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Optimal Bayesian Control of the Proportion of Defectives in a Manufacturing Process

Viliam Makis, Farnoosh Naderkhani, Leila Jafari

Abstract—In this paper, we present a model and an algorithm for the calculation of the optimal control limit, average cost, sample size, and the sampling interval for an optimal Bayesian chart to control the proportion of defective items produced using a semi-Markov decision process approach. Traditional p-chart has been widely used for controlling the proportion of defectives in various kinds of production processes for many years. It is well known that traditional non-Bayesian charts are not optimal, but very few optimal Bayesian control charts have been developed in the literature, mostly considering finite horizon. The objective of this paper is to develop a fast computational algorithm to obtain the optimal parameters of a Bayesian p-chart. The decision problem is formulated in the partially observable framework and the developed algorithm is illustrated by a numerical example.

Keywords—Bayesian control chart, semi-Markov decision process, quality control, partially observable process.

I. Introduction

Control charts are powerful tools used to control the process parameters to ensure the process stability, identify the occurrence of assignable causes and reduce the proportion of defectives produced. The application of the control charts has not been limited to manufacturing. They have been applied in various areas such as healthcare [1], condition-based maintenance [2], and financial engineering [3]. The traditional control chart procedures take samples from the process output at equally spaced sampling epochs and calculate the appropriate value of the statistic to control a particular process parameter. If the value of the statistic is beyond the control limits, the process is stopped and the search for assignable causes of variation is initiated. The control charts can be categorized as Bayesian or non-Bayesian, traditional charts. Very few Bayesian control charts have been developed in the literature. Reference [4] presented a control policy for an np-chart with a variable sampling interval such that a sample is taken at the instant the prior probability of the process being out of control reaches a certain predetermined constant level. He did not prove that such a policy is optimal. The optimality of this kind of policy for a simple, one-cycle control problem was proved by [5]. Further development in the area on an economic design of an attribute np-chart using a variable sample size can be found in [6].

The Bayesian approach focuses on determining the optimal control policy based on the posterior probability that the process is out of control, updated after each sample using Bayes' theorem. The models in [7]-[10], not directly linked to a control chart design, are the early contributions to the Bayesian process control. It was shown in [11] and [12] that non-Bayesian techniques are not optimal and it was suggested that in the general case, the action decision, sample size, and the sampling interval should be determined based on the probability that the process is in the out-of-control state. The papers [13]-[17] and [4] are the contributions dealing directly with the Bayesian control chart design issues. References [14], [15] and [17] have considered a finite horizon, univariate Bayesian chart, showed some results partially characterizing the optimal policy, and presented computational algorithms for finite production runs. It was shown in [13] that for a given sample size and sampling interval, a control limit policy with the optimal control limit dependent on the number of remaining periods is optimal for a finite-horizon Bayesian p-chart.

The optimality of a multivariate Bayesian control chart for the process mean was proved in [18] considering an infinite time horizon. In [19], the structure of the optimal Bayesian control policy for the process mean was established considering finite time horizon and multivariate observations. There have been several approaches to the traditional design of control charts, namely statistical, economic, and economic-statistical design approaches. A statistical design may have poor economic performance and, alternatively, an economic design that does not consider any statistical objectives may result in an excessive number of false alarms that may introduce extra variability into the process and destroy confidence in the control procedure [20]. Economic-statistical design considers both the statistical and economic requirements. It has been shown both for the univariate (e.g. [21]) and multivariate charts (e.g. [22]) that the additional statistical constraints have little effect on the cost, and thus both economic and statistical objectives can be met simultaneously, without too much compromise. The traditional approach to a control chart design with fixed or varying parameters (see a review paper [23] of the basic adaptive control charts) considers the classical control chart framework with the objective to determine the values of the chart parameters, namely, the sample size, sampling interval, and the control limits satisfying the economic and/or statistical requirements.

The optimality of the Bayesian p-chart for an infinite time horizon was proved in [24]. The λ-maximization technique was used to derive the optimality equation for the value function considering both the infinite time horizon as well as the m-stage stopping problem and the structure of the optimal policy was determined by analyzing the optimality equations. It was found that the algorithm based on the optimization

V. Makis is with the Department of Mechanical and Industrial Engineering, University of Toronto, Ontario, Canada (e-mail:makis@mie.utoronto.ca).
F. Naderkhani ZG and L. Jafari are with University of Toronto.

and calculation of the value function is quite slow because it requires a large number of computationally intensive iterations.

