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A Hyperexponential Approximation to Finite-Time and Infinite-Time Ruin Probabilities of Compound Poisson Processes

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***Abstract*—This article considers the problem of evaluating infinite-time (or finite-time) ruin probability under a given compound Poisson surplus process by approximating the claim size distribution by a finite mixture exponential, say Hyperexponential, distribution. It restates the infinite-time (or finite-time) ruin probability as a solvable ordinary differential equation (or a partial differential equation). Application of our findings has been given through a simulation study.

***Keywords*—Ruin probability, compound Poisson processes, mixture exponential (hyperexponential) distribution, heavy-tailed distributions.

I. Introduction

CONSIDER the following compound Poisson process

$$U_t = u + ct - \sum_{j=1}^{N(t)} X_j, \quad (1)$$

where $X_1, X_2, \cdots$ are a sequence of i.i.d. random variables with common density function $f_X(\cdot)$, $N(t)$ is a Poisson process with intensity rate λ, u and c stand for initial wealth/reserve and premium of the process, respectively.

The finite-time and infinite-time ruin probabilities for the above compound Poisson process are, respectively, denoted by $\psi(u;T)$ and $\psi(u)$ and defined by

$$\begin{aligned} \psi(u;T) &= P(\tau_u \leq T) \\ \psi(u) &= P(\tau_u < \infty), \end{aligned} \quad (2)$$

where τ_u is the hitting time, i.e., $\tau_u := \inf\{t : U_t \leq 0 | U_0 = u\}$.

Reference [15], among others, established that an infinite-time ruin probability $\psi(u)$ under a compound Poisson process can be restated as the following integro-differential equation

$$c\tilde{\psi}^{(1)}(u) - \lambda\tilde{\psi}(u) + \lambda \int_0^u \tilde{\psi}(u-x) f_X(x)dx = 0, \quad (3)$$

where $\tilde{\psi}(u) = 1 - \psi(u)$ and $\lim_{u\to\infty} \psi(u) = 0$.

Reference [18] showed that a finite-time ruin probability $\psi(u;T)$ under a compound Poisson process can be restated as the following partial integro-differential equation

$$c\left(\frac{\partial\tilde{\psi}(u;T)}{\partial u} - \frac{\partial\tilde{\psi}(u;T)}{\partial T}\right) - \lambda\tilde{\psi}(u;T) + \lambda \int_0^u \tilde{\psi}(u-x;T) f_X(x)dx = 0, \quad (4)$$

where $\tilde{\psi}(u;T) = 1 - \psi(u;T)$ $\lim_{u\to\infty} \psi(u;T) = 0$ for all $T > 0$ and $\psi(u;0) = 0$ for all $u \geq 0$.

Since the compound Poisson surplus process plays a vital role in many actuarial models, several authors study ruin probability under surplus process (1). An excellent review for infinite-time ruin probability can be found in [3]. For finite-time ruin probability: [18] showed that for exponential claim size distribution, partial integro-differential equation (4) can be transformed into a second-order partial differential equation. Reference [1] considered a compound Poisson surplus process with constant force of real interest. Then, they restated finite-time ruin probability $\psi(u;T)$ as a gamma series expansion. Reference [14] provided a global Lagrange type approximation in the z-space for $\psi(u;T)$ under surplus process (1). References [2] and [24] employed the Padé approximant method to approximate $\psi(u;T)$ under surplus process (1).

This article in the first step approximates claim size density function $f_X(\cdot)$ with a finite mixture exponential, say Hyperexponential, density function $f_X^*(\cdot)$. Then, it transforms two integro-differential equations (3) and (4), respectively, into an ordinary differential equation (ODE) and a partial differential equation (PDE). A simulation study has been conducted to show practical application of our findings.

The rest of this article is organized as follows: Some mathematical background for the problem has been collected in Section II. Section III provides the main contribution of this article. Applications of the results have been given in Section IV.

II. Preliminaries

From hereafter now, we set $\sum_{j=a}^{b} A_j = 0$, for $b < a$.

The following recalls the exponential type T functions which play a vital role in the rest of this article.

Definition 1. *An $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ function f is said to be of exponential type T on $\mathbb{C}$ if there are positive constants M and T such that $|f(\omega)| \leq M \exp\{T|\omega|\}$, for $\omega \in \mathbb{C}$.*

The Fourier transforms of exponential type functions are continuous functions which are infinitely differentiable everywhere and are given by a Taylor series expansion over every compact interval, see [6], [25] for more details.

From the Hausdorff-Young Theorem, one can observe that if $\{s_n\}$ is a sequence of functions converging, in $L^2(\mathbb{R})$ sense, to s. Then, the Fourier transforms of s_n converge, in

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$L^2(\mathbb{R})$ sense, to the Fourier transform of s, see [17] for more details. Using [13]'s method, [12] showed that most of the common distributions do have characteristic functions that can be extend to meromorphic functions.

The following from [16], [21] recalls Hausdorff-Young inequality for the Laplace transform:

Lemma 1. *Suppose $h(\cdot)$ is a given and nonnegative function that $f \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$. Then, $||h||_2 \leq \frac{1}{\sqrt{\pi}}||\mathcal{L}(h)||_2$, where $\mathcal{L}(h)$ stands for the Laplace transform.*

The Schwarz integrability condition states that in situation that all partial derivatives of a bivariate function exist and are continuous, one may change order of partial derivatives, see [4] for more details.

The following lemma provides useful results for the next section.

Lemma 2. *Suppose $k(\cdot)$ is a given and differentiable function and $y(\cdot)$ is an unknown function that satisfy*

$$\int_0^x y(t)\left(\sum_{i=1}^n \omega_i\mu_i e^{-\mu_i(x-t)}\right)dt \;=\; k(x),\ x \geq 0, \quad (5)$$

where ω_i, μ_i and μ_i are some given and nonnegative constants. Then, the above integral equation can be transformed into differential equation

$$\begin{aligned} 0 \;=\; & \sum_{i=1}^n \omega_i\mu_i y^{(n-1)}(x) + \sum_{i=1}^n\sum_{i\neq i}^n \omega_i\mu_i\mu_j y^{(n-2)}(x) \\ & -\sum_{i=1}^n\sum_{i\neq i}^n\sum_{k>j,\neq i}^n \omega_i\mu_i\mu_j\mu_k y^{(n-3)}(x) \\ & +\sum_{i=1}^n\sum_{i\neq i}^n\sum_{k>j,\neq i}^n\sum_{l>k,\neq i}^n \omega_i\mu_i\mu_j\mu_k\mu_l y^{(n-4)}(x) \\ & -\cdots + (-1)^n \sum_{i=1}^n\prod_{j\neq i}^n \mu_j y^{(0)}(x) \\ & -k^{(n)}(x) - \sum_{i=1}^n \mu_i k^{(n-1)}(x) - \sum_{i=1}^n\sum_{j\neq i}^n \mu_i\mu_j k^{(n-2)}(x) \\ & -\sum_{i=1}^n\sum_{j\neq i}^n\sum_{k>j\neq i}^n \mu_i\mu_j\mu_k k^{(n-3)}(x) - \cdots - \prod_{i=1}^n \mu_i k^{(0)}(x). \end{aligned}$$

