An Analytical Method for Solving General Riccati Equation

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Abstract—In this paper, the general Riccati equation is analytically solved by a new transformation. By the method developed, looking at the transformed equation, whether or not an explicit solution can be obtained is readily determined. Since the present method does not require a proper solution for the general solution, it is especially suitable for equations whose proper solutions cannot be seen at first glance. Since the transformed second order linear equation obtained by the present transformation has the simplest form that it can have, it is immediately seen whether or not the original equation can be solved analytically. The present method is exemplified by several examples.

Keywords—Riccati Equation, ordinary differential equation, nonlinear differential equation, analytical solution, proper solution.

I. Introduction

THE generalized Riccati equation is defined by

$$\frac{dy}{dx} + P(x)y + Q(x)y^2 - R(x) = 0 \tag{1}$$

Here, $P(x)$, $Q(x)$, and $R(x)$ are arbitrary functions of x. This equation is widely encountered in analytical mechanics, engineering and other fields. Therefore, depending on the functions $P(x)$, $Q(x)$ and $R(x)$, several methods have been developed to solve various types of Riccati equations [3], [4].

If $(x) = 0$, then the transformation $y = 1/z$ reduces (1) into a first order linear equation. For the case of $R(x) \neq 0$, several methods are available in the literature. The equation in this case is also known as Bernoulli equation.

The classical method for solving Riccati equation makes the transformation $\bar{y} = y_0 + 1/y$, where y_0 is a known solution of the equation. However, it is not possible to see a proper solution to the equation at every time. Therefore, this method has a limited usage. Among the other methods, Rao transformation and Allen-Stein transformation can be mentioned. The basic idea here is to bring the main equation into a separable form [1], [7], [8]. However, the differentiability condition is a serious problem with these methods.

Harko et al. have investigated the Riccati Equation and developed a restricted analytical solution [2]. Since the method requires specific conditions among the coefficients in the equation, it cannot be considered a general method. Sugai transformed Riccati equation into a second order differential equation by suggesting a new transformation [11]. Since the transformed equation is more complicated and unsolvable in most cases, it is also not general and applicable. Rao and Ukidave reduced Riccati equation into a separable form under restricted condition [9]. It is of no importance in respect of engineering application. Siller also investigated a separability condition of the equation [10].

Integrability condition for the Riccati equation has been studied by Mak and Harko, and a new method to generate analytical solutions of the Riccati equation was presented [5]. Mortici gives a new method of the variation of constants which leads directly to an equation with separable variables. This method also imposes several restrictions on the solution. Therefore, it cannot also be considered a general method [6].

When all methods are investigated, we observe that no method explains the question about whether or not the analytical solution is obtained in explicit, implicit, or power series form. In addition to finding the analytical solution to the problem for arbitrary values of $P(x)$, $Q(x)$, and $R(x)$, the present method answers these important questions.

II. Special Type Riccati Equation

In order to solve (1), consider a new transformation in the form

$$\bar{y} = f(x)e^{\int g(x)y(x)dx} \tag{2}$$

where (x), $g(x)$ are functions to be determined in a convenient manner. We differentiate both sides of (2) twice:

$$\bar{y}' = (f' + fgy)e^{\int g(x)y(x)dx} \tag{3}$$

and

$$\bar{y}'' = (fgy' + 2f'gy + fg'y + fg^2y^2 + f'')e^{\int g(x)y(x)dx} \tag{4}$$

Equation (4) can be written as

$$y' + \left[\frac{2f'}{f} + \frac{g'}{g}\right]y + gy^2 + \frac{f''}{fg} = \frac{1}{fg}\bar{y}''e^{-\int g(x)y(x)dx} \tag{5}$$

Recall that the left-hand side of (5) has the form of Riccati equation. Comparing (1) and (5), we have

$$\left[\frac{2f'}{f} + \frac{g'}{g}\right] = P(x) \tag{6a}$$

$$g(x) = Q(x) \tag{6b}$$

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$$\frac{f''}{fg} = -R(x) \tag{6c}$$

If we wish to solve an equation of the form

$$y' + \left[\frac{2f'}{f} + \frac{g'}{g}\right]y + gy^2 + \frac{f''}{fg} = 0 \tag{7}$$

then, by (5), in the first place, we can have

$$\bar{y}'' = 0 \;\rightarrow\; \bar{y} = ax + b \tag{8}$$

The functions f and g are to be obtained such that (6) are satisfied. Now, using the inverse transformation from (2), we obtain

$$y = \frac{1}{g}\frac{d}{dx}\left(\ln\frac{\bar{y}}{f}\right) \tag{9}$$

We recall that, for some simple types of Riccati equations, the solution is obtained without having any difficulty in solving the second order equation ((8)). In the other words, the solution is found without solving a complex second order differential equation with variable coefficients. The examples that are numbered from 1 to 4 illustrate the method.

