 x^2 + \alpha_2 x + \alpha_3 = 0 \qquad (11)$$

in which

$$\alpha_1 = -24, \alpha_2 = 384, \alpha_3 = -672 \ln 2.Z^3{}_{\mathrm{Re}\,c} \qquad (12)$$

The following simple method for solving the quartic equation in a standard form is worth noting amongst various other methods. We shall resolve this into two quadratic factors, which, since there is no x^3 term, must be of the form [4]

$$(x^2 + ax + b)(x^2 - ax + c) = 0 \qquad (13)$$

Comparing coefficients of these two expressions, we have:

$$\alpha_1 = b + c - a^2, \alpha_2 = ac - ab, \alpha_3 = b.c \qquad (14)$$

So

$$b = \frac{1}{2}(\alpha_1 + a^2 - \frac{\alpha_2}{a}) \qquad (15)$$

$$c = \frac{1}{2}(\alpha_1 + a^2 + \frac{\alpha_2}{a}) \qquad (16)$$

The product of these is α_3, so

$$4\alpha_3 = (\alpha_1 + a^2)^2 - \frac{\alpha_2{}^2}{a^2} = 0 \qquad (17)$$

This is a bi-cubical in a^3:

$$\alpha^6 + 2\alpha_1 a^4 + (\alpha_1{}^2 - 4\alpha_3)a^2 - \alpha_2{}^2 = 0 \qquad (18)$$

Denoting

$$a^2 = u \qquad (19)$$

Then, (18) may be transformed into a cubic equation:

$$u^3 + 2a_1 u^2 + (\alpha_1{}^2 - 4\alpha_3)u - \alpha_2{}^2 = 0 \qquad (20)$$

To simplify the solution of (20), it is recommended to transform it into a standard form by removing u^2 term. Putting

$$u = z - (-\frac{48}{3}) = z + 16 \qquad (21)$$

The resulted standard cubic equation has the following form:

$$z^3 + pz + q = 0 \qquad (22)$$

where:

$$p = 192(-1 + 14 \ln 2 + 6 \ln Z_{\mathrm{Re}\,c}) \qquad (23)$$

$$q = 1024(-143 + 42 \ln 2 + 18 \ln Z_{\mathrm{Re}\,c}) \qquad (24)$$

The discriminant of (22) is

$$\Delta = 4p^3 + 27q^2 = 0 \qquad (25)$$

Analysis was made for series values of y'. It has been found that the discriminant is positive, i.e. $\Delta > 0$ for values of $y' > 0.007$, this means that (25) has only one real root. The practical range of y' lying between 0.05 and 2.956 [6] is appropriate to $Z_{\mathrm{Re}\,c} = 0.00637$ and 1.678, respectively.

A: Discussion

1. Determination the Sign of Variant p:

When

$$p < 0 \Rightarrow -1 + 14 \ln 2 + 6 \ln Z_{\mathrm{Re}\,c} < 0 \Rightarrow Z_{\mathrm{Re}\,c} < e^{\frac{1-14\ln 2}{6}}$$

or

$$Z_{\mathrm{Re}\,c} < 0.23441 \qquad (26)$$

2. Determination the Sign of Variant q:

When

$$q < 0 \Rightarrow -143 + 42 \ln 2 + 18 \ln Z_{\mathrm{Re}\,c} < 0 \Rightarrow Z_{\mathrm{Re}\,c} < e^{\frac{143-42\ln 2}{18}}$$

or

$$Z_{\mathrm{Re}\,c} < 559.532 \qquad (27)$$

Since $Z_{\mathrm{Re}\,c}$ is less than 559.532 for all analyzed y' as we mentioned earlier, q is always negative. Determination the root of (25) for $Z_{\mathrm{Re}\,c} > 0.23441$ to find the root of (25) for $Z_{\mathrm{Re}\,c} > 0.23441$ i.e.; p is positive, we seek values of z and θ by comparing (25) with a hyperbolic sin function

$$4 \sinh^3 \theta + 3 \sinh \theta - \sinh 3\theta = 0 \qquad (28)$$

From the comparison, we obtain

$$\sinh 3\theta = \frac{143 - 42 \ln 2 - 18 \ln Z_{\mathrm{Re}\,c}}{(-1 + 14 \ln 2 + 6 \ln Z_{\mathrm{Re}\,c})^{3/2}} \qquad (29)$$

$$z = 16 \sinh \theta \sqrt{-1 + 14 \ln 2 + 6 \ln Z_{\mathrm{Re}\,c}} \qquad (30)$$

in which

$$\sinh \theta = \frac{e^{\theta} - e^{-\theta}}{2} \qquad (31)$$

$$\theta = \frac{1}{3}\ln(\sinh 3\theta + \cosh 3\theta) \quad (32)$$

$$\cosh 3\theta = \sqrt{1 + \sinh^2 3\theta} \quad (33)$$

Substituting (30) into (21) yields

$$u = 16(1 + \sinh\theta\sqrt{-1 + 14\ln 2 + 6\ln Z_{\mathrm{Re}c}}) \quad (34)$$

Substituting (34) into (16) yields

$$a = 4\sqrt{(1 + \sinh\theta\sqrt{-1 + 14\ln 2 + 6\ln Z_{\mathrm{Re}c}})} \quad (35)$$

Then, we need to determine the root of (25) for $Z_{\mathrm{Re}c} < 0.23441$.