Proof. For $n = 1$ see [18]. For $n > 1$, set $A_i = \omega_i\mu_i$ and $h_i(x) = \int_0^x y(t)\exp\{-\mu_i(x-t)\}dt$. Using the fact that the n^{th} derivatives $h_i(x)$ with respect to x is $h_i^{(n)}(x) = (-\mu_i)^n h_i(x) + \sum_{j=0}^{n-1}(-\mu_i)^{n-1-j}y^{(j)}(x)$, one may restate all first n derivatives of (5) as the following system of equation.

$$\begin{cases} k^{(n)}(x) = \sum_{i=1}^n A_i\left[y^{(n-1)}(x) - \mu_i y^{(n-2)}(x) + \cdots \right. \\ \qquad\qquad \left. +(-\mu_i)^{n-1}y^{(0)}(x) + (-\mu_i)^n h_i(x)\right], \\ k^{(n-1)}(x) = \sum_{i=1}^n A_i\left[y^{(n-2)}(x) - \mu_i y^{(n-3)}(x) + \cdots \right. \\ \qquad\qquad \left. +(-\mu_i)^{n-2}y^{(0)}(x) + (-\mu_i)^{n-1} h_i(x)\right], \\ k^{(n-2)}(x) = \sum_{i=1}^n A_i\left[y^{(n-3)}(x) - \mu_i y^{(n-4)}(x) + \cdots \right. \\ \qquad\qquad \left. +(-\mu_i)^{n-3}y^{(0)}(x) + (-\mu_i)^{n-2} h_i(x)\right], \\ \vdots \qquad\qquad \vdots \quad \vdots \\ k^{(0)}(x) = \sum_{i=1}^n A_i h_i(x) \end{cases}$$

Multiplying both sides of the first equation by 1, the second equation by $\sum_{i=1}^n \mu_i$; the third equation by $\sum_{i=1}^n\sum_{j>i}^n \mu_i\mu_j$; the forth equation by $\sum_{i=1}^n\sum_{j>i}^n\sum_{k>j>i}^n \mu_i\mu_j\mu_k$; and so on until the last equation which multiplying its both sides by $\prod_{i=1}^n \mu_i$, then adding together all equations leads to the desired results. □

A. Hyperexponential Distributions

The Hyperexponential (or mixture exponential) distribution is characterized by the number of n exponential distributions with means $1/\mu_i$ and associated wight $\omega_i \in \mathbb{R}$ [1] (i.e. $\sum_{i=1}^n \omega_i = 1$). The density function for a $n-$component Hyperexponential distribution is given by

$$f_X^*(x) \;=\; \sum_{i=1}^n \omega_i\mu_i e^{-\mu_i x},\ x \geq 0. \quad (6)$$

Reference [7] showed that one may approximate a large class of distributions, including several heavy tail distributions such as Pareto and Weibull distributions, arbitrarily closely, by Hyperexponential distributions. Reference [8] established that a survival function at x^γ, for all $x > 0$, is a completely monotone function if and only if its corresponding density function is a mixture of Weibull distributions with fixed shape parameter $1/\gamma$. Reference [9] showed that any Weibull distribution with shape parameter less than 1 can be restated as a Hyperexponential distributions.

Using the Hausdorff-Young Theorem, the following provides error bound for approximating the claim size density function $f_X(\cdot)$ by Hyperexponential density function $f_X^*(\cdot)$, given by (6).

Lemma 3. *Suppose random claim size X is surplus process* (1) *has density function $f_X(\cdot)$ and characteristic function $\theta_X(\cdot)$. Moreover, suppose that characteristic function $\theta_X(\cdot)$ is (or can be extend to) a meromorphic function. Then,* (**1**) *density function of compound sum $S(t) = \sum_{i=1}^{N(t)} X_i$, say $f_{S(t)}(\cdot)$, can be approximated by density function $f_{S^*(t)}(\cdot)$, where $S(t) = \sum_{i=1}^{N(t)} Y_i$ and Y_i is a $n-$component Hyperexponential distribution;* (**2**) *Error bound for such approximation satisfies $||f_{S(t)} - f_{S^*(t)}||_2 \leq \lambda t e^{-\lambda t}||\theta_X - \theta_Y||_2$, where $\theta_Y(s) = \sum_{j=1}^n \omega_i\mu_j/(\mu_i + s\sqrt{-1})$.*

Proof. Using the Hausdorff-Young Theorem, one may can conclude that $||f_{S(t)} - f_{S^*(t)}||_2 \leq ||e^{\lambda t(\theta_X - 1)} - e^{\lambda t(\theta_Y - 1)}||_2$.

The rest of proof arrives by using the fact that ψ_X and θ_Y are (or can be extend to) two meromorphic functions. □

III. Ruin Probability

This section utilizes integro-differential Equations (3) and (4) to derive an approximate formula for the infinite (and finite)-time ruin probability of a compound Poisson process (1). We seek an analytical solution $\widetilde{\psi}(\cdot)$ which is an exponential type function. In the other word, we assume:

[1] The hyperexponential H_n setup falls into the more general framework of phase-type (PH) approximations. A main step of this article is that would be to allow the w_i in (6) to be negative; maybe this is already implicit in the paper, but it should be mentioned.

$|\widetilde{\psi}(\omega)| \leq Me^{T|\omega|}$, $\omega \in \mathbb{C}$, for some real numbers M and T in $\mathbb{R}$. If this assumption is not met, as might be the case if, for example, there are point masses in $\psi(\cdot)$, our method works, but our error bounds may not be valid any more.

The following theorem provides an $(n+1)-$order ODE for infinite-time ruin probability $\psi(\cdot)$ in the situation that claim size distribution X has been approximated by an $n-$component Hyperexponential density function $f_X^*(\cdot)$.

Theorem 1. *Suppose claim size density function $f_X(\cdot)$ has been approximated by an $n-$component Hyperexponential density function $f_X^*(\cdot)$. Then, infinite-time survivals probability $\tilde{\psi}(\cdot)$ of a compound Poisson process* (1) *can be approximated by infinite-time survivals probability $\tilde{\psi}_*(\cdot)$ which can be evaluated using the following $(n+1)-$order ODE.* $\sum_{i=1}^n \lambda\omega_i\mu_i\tilde{\psi}_*^{(n-1)}(u) + \sum_{i=1}^n\sum_{i\neq i}^n \lambda\omega_i\mu_i\mu_j\tilde{\psi}_*^{(n-2)}(u) - \sum_{i=1}^n\sum_{i\neq i}^n\sum_{k>j,\neq i}^n \lambda\omega_i\mu_i\mu_j\mu_k\tilde{\psi}_*^{(n-3)}(u) + \sum_{i=1}^n\sum_{i\neq i}^n\sum_{k>j,\neq i}^n\sum_{l>k,\neq i}^n \lambda\omega_i\mu_i\mu_j\mu_k\mu_l\tilde{\psi}_*^{(n-4)}(u) + \cdots + (-1)^n\sum_{i=1}^n\prod_{j\neq i}^n \mu_j\tilde{\psi}_*^{(0)}(u) - \left[\lambda\tilde{\psi}_*^{(n)}(u) - c\tilde{\psi}_*^{(n+1)}(u)\right] - \sum_{i=1}^n \mu_i\left[\lambda\tilde{\psi}_*^{(n-1)}(u) - c\tilde{\psi}_*^{(n)}(u)\right] - \sum_{i=1}^n\sum_{j\neq i}^n \mu_i\mu_j\left[\lambda\tilde{\psi}_*^{(n-2)}(u) - c\tilde{\psi}_*^{(n-1)}(u)\right] - \sum_{i=1}^n\sum_{j\neq i}^n\sum_{k>j\neq i}^n \mu_i\mu_j\mu_k\left[\lambda\tilde{\psi}_*^{(n-3)}(u) - c\tilde{\psi}_*^{(n-2)}(u)\right] - \cdots - \prod_{i=1}^n \mu_i\left[\lambda\tilde{\psi}_*^{(0)}(u) - c\tilde{\psi}_*^{(1)}(u)\right], = 0$ *with boundary conditions that satisfy* $c\tilde{\psi}_*^{(m)}(0) - \lambda\tilde{\psi}_*^{(m-1)}(0) + \lambda\sum_{j=0}^{m-2}\tilde{\psi}_*^{(j)}(0)f^{(m-2-j)}(0) = 0$, *for* $m = 1, \cdots, n$.