The objective of this paper is to develop a faster computational algorithm by modeling the posterior probability stochastic process as a semi-Markov decision process. The rest of the paper is organized as follows. The decision model is described in Section II. Section III is devoted to the formulation of the Bayesian decision problem in the semi-Markov framework. It is shown that the optimal parameters of the Bayesian p-chart can be found by iteratively solving a system of linear equations. An effective computational algorithm is presented in Section IV, which is illustrated by a numerical example in Section V. Conclusions are presented in Section VI.

II. Model Description

In this section, a brief review of a Bayesian control chart for the proportion of defectives is provided. The process is assumed to be in two states, namely, in an in-control state (state 0) and out-of-control state (state 1). Bayesian control chart for attributes monitors the posterior probability that the process has shifted to the out-of-control state given the history of the process, when the fraction defective is the quality parameter of interest. The process makes transition from state 0 to state 1 when an assignable cause occurs. It is assumed that the time between occurrences of two randomly assignable causes follows exponential distribution with the mean $\frac{1}{\theta}$. A sample of size n is taken at time $m\Delta$ for $m = 1, 2, ...,\Delta$ is the sampling interval, and the number of defectives denoted by D is observed. The proportion of defectives is a function of the state of the process. When the process is in control, the proportion of defectives is denoted by p_0, and when the process is out of control, the proportion of defectives is p_1.

Let $\pi(t) = P(X_t = 1 \mid S_t)$ denote the probability that the process is out of control at time t given the observations up to time t. By applying the Bayes' formula, [13] proved that $\pi(t)$ is given by:

$$\pi_{m\Delta} = \frac{\nu_1}{\nu_0 + \nu_1} \tag{1}$$

where

$$\nu_1 = \binom{n}{D} p_1^d (1 - p_1)^{n-D} [1 - (1 - \pi) e^{-\theta\Delta}] \tag{2}$$

and

$$\nu_0 = \binom{n}{D} p_0^d (1 - p_0)^{n-D} e^{-\theta\Delta} (1 - \pi) \tag{3}$$

We further need to simplify the posterior probability given by (1). We divide both numerator and denominator by $\binom{n}{d} p_1^D (1 - p_1)^{n-D}$. The posterior probability simplifies as:

$$\pi_{m\Delta} = \frac{1 - (1 - \pi_{(m-1)\Delta}) e^{-\theta\Delta}}{1 - (1 - \pi_{(m-1)\Delta}) e^{-\theta\Delta} + (\frac{p_0}{p_1})^D (\frac{1-p_0}{1-p_1})^{n-D} (1 - \pi_{(m-1)\Delta}) e^{-\theta\Delta}} \tag{4}$$

In this paper, we design the Bayesian control chart for monitoring the proportion of defectives in order to minimize the long-run expected average cost per unit time. It is assumed that the process starts in the in-control state and it remains in that state until an assignable cause occurs. We make a reasonable assumption that the proportion of defectives when the process is out of control is greater than the proportion of defectives when the process is in control, i.e., $p_1 > p_0$. We monitor the process at periodic sampling epochs $m\Delta$. If the posterior probability that the process is out of control exceeds the control threshold denoted by L, the process is stopped and a search for an assignable cause is started. There are two scenarios at this time: (i) there is a true alarm when the assignable cause is found and repair action is needed to fix the process, and (ii) there is a false alarm when the search for an assignable cause failed and the process continues without any corrective action.

From the renewal theory, for any stationary policy, determined by the sampling interval and control limit (Δ, L), the long-run expected average cost per unit time is calculated as the fraction of the expected total cost (TC) over the cycle length (CL) where a cycle is completed when either a corrective action is applied or when false alarm occurs. The objective is to minimize the long-run expected average cost per unit time given by $g(\Delta, L) = \frac{E_R(TC)}{E_R(TC)}$. In order to minimize the long-run expected average cost per unit time, the standard cost assumptions similar to [13] and [18] are considered in this paper as:

- A: cost associated with the search for an assignable cause.
- R: cost of the process repair.
- M: quality-related expected cost per unit time incurred due to the increased proportion of defectives when the process is out of control.
- a: fixed cost per sample.
- b: cost per unit sampled.

This completes our model assumptions and model description. In the next section, we formulate the proposed problem in the SMDP framework.