Example 1: We first try to solve the equation

$$y' + 2y + y^2 + 1 = 0 \tag{10}$$

whose solution can also be found by integration. Comparing (10) with (7) gives

$$\left[\frac{2f'}{f} + \frac{g'}{g}\right] = 2 \tag{11a}$$

$$g = 1 \tag{11b}$$

$$\frac{f''}{fg} = 1 \tag{11c}$$

Inserting $g = 1$ into (11a) and solving the equation yields

$$f = ce^x \tag{12}$$

Equation (11c) is automatically satisfied. Now, using (9), we have

$$y = \frac{1}{g}\frac{d}{dx}\left(\ln\frac{\bar{y}}{f}\right) = \frac{1}{1}\frac{d}{dx}\left(\ln\frac{ax+b}{ce^x}\right) \tag{13}$$

or

$$y = \frac{1}{x+\bar{c}} - 1, \;\; \bar{c} = b/a \tag{14}$$

It can be verified that (14) satisfies (10).

Example 2: We now solve the equation

$$y' + 5xy + y^2 + \frac{25}{4}x^2 + \frac{5}{2} = 0 \tag{15}$$

Comparison of (15) with (7) yields

$$\left[\frac{2f'}{f} + \frac{g'}{g}\right] = 5x \tag{16a}$$

$$g = 1 \tag{16b}$$

$$\frac{f''}{fg} = \frac{25}{4}x^2 + \frac{5}{2} \tag{16c}$$

The function f can be obtained via (16a) and (16b)

$$f = ce^{\frac{5}{4}x^2} \tag{17}$$

Using (9), we find

$$y = \frac{1}{g}\frac{d}{dx}\left(\ln\frac{\bar{y}}{f}\right) = \frac{1}{1}\frac{d}{dx}\left(\ln\frac{ax+b}{ce^{\frac{5}{4}x^2}}\right) \tag{18}$$

or

$$y = \frac{1}{(x+\bar{c})} - \frac{5}{2}x, \;\; \bar{c} = b/a \tag{19}$$

Again (19) satisfies (15).

Example 3: (20) is required to be solved.

$$y' + 2xy + x^2y^2 - \frac{1}{x^2} + \frac{2}{x^4} + 1 = 0 \tag{20}$$

Comparison of (20) with (7) yields

$$\left[\frac{2f'}{f} + \frac{g'}{g}\right] = 2x \tag{21a}$$

$$g = x^2 \tag{21b}$$

$$\frac{f''}{fg} = -\frac{1}{x^2} + \frac{2}{x^4} + 1 \tag{21c}$$

The function f can readily be obtained

$$f = c.\frac{1}{x}.e^{\frac{x^2}{2}} \tag{22}$$

Using (9), we obtain

$$y = \frac{1}{g}\frac{d}{dx}\left(\ln\frac{\bar{y}}{f}\right) = \frac{1}{x^2}\frac{d}{dx}\left(\ln\frac{ax+b}{c.x^{-1}.e^{\frac{x^2}{2}}}\right) \tag{23}$$

or

$$y = \frac{1}{x^2(x+\bar{c})} - \frac{1}{x} + \frac{1}{x^3} \;\; \bar{c} = b/a \tag{24}$$

In the examples ever seen, the examples satisfying the condition $\bar{R}(x) = f''/fg$ automatically have been chosen. Otherwise, the unknown function $f(x)$ must be found such that the equations must be simultaneously satisfied.

$$\bar{P}(x) = \left[\frac{2f'}{f} + \frac{g'}{g}\right] \tag{25a}$$

$$\bar{Q}(x) = g \tag{25b}$$

$$\bar{R}(x) = \frac{f''}{fg} \tag{25c}$$

However, it is not always possible to obtain a unique $f(x)$ that satisfies both (25a) and (25c). To show this case, we

finally consider fourth example in this section.