Proof. An application of Lemma (2) by changing $k(u) \mapsto -c\tilde{\psi}_*^{(1)}(u) + \lambda\tilde{\psi}_*(u)$, $y(u) \mapsto \tilde{\psi}_*(u)$, and $\omega_i \mapsto \lambda\omega_i$ lead to the desired result. □

Using the fact that $\tilde{\psi}_*(0) = 1 - \lambda E(X)/c$, (see [10], p. 104) the above boundary condition equation leads to:

$$\begin{aligned}
\tilde{\psi}_*^{(1)}(0) &= \tilde{\psi}_*(0)\frac{\lambda}{c}, \\
\tilde{\psi}_*^{(2)}(0) &= \tilde{\psi}_*(0)\left[(\frac{\lambda}{c})^2 - (\frac{\lambda}{c})f_X(0)\right], \\
\tilde{\psi}_*^{(3)}(0) &= \tilde{\psi}_*(0)\left[(\frac{\lambda}{c})^3 - 2f_X(0)(\frac{\lambda}{c})^2 - (\frac{\lambda}{c})f_X^{(1)}(0)\right], \\
\tilde{\psi}_*^{(4)}(0) &= \tilde{\psi}_*(0)\left[(\frac{\lambda}{c})^4 - 3f_X(0)(\frac{\lambda}{c})^3 + (\frac{\lambda}{c})^2[-2f_X^{(1)}(0) + f_X^2(0)]\right. \\
&\quad \left. -(\frac{\lambda}{c})f_X^{(2)}(0)\right], \\
\tilde{\psi}_*^{(5)}(0) &= \tilde{\psi}_*(0)\left[(\frac{\lambda}{c})^5 - 4(\frac{\lambda}{c})^4 f_X(0) + (\frac{\lambda}{c})^3[-3f_X^{(1)}(0) + 3f_X^2(0)]\right. \\
&\quad \left. +(\frac{\lambda}{c})^2[-2f_X^{(2)}(0) + 2f_X(0)f_X^{(1)}(0)] - (\frac{\lambda}{c})f_X^{(3)}(0)\right], \\
\tilde{\psi}_*^{(6)}(0) &= \tilde{\psi}_*(0)\left[(\frac{\lambda}{c})^6 - 5(\frac{\lambda}{c})^5 f_X(0) + (\frac{\lambda}{c})^4[6f_X^2(0) - 4f_X^{(1)}(0)]\right. \\
&\quad \left. +(\frac{\lambda}{c})^3[-3f_X^{(2)}(0) + 6f_X(0)f_X^{(1)}(0) - f_X^3(0)]\right] \\
&\quad +\tilde{\psi}_*(0)\left[(\frac{\lambda}{c})^2[2f_X^{(3)}(0) + 2f_X(0)f_X^{(2)}(0) + f_X^{(1)}(0)f_X^{(1)}(0)]\right. \\
&\quad \left. -(\frac{\lambda}{c})f_X^{(4)}(0)\right], \\
\tilde{\psi}_*^{(7)}(0) &= \tilde{\psi}_*(0)\left[(\frac{\lambda}{c})^7 - 6(\frac{\lambda}{c})^6 f_X(0) + (\frac{\lambda}{c})^5[10f_X^2(0) - 5f_X^{(1)}(0)]\right. \\
&\quad \left. +(\frac{\lambda}{c})^4[-4f_X^{(2)}(0) + 7f_X(0)f_X^{(1)}(0) - 4f_X^{(3)}(0)]\right] \\
&\quad +\tilde{\psi}_*(0)(\frac{\lambda}{c})^3\left[2f_X^{(3)}(0) + 2f_X(0)f_X^{(1)}(0) + 3f_X^{(1)}(0)f_X^{(1)}(0)\right. \\
&\quad \left. -f_X^{(3)}(0) + 4f_X(0)f_X^{(2)}(0) - 3f_X^2(0)f_X^{(1)}(0)\right] \\
&\quad +\tilde{\psi}_*(0)\left[(\frac{\lambda}{c})^2[-2f_X^{(4)}(0) + f_X(0)f_X^{(3)}(0) + 2f_X^{(1)}(0)f_X^{(2)}(0)\right. \\
&\quad \left. +f_X^{(3)}(0)] - (\frac{\lambda}{c})f_X^{(5)}(0)\right]
\end{aligned}$$

and so on.

The following provides error bound for approximating infinite-time survivals probability $\tilde{\psi}(\cdot)$ by $\tilde{\psi}_*(\cdot)$.

Theorem 2. *Suppose claim size density function $f_X(\cdot)$ has been approximated by an $n-$component Hyperexponential density function $f_X^*(\cdot)$. Then, the infinite-time survival probability $\tilde{\psi}(u)$ of compound Poisson process* (1) *can be approximated by $\tilde{\psi}_*(u)$, given by Theorem* (1), *and its error satisfies* $||\psi(u) - \psi_*(u)||_2 \leq \frac{c\lambda\tilde{\psi}(0)}{\sqrt{\pi}a_1^2}\left|\left|\varphi_X(s) - \sum_{j=1}^n \frac{\omega_i\mu_i}{\mu_i+s}\right|\right|_2$, *where* $a_1 = \sup\{\varphi_X(s), \sum_{j=1}^n \frac{\omega_i\mu_i}{\mu_i+s}\}$ *and $\varphi_X(s)$ stands for the characteristic function of random claim X.*

Proof. Application of the Hausdorff-Young for Laplace transform (Lemma 1) along with fact that $\mathcal{L}(g'(x); x; s) = s\mathcal{L}(g(x); x; s) - g(0)$ and $\mathcal{L}(\int_0^x (g(x-y)f(y)dy; x; s) = \mathcal{L}(g(x); x; s)\mathcal{L}(f(x); x; s)$, one may conclude that

$$\begin{aligned}
||\psi(u) - \psi_*(u)||_2 &\leq \frac{1}{\sqrt{\pi}}||\mathcal{L}(\tilde{\psi}) - \mathcal{L}(\tilde{\psi}_*)||_2 \\
&= \frac{1}{\sqrt{\pi}}\left|\left|\frac{c\tilde{\psi}(0)}{cu - \lambda + \lambda\mathcal{L}(f)} - \frac{c\tilde{\psi}(0)}{cu - \lambda + \lambda\mathcal{L}(f^*)}\right|\right|_2.
\end{aligned}$$

Application of inequality $||1/h_1 - 1/h_2||_2 \leq a^{-2}||h_1 - h_2||_2$, where $a = \sup\{h_1, h_2\}$, from [11] completes the desired proof. □

The following theorem provides an $(n+1)-$order PDE for finite-time ruin probability $\psi(\cdot)$ in the situation that claim size distribution X has been approximated by an $n-$component Hyperexponential distribution function.