III. Formulation of the Bayesian Control Problem in the SMDP Framework

In this section, we formulate and solve the control problem in the SMDP framework. In order to calculate the minimum value of the long-run expected average cost per unit time, for a fixed control limit $L \in (0, 1)$, the posterior probability interval $[0, 1]$ is partitioned into K sub-intervals. We obtain the posterior probability at sampling time $m\Delta$ assuming that the system has not failed by that time. For a fixed large K, the SMDP is defined to be in sate $1 \leq k \leq K$ if the coded value of the posterior probability is in the interval $[\frac{k-1}{K}, \frac{k}{K})$. If the posterior probability is below the control limit, the SMDP is in the set W_1, where $W_1 = \{i : 1 \leq i \leq K\}$. If the posterior probability exceeds the control limit, the process is stopped and full inspection is performed. At this point, the SMDP is in the set $W_2 = \{h : L \leq h \leq 1\}$. After the full system inspection, if true alarm occurs, the SMDP is defined to be in state r and repair action should be initiated. We denote the set $W_3 = \{r\}$. Otherwise, the SMDP is defined to be in state 0. Thus, the state space for SMDP formulation is given by $W = \{0 \cup W_1 \cup W_2 \cup W_3\}$.

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To minimize the long-run expected average cost, the SMDP is fully determined by the following quantities:

1) $P_{m,k}$ = the probability that the process will be in state k at the next decision epoch given the current state is $m \in W$.
2) τ_m = the expected sojourn time until the next decision epoch given the current state is $m \in W$.
3) C_m = the expected cost incurred until the next decision epoch given the current state is $m \in W$.

Using the quantities defined above, for a fixed control limit L and sampling interval Δ, the long-run expected average cost $g(\Delta, L)$ can be obtained by solving the following system of linear equations (see [25]):

$$\begin{aligned} u_m &= C_m - g(\Delta, L)\tau_m + \sum_{k \in W} P_{m,k} u_k, \quad \text{for each } m \in W \\ u_l &= 0. \quad \text{for some } l \in W \end{aligned} \tag{5}$$

The next subsections are devoted to explicitly computing the quantities $P_{m,k}, \tau_m, C_m$, defined above.

A. Calculation of Transition Probabilities

In this subsection, the formulae for the calculation of the transition probabilities are presented.

- For states i and k, the control limit:

$$P_{i,k} = P\Big(\frac{k-1}{K} < \pi_{m\Delta} \leq \frac{k}{K} \mid \pi_{(m-1)\Delta} = i\Big). \tag{6}$$

  Note that $\pi_{(m-1)\Delta}$ is the coded value of the posterior probability which is the mid-point of the corresponding interval.
- When the posterior probability exceeds the control limit, the process is stopped and full inspection is performed. In this case, the SMDP enters the inspection state and the transition probability is given by:

$$P_{i,h} = P\Big(\frac{h-1}{K} < \pi_{m\Delta} \leq \frac{h}{K} \mid i\Big). \tag{7}$$

- The process can make transition from inspection state either to state 0 if the false alarm occurred, or to the repair state r when a true alarm occurred. The transition probabilities for the two cases are given as:

$$\begin{aligned} P_{h,0} &= 1 - \frac{h-0.5}{K} \\ P_{h,r} &= \frac{h-0.5}{K} \end{aligned} \tag{8}$$

- When the process is out of control, the repair action must be performed to bring the process into the in control state. So, the transition probability is given by:

$$P_{r,0} = 1 \tag{9}$$

To calculate the transition probability in (6), we need to develop the posterior probability defined in (4) as:

$$\begin{aligned} &P_{i,k} = P\Big(\frac{k-1}{K} < \pi_{m\Delta} \leq \frac{k}{K} \mid \pi_{(m-1)\Delta} = i\Big) \\ &= P\Big(\frac{k-1}{K} < \\ &\frac{1-(1-\pi_{(m-1)\Delta})e^{-\theta\Delta}}{1-(1-\pi_{(m-1)\Delta})e^{-\theta\Delta} + (\frac{p_0}{p_1})^D(\frac{1-p_0}{1-p_1})^{n-D}(1-\pi_{(m-1)\Delta})e^{-\theta\Delta}} \\ &\leq \frac{k}{K} \mid i, X_{m\Delta} = 0\Big) \times P(X_{m\Delta} = 0 \mid i) \\ &+P\Big(\frac{k-1}{K} \\ &< \frac{1-(1-\pi_{(m-1)\Delta})e^{-\theta\Delta}}{1-(1-\pi_{(m-1)\Delta})e^{-\theta\Delta} + (\frac{p_0}{p_1})^D(\frac{1-p_0}{1-p_1})^{n-D}(1-\pi_{(m-1)\Delta})e^{-\theta\Delta}} \\ &\leq \frac{k}{K} \mid i, X_{m\Delta} = 1\Big) \times P(X_{m\Delta} = 1 \mid i) \end{aligned} \tag{10}$$