Example 4: We try to solve the equation

$$y' + \alpha x^2 y + \beta x y^2 + \bar{R}(x) = 0 \tag{26}$$

Using (25a) and (25b), we can write

$$\left[\frac{2f'}{f} + \frac{g'}{g}\right] = \alpha x^2 \tag{27a}$$

$$g = \beta x \tag{27b}$$

The function f can be obtained as

$$f = c.x^{-\frac{1}{2}}.e^{\frac{\alpha x^3}{6}} \tag{28}$$

We then have

$$\bar{R}(x) = \frac{f''}{fg} = \frac{\alpha^2}{4\beta} x^3 + \frac{3}{4\beta} x^{-3} + \frac{\alpha}{2\beta} \tag{29}$$

Thus, the present method gives a solution as long as the condition given in (29) is satisfied. The new form of (24) becomes

$$y' + \alpha x^2 y + \beta x y^2 + \frac{\alpha^2}{4\beta} x^3 + \frac{3}{4\beta} x^{-3} + \frac{\alpha}{2\beta} = 0 \tag{30}$$

The solution can be obtained in the following form

$$y = \frac{1}{g}\frac{d}{dx}\left(\ln\frac{\bar{y}}{f}\right) = \frac{1}{\beta x}\frac{d}{dx}\left(\ln\frac{ax+b}{c.x^{-\frac{1}{2}}.e^{\frac{\alpha x^3}{6}}}\right) \tag{31}$$

or

$$y = \frac{1}{\beta x(x+\bar{c})} - \frac{\alpha x}{2\beta} + \frac{1}{2\beta x^2} \quad \bar{c} = b/a \tag{32}$$

III. The Riccati Equation of General Type

The transformation in (2) prevents the free choice of the term $Q(x)$ in (1). So, the method can be valid only for special type Riccati equations. Now, we wish to remove this restriction to have the analytical solution of general type Riccati equation involving arbitrary $P(x)$, $Q(x)$, and $R(x)$. Again, we consider the same transformation

$$\bar{y} = f(x)e^{\int g(x)y(x)dx} \tag{33}$$

Differentiating (33) twice yields

$$\bar{y}' = (f' + fgy)e^{\int g(x)y(x)dx} \tag{34}$$

and

$$\bar{y}'' = (fgy' + 2f'gy + fg'y + fg^2y^2 + f'')e^{\int g(x)y(x)dx} \tag{35}$$

Equation (35) can be written as

$$y' + \left[\frac{2f'}{f} + \frac{g'}{g}\right]y + gy^2 + \frac{f''}{fg} = \frac{1}{fg}\bar{y}''e^{-\int g(x)y(x)dx} \tag{36}$$

In order to remove the restriction in Section II, we now assume that we seek the solution of the equation

$$y' + \left[\frac{2f'}{f} + \frac{g'}{g}\right]y + gy^2 + \frac{f''}{fg} = S(x) \tag{37}$$

where $S(x)$ is a function to be determined. Thus, (36) gives

$$\bar{y}'' = f.g.S(x).e^{\int g(x)y(x)dx} \tag{38}$$

However, by (33), we can also write

$$\bar{y}'' = gS(x)(\bar{y}) \tag{39}$$

Replacing $\bar{y}$ by $u(x)$, we obtain

$$u'' - gSu = 0 \tag{40}$$

Equation (40) is a linear second order ordinary differential equation whose general solution is generally obtained in power series. However, if gS is a constant or something that leads to the analytical solution of (40), then the explicit form of u or y can always be obtained.

Now, an attention should be paid to (40). According to this result, we are led to conclude that one of the main advantage of the transformation given by (33) is that it converts Riccati equation directly into the linear form of (40). Equation (40) has the simplest form of a second order differential equation. This form also provides us the knowledge of whether we can obtain the explicit solution of (37) or not. After (40) is solved via classical methods, the inverse transformation yields the solution as

$$y = \frac{1}{g(x)}\frac{d}{dx}\left(\ln\frac{u(x)}{f(x)}\right) \tag{41}$$

Since (41) does not involve any expression to be integrated, there exists no difficulty in obtaining the explicit form of $y(x)$.