Theorem 3. *Suppose claim size density function $f_X(\cdot)$ has been approximated by an $n-$component Hyperexponential density function $f_X^*(\cdot)$. Then, finite-time survivals probability $\tilde{\psi}(u;T)$ of a compound Poisson process* (1) *can be approximated by finite-time survivals probability $\tilde{\psi}_*(u;T)$ which can be evaluated using the following $(n+1)-$order PDE.*

$$\begin{aligned}
0 &= \sum_{i=1}^n \lambda\omega_i\mu_i\frac{\partial^{n-1}}{\partial u^{n-1}}\tilde{\psi}_*(u;T) + \sum_{i=1}^n\sum_{i\neq i}^n \lambda\omega_i\mu_i\mu_j\frac{\partial^{n-2}}{\partial u^{n-2}}\tilde{\psi}_*(u;T) \\
&\quad - \sum_{i=1}^n\sum_{i\neq i}^n\sum_{k>j,\neq i}^n \lambda\omega_i\mu_i\mu_j\mu_k\frac{\partial^{n-3}}{\partial u^{n-3}}\tilde{\psi}_*(u;T) \\
&\quad + \sum_{i=1}^n\sum_{i\neq i}^n\sum_{k>j,\neq i}^n\sum_{l>k,\neq i}^n \lambda\omega_i\mu_i\mu_j\mu_k\mu_l\frac{\partial^{n-4}}{\partial u^{n-4}}\tilde{\psi}_*(u;T) \\
&\quad - \cdots + (-1)^n\sum_{i=1}^n\prod_{j\neq i}^n \mu_j\frac{\partial^0}{\partial u^0}\tilde{\psi}_*(u;T) \\
&\quad - \left[\lambda\frac{\partial^n}{\partial u^n}\tilde{\psi}_*(u;T) - c\frac{\partial^{n+1}}{\partial u^{n+1}}\tilde{\psi}_*(u;T) + c\frac{\partial^{n+1}}{\partial T\partial u^n}\tilde{\psi}_*(u;T)\right] \\
&\quad - \sum_{i=1}^n \mu_i\left[\lambda\frac{\partial^{n-1}}{\partial u^{n-1}}\tilde{\psi}_*(u;T) - c\frac{\partial^n}{\partial u^n}\tilde{\psi}_*(u;T) + c\frac{\partial^n}{\partial T\partial u^{n-1}}\tilde{\psi}_*(u;T)\right] \\
&\quad - \sum_{i=1}^n\sum_{j\neq i}^n \mu_i\mu_j\left[\lambda\frac{\partial^{n-2}}{\partial u^{n-2}}\tilde{\psi}_*(u;T) - c\frac{\partial^{n-1}}{\partial u^{n-1}}\tilde{\psi}_*(u;T)\right. \\
&\qquad \left. +c\frac{\partial^{n-1}}{\partial T\partial u^{n-2}}\tilde{\psi}_*(u;T)\right] \\
&\quad - \sum_{i=1}^n\sum_{j\neq i}^n\sum_{k>j\neq i}^n \mu_i\mu_j\mu_k\left[\lambda\frac{\partial^{n-3}}{\partial u^{n-3}}\tilde{\psi}_*(u;T) - c\frac{\partial^{n-2}}{\partial u^{n-2}}\tilde{\psi}_*(u;T)\right. \\
&\qquad \left. +c\frac{\partial^{n-2}}{\partial T\partial u^{n-3}}\tilde{\psi}_*(u;T)\right] \\
&\quad - \cdots - \prod_{i=1}^n \mu_i\left[\lambda\frac{\partial^0}{\partial u^0}\tilde{\psi}_*(u;T) - c\frac{\partial^1}{\partial u^1}\tilde{\psi}_*(u;T) + c\frac{\partial^1}{\partial T^1}\tilde{\psi}_*(u;T)\right],
\end{aligned}$$

where $\tilde{\psi}_*^{(n)}(0;T) = \lim_{u\to 0}\frac{\partial^n}{\partial u^n}\tilde{\psi}_*(u;T)$ *and boundary conditions that satisfy* $c\tilde{\psi}_*^{(m)}(0;T) - c\frac{\partial}{\partial T}\tilde{\psi}_*^{(m-1)}(0;T) - \lambda\tilde{\psi}_*^{(m-1)}(0;T) + \lambda\sum_{j=0}^{m-2}\tilde{\psi}_*^{(j)}(0;T)f_X^{(n-2-j)}(0) = 0$, *for* $m = 1, \cdots, n$.

Proof. Using partial integro-differential equation (4) and the Schwarz integrability condition, one may change order of differentiation and obtain the above recursive formula for boundary conditions. An application of Lemma (2) by changing $k(u) \mapsto -c\frac{\partial}{\partial u}\tilde{\psi}_*(u;T) + c\frac{\partial}{\partial T}\tilde{\psi}_*(u;T) + \lambda\tilde{\psi}_*(\cdot)$, $y(u) \mapsto \tilde{\psi}_*(u;T)$, and $\omega_i \mapsto \lambda\omega_i$ lead to the desired result. □

Using the fact that $\tilde{\psi}_*(u;0) = 1$, $\tilde{\psi}_*(0;T) = \int_0^{cT} F_{S,T}(x)dx/(cT)$, and $F_{S,T}(x) = P(\sum_{j=1}^{N(T)} X_j \le x)$, for all $x \in \mathbb{R}^+$ (see [3], p. 121). One may compute the following from boundary conditions from recursive formula given by Theorem (3).