To simplify notation, let $(1-\pi_{(m-1)\Delta})e^{-\theta\Delta} = \alpha$ and denote $\pi_{(m-1)\Delta} = \pi$. Then, the first term in (10) can be written as:

$$\begin{aligned} &P_{i,k} = P\Big(\frac{k-1}{K} < \pi_{m\Delta} \leq \frac{k}{K} \mid \pi = i\Big) \\ &= P\Big(\frac{k-1}{K} < \frac{1-\alpha}{1-\alpha + [\frac{p_0(1-p_1)}{P_1(1-p_0)}]^D(\frac{1-p_0}{1-p_1})^n\alpha} \leq \frac{k}{K} | \pi, X_{m\Delta} = 0\Big) \\ &= P\Big(\frac{k-1}{K}[1-\alpha + [\frac{p_0(1-P_1)}{p_1(1-p_0)}]^D(\frac{1-p_0}{1-p_1})^n\alpha] < 1-\alpha \\ &\quad \leq \frac{k}{K}[1-\alpha + [\frac{p_0(1-p_1)}{p_1(1-p_0)}]^D(\frac{1-p_0}{1-p_1})^n\alpha] \,|\pi, X_{m\Delta} = 0\Big) \\ &= P\Big((\frac{k-1}{K})(1-\alpha) + (\frac{k-1}{K})[\frac{p_0(1-p_1)}{p_1(1-p_0)}]^D(\frac{1-p_0}{1-p_1})^n\alpha \\ &\quad < 1-\alpha \leq \\ &\quad (\frac{k}{K})(1-\alpha) + (\frac{k}{K})[\frac{p_0(1-p_1)}{p_1(1-p_0)}]^D(\frac{1-p_0}{1-p_1})^n\alpha|\pi, X_{m\Delta} = 0\Big) \\ &\quad = P\Big((\frac{k-1}{K})[\frac{p_0(1-p_1)}{p_1(1-p_0)}]^D(\frac{1-p_0}{1-p_1})^n\alpha \\ &\quad < (1-\alpha)(1-(\frac{k-1}{K})) \leq \\ &\quad (\frac{1-\alpha}{K}) + (\frac{k}{K})[\frac{p_0(1-p_1)}{p_1(1-p_0)}]^D(\frac{1-p_0}{1-p_1})^n\alpha \,|\, \pi, X_{m\Delta} = 0\Big) \\ &\quad = P\Big(\underbrace{[\frac{p_0(1-p_1)}{p_1(1-p_0)}]^D(\frac{1-p_0}{1-p_1})^n}_{\text{Term I}} \\ &\quad <(\frac{1-\alpha}{\alpha})((\frac{K-(k-1)}{k-1})) \leq \\ &\quad \underbrace{(\frac{1-\alpha}{\alpha(k-1)})+(\frac{k}{k-1})[\frac{p_0(1-p_1)}{p_1(1-p_0)}]^D(\frac{1-p_0}{1-p_1})^n}_{\text{Term II}}|\pi, X_{m\Delta} = 0\Big) \end{aligned} \tag{11}$$

By taking the logarithm of both sides, Term I and Term II are summarized as:

$$\begin{aligned} \text{Term I} &= \underbrace{[\frac{p_0(1-p_1)}{p_1(1-p_0)}]}_{<1}{}^d < \frac{(\frac{1-\alpha}{\alpha})(\frac{K-(k-1)}{k-1})}{\left(\frac{1-p_0}{1-p_1}\right)^n} \\ &= D > \frac{log(\frac{(\frac{1-\alpha}{\alpha})(\frac{K-(k-1)}{k-1})}{\left(\frac{1-P_0}{1-P_1}\right)^n})}{log(\frac{P_0(1-P_1)}{P_1(1-P_0)})}, \end{aligned} \tag{12}$$

and

$$\text{Term II} = [\underbrace{\frac{p_0(1-p_1)}{p_1(1-p_0)}}_{<1}]^D \geq \frac{(\frac{1-\alpha}{\alpha})(\frac{K-k}{k})}{\left(\frac{1-p_0}{1-p_1}\right)^n}$$
$$= D \leq \frac{log(\frac{(\frac{1-\alpha}{\alpha})(\frac{K-k}{k})}{\left(\frac{1-p_0}{1-p_1}\right)^n})}{log(\frac{p_0(1-p_1)}{p_1(1-p_0)})}. \quad (13)$$