Example 5: As a first example in this section, we solve the equation

$$y' + 2xy - y^2 - (1 + x^2) = 0 \tag{42}$$

Comparing (42) and (37), we have

$$g = -1 \tag{43a}$$

$$\left[\frac{2f'}{f} + \frac{g'}{g}\right] = 2x \tag{43b}$$

$$\frac{f''}{fg} = -(1 + x^2), \quad S(x) = 0 \tag{43c}$$

Solving (43a) and (43b) gives $f = ce^{x^2/2}$ (c=constant), and we can see immediately that f''/fg directly equals (x). Hence, $S(x)$ can be taken as zero. This means that $\bar{y}'' = 0$. The solution of $\bar{y}$ is given by $\bar{y} = ax + b$. Substituting this result into the inverse transformation, after some operations, we obtain

$$y = x - \frac{1}{x+\bar{c}} \quad (\bar{c} = constant) \tag{44}$$

It can be checked that (44) satisfies (42).

Example 6: We seek the solution of the equation

$$y' + 8xy + 4y^2 + 4x^2 - 3 = 0 \tag{45}$$

Comparing (37) and (45), we have

$$g = 4 \tag{46a}$$

$$\left[\frac{2f'}{f} + \frac{g'}{g}\right] = 8x \tag{46b}$$

Solving (46a) and (46b) gives $f = ce^{2x^2}$ (c=constant). The other condition yields

$$\frac{f''}{fg} = 4x^2 + 1, \ \ S(x) = 4 \tag{47}$$

The transformed equation in this case becomes

$$u'' - 16u = 0 \tag{48}$$

The characteristic equation of (48) is

$$m^2 - 16 = 0 \tag{49}$$

whose roots are $m_1 = 4$ and $m_2 = -4$. Thus, the solution of (48) has the form

$$u = c_1 e^{4x} + c_2 e^{-4x} \tag{50}$$

Using (41), we obtain $y(x)$ as

$$y = \frac{1}{g(x)} \frac{d}{dx}\left(\ln \frac{u(x)}{f(x)}\right) = \frac{1}{4} \frac{d}{dx}\left(\ln \frac{c_1 e^{4x} + c_2 e^{-4x}}{ce^{2x^2}}\right) \tag{51}$$

$$y = \frac{c_1 e^{4x} - c_2 e^{-4x}}{c_1 e^{4x} + c_2 e^{-4x}} - x \tag{52}$$

It can be verified that

$$y = \frac{e^{8x} - \bar{c}}{e^{8x} + \bar{c}} - x, \ \ (\bar{c} = c_2/c_1) \tag{53}$$

Note that if we take $c_1 e^{4x}$ as the solution of u, then we obtain the proper solution $y_p = 1 - x$. In addition, if we take $c_2 e^{-4x}$ only, then we find the other proper solution of (44) as $y_p = -1 - x$. Indeed, we can verify that these are proper solutions of (45). Thus, the method presented also gives the proper solutions of Riccati Equation. Then, when desired, one can use the famous transformation $y = \bar{S}(x) + 1/x$ to find the general solution. Here, $\bar{S}(x)$ is a proper solution.

Example 7: We now try to solve the equation

$$y' + \left(4 - \frac{1}{x}\right)y + \frac{1}{x}y^2 - x^2 + 4x = 0 \tag{54}$$

We recall that a proper solution of (54) is not readily seen. Therefore, the well-known classical method cannot be used. It

is obvious that (55a) and (55b) are valid:

$$g = \frac{1}{x} \tag{55a}$$

$$\left[\frac{2f'}{f} + \frac{g'}{g}\right] = 4 - \frac{1}{x} \tag{55b}$$

Solving (55a) and (55b) gives $f = ce^{2x}$ (c=constant). Now, noting that

$$\frac{f''}{fg} = 4x \tag{56}$$

from (37), we obtain $S(x)$ as

$$\frac{f''}{fg} - S(x) = -x^2 + 4x \implies 4x - S(x) = -x^2 + 4x \implies S(x) = x^2 \tag{57}$$