$$
\begin{aligned}
\tilde{\psi}_*^{(1)}(0;T) &= \frac{\lambda}{c}\tilde{\psi}_*(0;T) + \frac{\partial}{\partial T}\tilde{\psi}_*(0;T),\\
\tilde{\psi}_*^{(2)}(0;T) &= \tilde{\psi}_*(0;T)\left[(\frac{\lambda}{c})^2 - (\frac{\lambda}{c})f_X(0)\right] + \frac{\partial}{\partial T}\tilde{\psi}_*^{(1)}(0;T),\\
\tilde{\psi}_*^{(3)}(0;T) &= \tilde{\psi}_*(0;T)\left[(\frac{\lambda}{c})^3 - 2f_X(0)(\frac{\lambda}{c})^2 - (\frac{\lambda}{c})f_X^{(1)}(0)\right]\\
&\quad + \frac{\partial}{\partial T}\tilde{\psi}_*^{(2)}(0;T),\\
\tilde{\psi}_*^{(4)}(0;T) &= \tilde{\psi}_*(0;T)\Big[(\frac{\lambda}{c})^4 - 3f_X(0)(\frac{\lambda}{c})^3\\
&\quad + (\frac{\lambda}{c})^2[-2f_X^{(1)}(0) + f_X^2(0)] - (\frac{\lambda}{c})f_X^{(2)}(0)\Big]\\
&\quad + \frac{\partial}{\partial T}\tilde{\psi}_*^{(3)}(0;T),\\
\tilde{\psi}_*^{(5)}(0;T) &= \tilde{\psi}_*(0;T)\Big[(\frac{\lambda}{c})^5 - 4(\frac{\lambda}{c})^4 f_X(0)\\
&\quad + (\frac{\lambda}{c})^3[-3f_X^{(1)}(0) + 3f_X^2(0)] + (\frac{\lambda}{c})^2[-2f_X^{(2)}(0)\\
&\quad + 2f_X(0)f_X^{(1)}(0)] - (\frac{\lambda}{c})f_X^{(3)}(0)\Big] + \frac{\partial}{\partial T}\tilde{\psi}_*^{(4)}(0;T),\\
\tilde{\psi}_*^{(6)}(0;T) &= \tilde{\psi}_*(0;T)\Big[(\frac{\lambda}{c})^6 - 5(\frac{\lambda}{c})^5 f_X(0)\\
&\quad + (\frac{\lambda}{c})^4[6f_X^2(0) - 4f_X^{(1)}(0)]\\
&\quad + (\frac{\lambda}{c})^3[-3f_X^{(2)}(0) + 6f_X(0)f_X^{(1)}(0) - f_X^3(0)]\Big]\\
&\quad + \tilde{\psi}_*(0;T)\Big[(\frac{\lambda}{c})^2[2f_X^{(3)}(0) + 2f_X(0)f_X^{(2)}(0)\\
&\quad + f_X^{(1)}(0)f_X^{(1)}(0)] - (\frac{\lambda}{c})f_X^{(4)}(0)\Big] + \frac{\partial}{\partial T}\tilde{\psi}_*^{(5)}(0;T),\\
\tilde{\psi}_*^{(7)}(0;T) &= \tilde{\psi}_*(0;T)\Big[(\frac{\lambda}{c})^7 - 6(\frac{\lambda}{c})^6 f_X(0)\\
&\quad + (\frac{\lambda}{c})^5[10f_X^2(0) - 5f_X^{(1)}(0)]\\
&\quad + (\frac{\lambda}{c})^4[-4f_X^{(2)}(0) + 7f_X(0)f_X^{(1)}(0) - 4f_X^{(3)}(0)]\Big]\\
&\quad + \tilde{\psi}_*(0;T)(\frac{\lambda}{c})^3\Big[2f_X^{(3)}(0) + 2f_X(0)f_X^{(1)}(0)\\
&\quad + 3f_X^{(1)}(0)f_X^{(1)}(0) - f_X^{(3)}(0) + 4f_X(0)f_X^{(2)}(0)\\
&\quad - 3f_X^2(0)f_X^{(1)}(0)\Big]\\
&\quad + \tilde{\psi}_*(0;T)\Big[(\frac{\lambda}{c})^2[-2f_X^{(4)}(0) + f_X(0)f_X^{(3)}(0)\\
&\quad + 2f_X^{(1)}(0)f_X^{(2)}(0) + f_X^{(3)}(0)] - (\frac{\lambda}{c})f_X^{(5)}(0)\Big]\\
&\quad + \frac{\partial}{\partial T}\tilde{\psi}_*^{(6)}(0;T),
\end{aligned}
$$

where $\tilde{\psi}_*^{(n)}(0;T) = \lim_{u\to 0}\frac{\partial^n}{\partial u^n}\tilde{\psi}_*(u;T)$.

Using the central limit theorem for compound sum $\sum_{i=1}^{N(t)} X_i$ (see [10], §2.5, or [22], §1.9), one may provide the following approximation for expression $\tilde{\psi}_*(0;T) = \int_0^{cT} F_{S,T}(x)dx/(cT)$

$$\tilde{\psi}_*(0;T) \approx \frac{1}{cT}\int_0^{cT}\Phi\left(\frac{x - \lambda T m_1}{\sqrt{\lambda T m_2}}\right)dx,$$

where $m_i = E(X^i)$, for $i = 1, 2$, and $\Phi(\cdot)$ stands for cumulative distribution function for standard normal distribution, see [10], §2.5 ,or [22], §1.9, for other parametric approximation approaches and [19] for a nonparametric approximation approach. For heavy tailed random claim size X that the ordinary central limit theorem does not work properly. One has to employ an appropriated version of the central limit theorem, see [20], [5], among others, for more details.

The following provides error bound for approximating finite-time survivals probability $\tilde{\psi}(u;T)$ by $\tilde{\psi}_*(u;T)$.

Theorem 4. *Suppose claim size density function $f_X(\cdot)$ has been approximated by an $n-$component Hyperexponential density function $f_X^*(\cdot)$. Then, the infinite-time survival probability $\tilde{\psi}(u;T)$ of compound Poisson process* (1) *can be approximated by $\tilde{\psi}_*(u;T)$, given by Theorem* (3)*, and its error satisfies*

$$
\begin{aligned}
||Error||_2 &\le \frac{\lambda}{\sqrt{\pi}}\left[-\frac{c\tilde{\psi}(0;T)}{a_1^2} + \frac{a_2 T}{c} - c\tilde{\psi}(0;T)a_1^2a_2^2a_3\right]\\
&\quad \times \left\|\varphi_X(s) - \sum_{j=1}^{n}\frac{\omega_i\mu_i}{\mu_i + s}\right\|_2,
\end{aligned}
$$

where $Error = \psi(u;T) - \psi_*(u;T)$, $a_1 = \sup\{\varphi_X(s), \sum_{j=1}^n \frac{\omega_i\mu_i}{\mu_i+s}\}$, $a_2 = \sup\{1/s - c/A(s), 1/s - c/A_*(s)\}$, $a_2 = \sup\{e^{A(s)/cT}, e^{A_*(s)/cT}\}$, $A(s) = cs - \lambda + \lambda\varphi_X(s)$, $A_*(s) = cs - \lambda + \lambda\sum_{j=1}^n \omega_i\mu_i/(\mu_i + s)$, and $\varphi_X(s)$ stands for the characteristic function of random claim X.