So, the transition probability in (11) can be summarized as:

$$P_{i,k} = P\Big(\frac{k-1}{K} < \pi_{m\Delta} \leq \frac{k}{K} \mid \pi = i\Big) =$$
$$P\Big(\Big\lfloor\frac{log(\frac{(\frac{1-\alpha}{\alpha})(\frac{K-(k-1)}{k-1})}{\left(\frac{1-p_0}{1-p_1}\right)^n})}{log(\frac{p_0(1-p_1)}{p_1(1-p_0)})}\Big\rfloor < D \leq \Big\lfloor\frac{log(\frac{(\frac{1-\alpha}{\alpha})(\frac{K-k}{k})}{\left(\frac{1-p_0}{1-p_1}\right)^n})}{log(\frac{p_0(1-p_1)}{p_1(1-p_0)})}\Big\rfloor \Big| i, 0\Big). \quad (14)$$

The number of defectives in a sample follows binomial distribution with mean np_0 when the process is in control and np_1 when the process is out of control. For $\xi = 0, 1$, the first and third term in (10) can be calculated as:

$$\begin{aligned} & P(A < D \leq B \mid \pi = i, X_{m\Delta} = \xi) \\ = \; & P(D \leq B \mid \pi = i, X_{m\Delta} = \xi) \\ - \; & P(D \leq A \mid \pi = i, X_{m\Delta} = \xi) \\ = \; & \sum_{j=0}^{B} \binom{n}{j} p_\xi^j (1-p_\xi)^{n-j} - \sum_{j=0}^{A} \binom{n}{j} p_\xi^j (1-p_\xi)^{n-j}. \end{aligned} \quad (15)$$

The second term in (10) is calculated as:

$$\begin{aligned} P(X_{m\Delta} = 0 \mid \pi = i) &= P(X_{m\Delta} = 0 \mid i, X_0 = 0)(1-\pi) \\ &+ \; P(X_{m\Delta} = 0 \mid i, X_0 = 1)\pi \\ &= \; e^{-\theta\Delta}(1-\pi) \end{aligned} \quad (16)$$

Finally, the last term in (10) is given by:

$$\begin{aligned} P(X_{m\Delta} = 1 \mid \pi = i) &= P(X_{m\Delta} = 1 \mid i, X_0 = 0)(1-\pi) \\ &+ \; P(X_{m\Delta} = 1 \mid i, X_0 = 1)\pi \\ &= \; (1 - e^{-\theta\Delta})(1-\pi) + \pi \end{aligned} \quad (17)$$

This completes the calculation of transition probability defined in (6). A similar approach will be used to calculate the transition probability given by (7) which is omitted here to save the space.

In the next section, the calculation of the average cost and the sojourn times in the SMDP framework is presented.

IV. Calculation of the Expected Costs and sojourn times

In this section, we derive the formulae for the expected costs and the sojourn times.

- For state i, the average cost is calculated as:

$$\begin{aligned} C_i = E(Cost \mid i) &= a + bn + E(\int_0^\Delta (MI_{\{X_s=1\}} \mid i)\; ds) \\ &= \; a + bn + \int_0^\Delta M \times P(X_s = 1 \mid i)\; ds \\ &= \; a + bn + M\int_0^\Delta [(1-e^{-\theta s})(1-\pi) + \pi] ds, \end{aligned} \quad (18)$$

  and the sojourn time is:

$$\tau_i = \Delta \quad (19)$$

- The average cost in inspection state is given by:

$$C_h = E(Cost \mid h) = A, \quad (20)$$

  and the sojourn time is:

$$\tau_h = T_I, \quad (21)$$

  where T_I denotes the time required for performing the inspection.
- Finally, the average cost in state r is given by:

$$C_r = E(Cost \mid r) = R, \quad (22)$$

  and the sojourn time is:

$$\tau_r = T_R, \quad (23)$$

  where T_R is the repair time of the process.

This completes the calculation of the required components for SMDP. In the next section, we present numerical examples to illustrate the developed SMDP algorithm.