Hence, (40) reads

$$u'' - xu = 0 \tag{58}$$

Equation (58) is Airy equation, and its solution is given by

$$\bar{y} = u = c_1 AiryAi(x) + c_2 AiryBi(x) \tag{59}$$

Here, the functions $AiryAi(x)$ and $AiryBi(x)$ are given as

$$AiryAi(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2.3.5.6...(3n-3).(3n-1).3n}$$

$$AiryBi(x) = x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3.4.6.7...(3n-2).3n.(3n+1)} \tag{60}$$

Their derivatives are given by

$$(AiryAi(x))' = \sum_{n=1}^{\infty} \frac{x^{3n-1}}{2.3.5.6...(3n-3).(3n-1)}$$

$$(AiryBi(x))' = 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{3.4.5.6.7...(3n-2).3n} \tag{61}$$

Hence, using the inverse transformation, we obtain

$$y = x\left(\frac{(AiryAi(x))' + c.(AiryBi(x))'}{AiryAi(x) + c.AiryBi(x)} - 2\right) \tag{62}$$

where c is a constant.

Example 8: We try to solve the equation which has been studied by Mortici [6]

$$y' - \frac{\beta}{x}y - \alpha y^2 - \frac{\gamma}{x^2} = 0 \tag{63}$$

Comparing (37) and (63), we have

$$g = -\alpha \tag{64a}$$

$$\left[\frac{2f'}{f} + \frac{g'}{g}\right] = -\frac{\beta}{x} \tag{64b}$$

Solving (64a) and (64b) gives $f = cx^{-\beta/2}$. (c=constant). Noting that

$$\frac{f''}{fg} = \frac{-(\beta^2+2\beta)}{4\alpha}\cdot\frac{1}{x^2} \tag{64c}$$

from (37), we obtain $S(x)$ as

$$\frac{f''}{fg} - S(x) = -\frac{\gamma}{2} \implies \frac{-(\beta^2+2\beta)}{4\alpha}\cdot\frac{1}{x^2} - S(x) = -\frac{\gamma}{2} \tag{65}$$

$$S(x) = (\gamma - \frac{(\beta^2+2\beta)}{4\alpha})\frac{1}{x^2} \tag{66}$$

We have the transformed equation as (67) via (40):

$$u'' - (-\alpha)(\gamma - \frac{(\beta^2+2\beta)}{4\alpha})\frac{1}{x^2}\, u = 0 \tag{67}$$

or

$$u'' + (\frac{4\gamma\alpha-(\beta^2+2\beta)}{4})\frac{1}{x^2}\, u = 0 \tag{68}$$

After this point, some cases can be considered.

Case 1: If we assume $4\gamma\alpha = (\beta^2 + 2\beta)$, the transformed equation in this case becomes

$$u'' = 0 \tag{69}$$

The form of u must be $u = ax + b$. Hence, (41) gives

$$y = \frac{1}{g}\frac{d}{dx}\left(\ln\frac{u}{f}\right) = -\frac{1}{\alpha}\frac{d}{dx}\left(\ln\frac{ax+b}{cx^{-\beta/2}}\right) \tag{70}$$

or

$$y = \frac{-1}{\alpha(x+\bar{c})} - \frac{\beta}{2\alpha x} \qquad \bar{c} = b/a \tag{71}$$

Case 2: If we assume $4\gamma\alpha \ne (\beta^2 + 2\beta)$ as a general case, then we can put $k = \frac{4\gamma\alpha-(\beta^2+2\beta)}{4} = const$. The transformed equation in this case becomes

$$u'' + \frac{k}{x^2}\, u = 0 \tag{72}$$

This is an Euler-Cauchy Equation which has the general form.

$$u'' + \frac{A}{x}u' + \frac{B}{x^2}\, u = 0 \tag{73}$$

This equation can be transformed into the linear form as

$$Y'' + (A-1)Y' + BY = 0, \qquad (A = 0, B = k) \tag{74}$$

The characteristic equation of (74) is

$$m^2 - m + k = 0 \tag{75}$$

whose roots are m_1 and m_2. m_1 and m_2 can be real, complex, or repeated. The form of solution depends on the types of the roots.