For $Z_{\mathrm{Re}c} < 0.23441$, i.e. p is negative, we seek values of z and θ by comparing (25) with a hyperbolic cosine function

$$4\cosh^3\theta - 3\cosh\theta - \cosh 3\theta = 0 \quad (36)$$

$$\cosh 3\theta = \frac{-143 + 42\ln 2 + 18\ln Z_{\mathrm{Re}c}}{(-1 + 14\ln 2 - 6\ln Z_{\mathrm{Re}c})^{3/2}} \quad (37)$$

From the comparison, we obtain

$$z = 16\cosh\theta\sqrt{1 - 14\ln 2 - 6\ln Z_{\mathrm{Re}c}} \quad (38)$$

in which

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2} \quad (39)$$

$$\theta = \frac{1}{3}\ln(\sinh 3\theta + \cosh 3\theta) \quad (40)$$

$$\sinh 3\theta = \sqrt{\cosh^2 3\theta - 1} \quad (41)$$

Substituting (38) into (21) yields

$$u = 16(1 + \cosh\theta\sqrt{1 - 14\ln 2 - 6\ln Z_{\mathrm{Re}c}}) \quad (42)$$

Substituting (42) into (16) yields

$$a = 4\sqrt{(1 + \cosh\theta\sqrt{1 - 14\ln 2 - 6\ln Z_{\mathrm{Re}c}})} \quad (43)$$

Returning to (13), we find:

$$x^2 + ax + b = 0 \quad (44)$$

or

$$x^2 - ax + c = 0 \quad (45)$$

An analysis has been made using EXCEL Software for all selected values of y'. From this analysis, it has been found that (45) gives imaginary roots, because the discriminant of this equation is negative. In contrast, the discriminant of (44) is positive for $a < 10.88$, which is appropriate for all analyzed values of y'. Therefore, we have two roots for (44). One of these roots gives very small values of y', which are out the scope of this analysis, so this root was neglected. The other root gives reliable values of y', which are closed to those assumed. This root is given by:

$$x = \frac{1}{2}\left[-a + \sqrt{48 - a^2 + \frac{768}{a}}\right] \quad (46)$$

Substituting x from (46) into (6) yields a simple formula for calculating the non-dimensional depth in rectangular channel:

$$y' = e^{\frac{1}{2}\left[-a + \sqrt{48 - a^2 + \frac{768}{a}}\right]} \quad (47)$$

III. Evaluation of Accuracy

To evaluate the accuracy of the abovementioned method, the relative errors using this method are given in Table I. From this table, it can be seen that the relative errors involved in the use of (47) are very small for practical range of y'. These errors are a result of omitting terms of expanded series containing powers more than 4. It is evident that, for large values of y' ($y' = 7.3$) and very small values of y' ($y' = 0.03$) which do not exist in practice, it produces errors up to - 4.75%. On the other hand, the proposed method is widely applicable ($y' = 10$), and the relative error involved in use of (47)"is -8.06%.

IV. Numerical Examples

It is decided to demonstrate the proposed solution through numerical examples:

A. Example 1

Find the normal depth of flow in a rectangular channel having longitudinal slope of 0.0001 and bed width $10m$, when it carries at $20m^3/\sec$ and $n = 0.012$

Solution:

- Calculate $Z_{\mathrm{Re}c}$ from (3), $Z_{\mathrm{Re}c} = 0.05171$
- Since $Z_{\mathrm{Re}c} < 0.23441$, we use (29), (31)-(33), (35), (46), and (47) to calculate each of the following parameters respectively:

$\cosh 3\theta = 6.12263$, $\sinh 3\theta = 6.04041$, $\theta = 0.8328$, $\cosh\theta = 1.3673$, $a = 9.04873$, $x = -0.95385$, $y' = 0.19263$

Lastly, we find the normal depth:

$$y = y'.b = 1.9263\,m$$

The exact answer is 1.9264 m and the relative error is $\varepsilon = -0.002\%$

TABLE I
RELATIVE ERROR OF USING PROPOSED SOLUTION FOR UNIFORM DEPTH IN RECTANGULAR CHANNEL

Non- Dimensional Normal Depth y'	Parameter $Z_{\mathrm{Re}\,c}$	Present Formula (47) Non-Dimensional Normal Depth y'	Relative Error $\varepsilon\%$
0.03	0.0028	0.0860	-4.750
0.05	.00637	0.049	-1.586
0.1	0.0191	0.0998	-0.217
0.56	0.2306	0.560	0.002
0.7	0.3078	0.700	0.002
1.0	0.4808	1.000	0.0004
1.5	0.7800	1.499	-0.029
2.0	1.0858	1.997	-0.123
2.5	1.3947	2.493	-0.296
2.956	1.6779	2.941	-0.530
5	2.9560	4.891	-2.185
7.3	4.4000	6.953	-4.750

B. Example 2

A discharge $3m^3/\sec$ in a rectangular 1.5-m width channel having set to a slope of 0.0004. Find the normal depth of flow if $n = 0.015$.

Solution:

- Calculate $Z_{\mathrm{Re}\,c}$ from (3), $Z_{\mathrm{Re}\,c} = 0.76314$.
- Since $Z_{\mathrm{Re}\,c} > 0.23441$, we use (37), (39)-(41), (43), (46), and (47) to calculate each of the following parameters respectively:

$$\sinh 3\theta = 6.30,\ \cosh 3\theta = 6.38,\ \theta = 0.84668,$$
$$\sinh\theta = 095153,\ a = 7.51771,\ x = 1.0796,\ y' = 1.47176$$

Lastly, we find the normal depth:

$$y = y'.b = 2.20764\,m$$

The exact answer is 2.2082 m and the relative error is $\varepsilon = -0.026\%$.

V. CONCLUSION

Semi-analytical equation has been derived for calculating normal depth in rectangular open channels. The proposed solution was achieved by generalizing Manning's equation, then expanding it using Maclaurin's series. The accuracy of proposed solution and practical range of applicability are shown in Table I. It is worth noting that the derived solution is simple, easy-to-use, and useful for irrigation engineer.

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