V. Numerical Examples

In this section, numerical examples are provided to show the performance of the proposed model. The numerical example is provided in two parts: in part (i), four sets of parameters are considered and the program is run for these four sets to find the optimal values for the long-run expected average cost as well as the control chart parameters, and in part (ii), the proposed Bayesian control chart performance is compared with the results in [13] using traditional p-chart.

Part (i): Four sets of different input parameters are considered in Table I. We also assume that $T_I = T_R = 1$.

TABLE I
Sets of Parameters

Sets	θ	a	b	p_0	p_1	M	A	R
I	0.01	1	0.1	0.05	0.2	100	500	250
II	0.01	5	1	0.05	0.2	100	500	250
III	0.01	1	1	0.05	0.2	100	500	250
IV	0.01	5	0.1	0.05	0.2	100	500	250

The interval $[0, 1]$ of the posterior probability is discretized into $K = 50$ subintervals. The computational algorithm was very fast, the results were obtained in 2.3208 seconds for each run on an Intel Core (TM) i5 CPU with 2.27 GHz. By applying policy iteration algorithm explained in Section III, the optimal sample size, optimal control limit and optimal sampling interval are obtained and the results are given in Table II for all sets. The results for sets I and II show that

TABLE II
RESULTS FOR THE OPTIMAL BAYESIAN CONTROL CHART FOR ALL SETS

Sets	Optimal control limit (L)	Sample size (n)	Sampling interval (Δ)	Average cost (g)
I	0.55	60	4	13.9620
II	0.35	30	8	18.8186
III	0.35	30	8	18.3291
IV	0.55	60	4	14.7902

when the sampling costs are increased (set II), the average cost will be increased and the control limit decreases significantly. These results can be explained as: When the sampling is costly, the sample size is decreased and the control limit decreases to compensate for the lower control chart sensitivity. On the other hand, the sampling interval is increased due to costly sampling.

The results for sets I and IV are similar, except the total average cost which is higher for set IV. The input parameters for both sets are the same except the fixed cost per sample which is higher for set IV than for set I and it leads to the increase in the total average cost. But it does not affect the other control parameters such as the control limit and the sampling interval because it is a fixed cost which does not depend on the sample size.

Part (ii): In this part, the results of the Bayesian p-chart are compared with the results obtained by traditional p-chart (see [13]). To make a fair comparison, the same input parameters as in [13] are considered in this paper. For the set

TABLE III
SET OF PARAMETERS

θ	p_0	p_1	M	A	R
0.01	0.05	0.2	100	500	250

of parameters in Table III, the sampling costs are not taken into account and the times to inspect and repair the process are negligible. For given sampling intervals $\Delta = 4, \Delta = 6, \Delta = 8$, and sample size $n = 5$, the results are obtained in Table IV.

TABLE IV
COMPARISON RESULTS OF THE PRESENTED MODEL WITH THE TRADITIONAL p-CHART

Δ	Proposed approach	λ-MAX algorithm	Traditional p-chart
4	19.5650	19.12	22.12
6	20.9460	20.98	26.18
8	22.9614	22.30	29.46

The average cost results obtained using the presented approach and the λ-maximization algorithm are very close, but the computational times for the SMDP algorithm developed in this paper are considerably lower when compared with the λ-maximization algorithm for which the computational times were about 5 times higher on average. For a set of given parameters, each SMDP run took only 2.3208 seconds which is very fast. It is also shown that the average cost of the proposed model is considerably lower than the cost for the traditional p-chart.

VI. CONCLUSIONS

In this paper, we have developed a fast computational algorithm for the design of a Bayesian control chart for monitoring the proportion of defectives. The process is monitored at periodic sampling epochs and the posterior probability of the process being out of control is updated at each sampling epoch after collecting a new sample. If the posterior probability exceeds the control limit, the chart signals, the process is stopped, and a search for an assignable cause is initiated. If true alarm occurs, the process is repaired, otherwise it continues without any action. The cost components of the process include quality cost, sampling, repair and inspection costs, which are considered for designing the proposed optimal Bayesian p-chart. The objective is to find the optimal values of the control chart parameters, namely, the sampling interval, sample size, and the control limit to minimize the long-run expected average cost per unit time.The problem is formulated and solved in SMDP framework and numerical examples illustrate the effectiveness of the proposed model and a high speed of the computational algorithm. In addition, the developed Bayesian control chart is compared with the traditional p control chart and the results show that the proposed chart has a considerably lower expected average cost per unit time and hence, it is a highly effective tool for controlling the proportion of defectives in modern manufacturing processes.

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