Proof. Taking the Laplace transform from both sides of (4) leads to the following first-order PDE

$$A(s)\mathcal{L}(\tilde{\psi}(u;T);u;s) - c\tilde{\psi}(0;T) - c\frac{\partial}{\partial T}\mathcal{L}(\tilde{\psi}(u;T);u;s) = 0,$$

where $\mathcal{L}(\tilde{\psi}(u;1);u;s) = 1/s$. Therefore, the Laplace transform of finite-time ruin probability for compound Poisson process (1) is

$$\mathcal{L}(\tilde{\psi}(u;T);u;s) = \frac{c\tilde{\psi}(0;T)}{A(s)} + \left(\frac{1}{s} - \frac{c\tilde{\psi}(0;T)}{A(s)}\right)e^{A(s)/cT}.$$

The above finding along with an application of the Hausdorff-Young for Laplace transform (Lemma 1) lead to

$$
\begin{aligned}
||E||_2 &\le \frac{1}{\sqrt{\pi}}||\mathcal{L}(\tilde{\psi}(u;T);u;s) - \mathcal{L}(\tilde{\psi}_*(u;T);u;s)||_2\\
&= \frac{1}{\sqrt{\pi}}\Bigg\|\frac{c\tilde{\psi}(0;T)}{A(s)} + \left(\frac{1}{s} - \frac{c\tilde{\psi}(0;T)}{A(s)}\right)e^{A(s)/cT}\\
&\quad - \frac{c\tilde{\psi}(0;T)}{A_*(s)} - \left(\frac{1}{s} - \frac{c\tilde{\psi}(0;T)}{A_*(s)}\right)e^{A_*(s)/cT}\Bigg\|_2\\
&\le \frac{c\tilde{\psi}(0;T)}{\sqrt{\pi}a_1^2}||A(s) - A_*(s)||_2 + \frac{b}{\sqrt{\pi}}\left\|\frac{TA(s)}{c} - \frac{TA_*(s)}{c}\right\|_2\\
&\quad + \frac{b}{\sqrt{\pi}}\left\|\ln(\frac{1}{s} - \frac{c\tilde{\psi}(0;T)}{A(s)}) - \ln(\frac{1}{s} - \frac{c\tilde{\psi}(0;T)}{A_*(s)})\right\|_2,
\end{aligned}
$$

where $E = \psi(u;T) - \psi_*(u;T)$, the second inequality arrives by application of inequality $||1/h_1 - 1/h_2||_2 \le a^{-2}||h_1 - h_2||_2$, and $a = \sup\{h_1, h_2\}$, from [11], triangle inequality, and the Mean value theorem (i.e., $(\exp\{A(s)/cT + \ln(\frac{1}{s} - \frac{c\tilde{\psi}(0;T)}{A(s)})\} - \exp\{A_*(s)/cT + \ln(\frac{1}{s} - \frac{c\tilde{\psi}(0;T)}{A_*(s)})\})/(A(s)/cT + \ln(\frac{1}{s} - \frac{c\tilde{\psi}(0;T)}{A(s)}) - A_*(s)/cT - \ln(\frac{1}{s} - \frac{c\tilde{\psi}(0;T)}{A_*(s)})) \le b$ where $b = \sup\{A(s)/cT + \ln(\frac{1}{s} - \frac{c\tilde{\psi}(0;T)}{A(s)}), A_*(s)/cT + \ln(\frac{1}{s} - \frac{c\tilde{\psi}(0;T)}{A_*(s)})\}$).

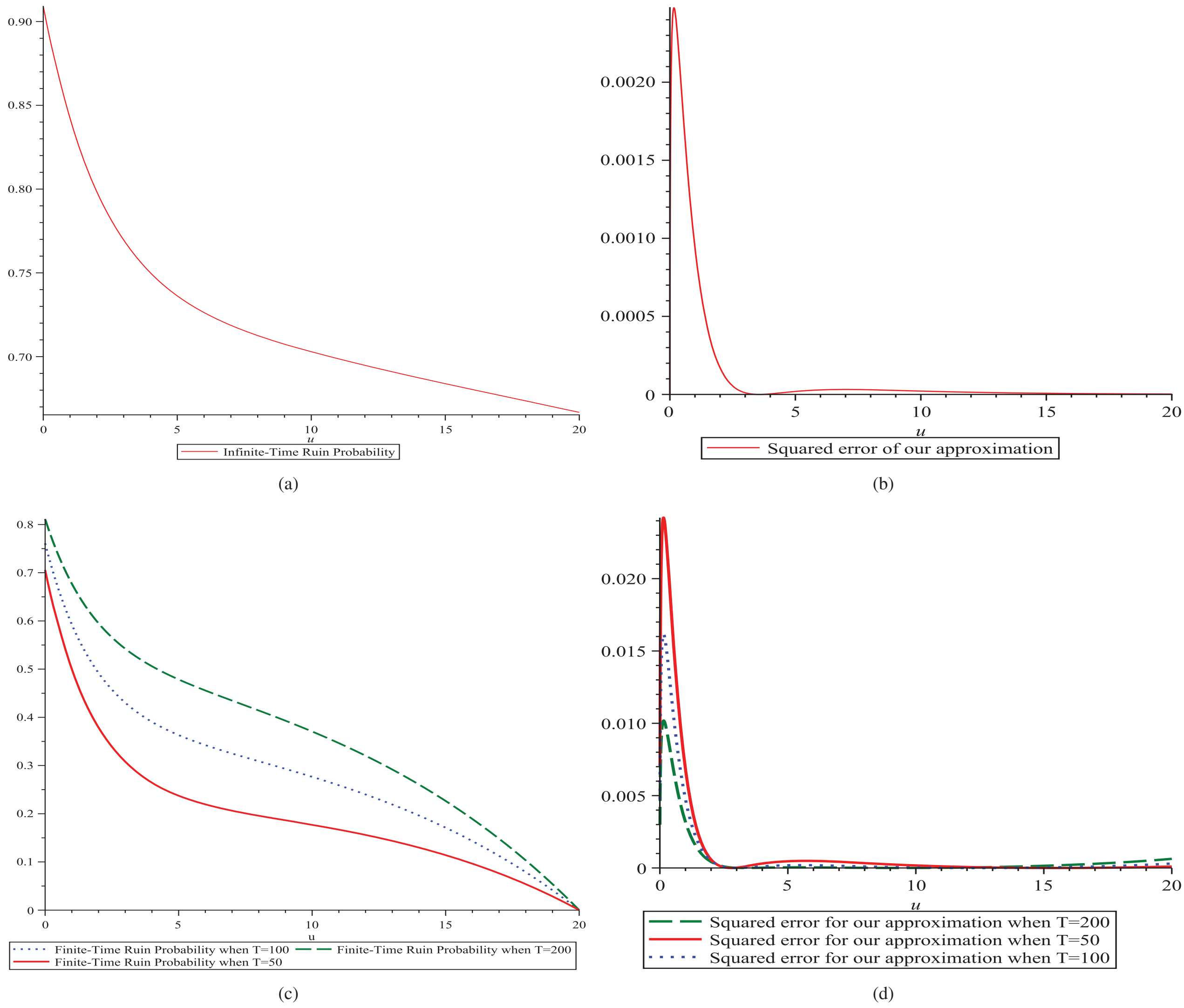

Fig. 1 (a) Behavior for approximated infinite-time ruin probability $\psi_*(u)$; (b) Squared error for approximated infinite-time ruin probability $\psi_*(u)$; (c) Behavior for approximated finite-time ruin probability $\psi_*(u;T)$, for $T = 50, 100, 200$; and (d) Squared error for approximated finite-time ruin probability, for $T = 50, 100, 200$

Application of inequality $||\ln h_1 - \ln h_2||_2 \leq ||h_1 - h_2||_2/a$, where $a = \sup\{h_1, h_2\}$, from [11] completes the desired proof. □

IV. Simulation Study

Consider compound Poisson process (1) with intensity rate $\lambda = 1$ and premium $c = 1.1$. This section conducts two simulation studies to show practical application the about findings.