Case 2a: If $(1 - 4k) > 0$, $m_1 \ne m_2$ and $m_1, m_2 \in \mathbb{R}$. $m_{1,2} = \frac{1\pm\sqrt{1-4k}}{2}$

The solution for $u(x)$ in this case is

$$u = c_1 x^{m_1} + c_2 x^{m_2} \tag{76}$$

Then, the solution for $y(x)$ is given by

$$y = \frac{1}{g}\frac{d}{dx}\left(\ln\frac{u}{f}\right) = -\frac{1}{\alpha}\frac{d}{dx}\left(\ln\frac{c_1 x^{m_1}+c_2 x^{m_2}}{cx^{-\beta/2}}\right) \tag{77}$$

or

$$y = -\frac{1}{\alpha x}\left(\frac{c_1 m_1 x^{m_1}+c_2 m_2 x^{m_2}}{c_1 x^{m_1}+c_2 x^{m_2}} + \frac{\beta}{2}\right) \tag{78}$$

or

$$y = -\frac{1}{\alpha x}\left(\frac{m_1 x^{m_1}+\bar{c}m_2 x^{m_2}}{x^{m_1}+\bar{c}x^{m_2}} + \frac{\beta}{2}\right) \qquad \bar{c} = c_2/c_1 \tag{79}$$

Case 2b: If $(1 - 4k) = 0$ $m_1 = m_2 = m$ and $m \in \mathbb{R}$. The repeated root is $m = \frac{1}{2}$

The solution for $u(x)$ is

$$u = c_1 x^m + c_2 x^m \ln x \tag{80}$$

The solution in this case is obtained as

$$y = \frac{1}{g}\frac{d}{dx}\left(\ln\frac{u}{f}\right) = -\frac{1}{\alpha}\frac{d}{dx}\left(\ln\frac{c_1 x^m+c_2 x^m \ln x}{cx^{-\beta/2}}\right) \tag{81}$$

or

$$y = -\frac{1}{\alpha}\left(\frac{c_1 m x^{m-1}+c_2 x^{m-1}(1+m\ln x)}{c_1 x^m+c_2 x^m \ln x} + \frac{\beta}{2x}\right) \tag{82}$$

or

$$y = -\frac{1}{2\alpha x}\left(1 + \frac{2}{\bar{c}+\ln x} + \beta\right), (m = 1/2 \text{ and } \bar{c} = c_1/c_2) \tag{83}$$

or

$$y = -\frac{\beta+1}{2\alpha x} - \frac{1}{x(\alpha\bar{c}+\alpha\ln x)} \tag{84}$$

Please note that (84) is equal to Mortici's solution [6] in this case.

Case 2c: $m_1, m_2 \notin \mathbb{R}$, $m_1 = a + bi$ and $m_2 = a - bi$, where $i = \sqrt{-1}$. The solution for $u(x)$ is

$$u = e^{-ax}(c_1 \cos bx + c_2 \sin bx) \tag{85}$$

Hence, the solution for $y(x)$ is finally obtained as

$$y = \frac{1}{g}\frac{d}{dx}\left(\ln\frac{u}{f}\right) = -\frac{1}{\alpha}\frac{d}{dx}\left(\ln\frac{e^{-ax}(c_1 \cos bx+c_2 \sin bx)}{cx^{-\beta/2}}\right) \tag{86}$$

or

$$y = -\frac{1}{\alpha}\left(\frac{(b\bar{c}-a)\cos bx-(a\bar{c}+b)\sin bx}{\cos bx+\bar{c}\sin bx} + \frac{\beta}{2x}\right), \; (\bar{c} = c_2/c_1) \tag{87}$$

IV. RESULT AND DISCUSSION

In the present work, we have proposed a new transformation which reduces Riccati equation of general type into a second order linear homogeneous differential equation which is readily solvable. The reduced equation has the simplest form of second order. Therefore, there is no need for extra operation to reduce the transformed equation. Looking at

the form of gS, we can immediately see whether the solution is expressed in explicit or implicit form. Since we do not need any proper solution already required for the analytical solution, the present method is further simple and can be preferred in teaching the solutions of Riccati differential equation. Since the method does not put any condition on the functions involved in the equation, the method is very general.

According to the present method, proper solutions of Riccati equation can also be obtained. Thus, using the conventional method of transforming the original equation into a linear first order differential equation can be followed. This situation is demonstrated in Example 6.

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