Example 1. *Suppose random claim X in compound Poisson process* (1) *has been distributed according to Weibull(0.3,9.26053). Reference [7] using a three-moment matching algorithm showed that density function of random claim X can be approximated by the following 2-component Hyperexponential density function*

$$f_X^*(x) = 0.000095e^{-0.019x} + 1.348225e^{-1.355x}. \quad (7)$$

For infinite-time ruin probability: Application of Theorem (1) leads to the following second order ODE

$$1.1\tilde{\psi}_*^{(3)}(u) + 0.5114\tilde{\psi}_*^{(2)}(u) + 0.0026395\tilde{\psi}_*^{(1)}(u) = 0$$

with initial conditions $\tilde{\psi}_*(0) = 0.0909090909$, $\lim_{u\to 0}\tilde{\psi}_*^{(1)}(u) = 0.08264462809$, and $\lim_{u\to 0}\tilde{\psi}_*^{(2)}(u) = -0.03629992491$.

Solving the above ODE, one may approximate finite-time survival probability $\tilde{\psi}(u)$ of compound Poisson process (1) by $\tilde{\psi}_*(u) = 0.9974815963 - 0.2101123939e^{-0.01502369720u} - 0.2873692025e^{-0.8589763028u}$.

Figs. 1 (a) and (b) illustrate behavior for such approximated infinite-time ruin probability and its corresponding squared

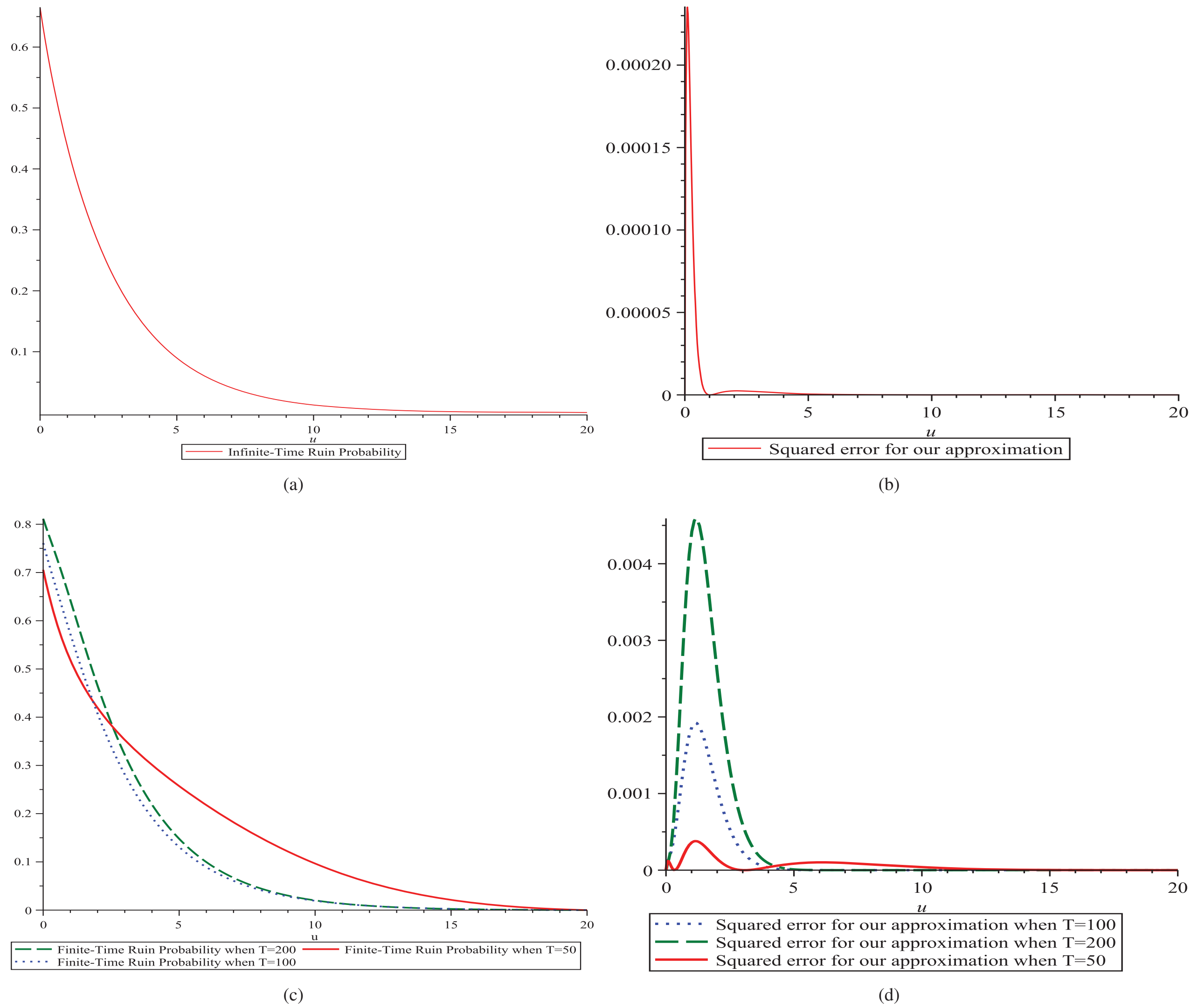

Fig. 2 (a) Behavior for approximated infinite-time ruin probability $\psi_*(u)$; (b) Squared error for approximated infinite-time ruin probability $\psi_*(u)$; (c) Behavior for approximated finite-time ruin probability $\psi_*(u;T)$, for $T = 50, 100, 200$; and (d) Squared error for approximated finite-time ruin probability, for $T = 50, 100, 200$

error, respectively. Such error has been evaluated by substitution approximated infinite-time ruin probability in integro-differential equation (3).

For finite-time ruin probability: Application of Theorem (3) leads to the following PDE $1.1\frac{\partial^3}{\partial u^3}\tilde{\psi}_*(u;T)+0.5114\frac{\partial^2}{\partial u^2}\tilde{\psi}_*(u;T)-0.0539995\frac{\partial^1}{\partial u^1}\tilde{\psi}_*(u;T)-1.1\frac{\partial^3}{\partial u^2\partial T}\tilde{\psi}_*(u;T)-1.5114\frac{\partial^2}{\partial u\partial T}\tilde{\psi}_*(u;T)-0.0283195\frac{\partial}{\partial T}\tilde{\psi}_*(u;T) = 0$with initial conditions $\tilde{\psi}_*(u,0) = 0$, $\tilde{\psi}_*(0;T) = \beta(T)$, $\lim_{u\to 20}\tilde{\psi}_*(u;T) = 1$, $\lim_{u\to 0}\frac{\partial}{\partial u}\tilde{\psi}_*(u;T) = 0.9091\beta(T) + \frac{\partial}{\partial T}\beta(T)$, where $\beta(T) = \frac{1}{1.1T}\int_0^{1.1T}\Phi\left(\frac{x-T}{\sqrt{29.36T}}\right)dx$.

Solving the above PDE, one may approximate finite-time survival probability $\tilde{\psi}(u;T)$ of compound Poisson process (1) by $\tilde{\psi}_*(u;T)$, that its behavior (for $T = 50, 100, 200$) has been illustrated in Fig. 2 (c). Fig. 2 (d) illustrates squared error of our approximation (for $T = 50, 100, 200$).

As one may observe, our squared error is less than 0.0025 and 0.025, for approximating infinite-time and finite-time ruin probability, respectively. Such error maybe reduced by increasing number of component in Hyperexponential density function.

Example 2. *Suppose random claim X in compound Poisson process (1) has been distributed according to Gamma(0.7310,1). Reference [24] using the Padé approximant method showed that density function of random claim X can be approximated by by the following 3-component Hyperexponential density function*

$$f_X^*(x) = 0.8099e^{-3.2398x} + 0.3616e^{-1.4465x} + 0.5198e^{-1.0396x}. \tag{8}$$

For infinite-time ruin probability: Application of Theorem (1) leads to the following second order ODE $1.1\tilde{\psi}_*^{(4)}(u) +$

$5.29849\tilde{\psi}_*^{(3)}(u) + 6.479507012\tilde{\psi}_*^{(2)}(u) + 1.797919457\tilde{\psi}_*^{(1)}(u) = 0$with initial conditions $\tilde{\psi}_*(0) = 0.3354861821$, $\lim_{u\to 0}\tilde{\psi}_*^{(1)}(u) = 0.3049874383$, $\lim_{u\to 0}\tilde{\psi}_*^{(2)}(u) = -0.2385743202$, and $\lim_{u\to 20}\tilde{\psi}_*(u) = 1$.

Solving the above ODE, one may approximate infinite-time survival probability $\tilde{\psi}(u)$ of compound Poisson process (1) by $\tilde{\psi}_*(u) = 1 - 0.013037e^{-3.073097u} - 0.008568e^{-1.349630u} - 0.642908e^{-0.394082u}$. Figs. 2 (a) and (b) illustrate behavior for such approximated infinite-time ruin probability and its corresponding squared error, respectively. Such error has been evaluated by substitution approximated infinite-time ruin probability in integro-differential equation (3).

For finite-time ruin probability: Application of Theorem (3) leads to the following PDE $1.1\frac{\partial^4}{\partial u^4}\tilde{\psi}_*(u;T) + 5.29849\frac{\partial^3}{\partial u^3}\tilde{\psi}_*(u;T) + 6.479507012\frac{\partial^2}{\partial u^2}\tilde{\psi}_*(u;T) + 1.797919457\frac{\partial}{\partial u}\tilde{\psi}_*(u;T) - 5.359146\frac{\partial}{\partial T}\tilde{\psi}_*(u;T) - 10.5141\frac{\partial^2}{\partial u\partial T}\tilde{\psi}_*(u;T) - 6.2985\frac{\partial^3}{\partial u^2\partial T}\tilde{\psi}_*(u;T) - 1.1\frac{\partial^4}{\partial u^3\partial T}\tilde{\psi}_*(u;T) = 0$ with initial conditions $\tilde{\psi}_*(u,0) = 0$, $\tilde{\psi}_*(0;T) = \beta(T)$, $\lim_{u\to 20}\tilde{\psi}_*(u;T) = 1$, $\lim_{u\to 0}\frac{\partial}{\partial u}\tilde{\psi}_*(u;T) = 0.9091\beta(T) + \frac{\partial}{\partial T}\beta(T)$, $\lim_{u\to 0}\frac{\partial}{\partial u}\tilde{\psi}_*(u;T) = -0.71113\beta(T) + 0.9091\frac{\partial}{\partial T}\beta(T) + \frac{\partial^2}{\partial T^2}\beta(T)$,where $\beta(T) = \frac{1}{1.1T}\int_0^{1.1T}\Phi\left(\frac{x-0.7309651999T}{\sqrt{0.7309651995T}}\right)dx$.

Solving the above PDE, one may approximate finite-time survival probability $\tilde{\psi}(u;T)$ of compound Poisson process (1) by $\tilde{\psi}_*(u;T)$, that its behavior (for $T = 50, 100, 200$) has been illustrated in Fig. 2 (c). Fig. 2 (d) of illustrates squared error of our approximation (for $T = 50, 100, 200$).

As one may observe, our squared error is less than 0.00025 and 0.005, for approximating infinite-time and finite-time ruin probability, respectively. Such error maybe reduced by increasing number of component in Hyperexponential density function.

It worthwhile to mention that: A given density function (or a density function corresponding to a given data set) can be approximated by a Hyperexponential distribution using a Matlab package called "bayesf", see [23] for more details.

V. Conclusion and Suggestions

This article approximates claim size density function $f_X(\cdot)$ by a $n-$component Hyperexponential density function $f_X^*(\cdot)$. Then, it restates the problem of finding an infinite-time (or finite-time) ruin probability as a $(n+1)-$order ordinary differential equation (or a partial differential equation for finite-time ruin probability). Application of our findings has been given though a simulation study.

Certainly the following generalized Hyperexponential distribution can be provided a more accurate approximation in the situation that the true density function (or recorded data) has more than one mode.

$$g_X^{GHE}(x) = \sum_{i=1}^{n}\omega_i\mu_i e^{-\mu_i(x-b_i)}I_{[b_i,\infty)}(x). \quad (9)$$

In such situation the finite and infinite ruin probabilities can be evaluated using the following lemma.

Lemma 4. *Suppose claim size density function $f_X(\cdot)$ has been approximated by generalized Hyperexponential distribution $g_X^{GHE}(\cdot)$. The survival probability can be found by the following two inverse Laplace transforms.*

(i) *The Laplace transform of the infinite-time survival probability can be found by the following equation*

$$\mathcal{L}\left(\tilde{\psi}(u);u;s\right) = \frac{c\tilde{\psi}(0)}{cs - \lambda + \lambda\sum_{i=1}^{k}\frac{\omega_i\mu_i}{\mu_i+s}e^{-sb_i}}$$

(ii) *The Laplace transform of the finite-time survival probability can be found by the following equation*

$$\mathcal{L}\left(\tilde{\psi}(u;T);u;s\right) = \frac{c\tilde{\psi}(0;T)}{A^{**}(s)} + \left(\frac{1}{s} - \frac{c\tilde{\psi}(0;T)}{A^{**}(s)}\right)e^{A^{**}(s)/cT},$$

where $A^{**}(s) = cs - \lambda + \lambda\sum_{i=1}^{k}\frac{\omega_i\mu_i}{\mu_i+s}e^{-sb_i}$.

Proof. The desired result arrives by taking a Laplace transform from both sides of equations (3) and (4) and solving corresponding first-order PDE with boundary condition $\tilde{\psi}(u;0) = 1$ or $\mathcal{L}(\tilde{\psi}(u;0);u;s) = 1/s$. □

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