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Variational Evolutionary Splines for Solving a Model of Temporomandibular Disorders

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***Abstract*—The aim of this work is to modelize the occlusion of a person with temporomandibular disorders as an evolutionary equation and approach its solution by the construction and characterizing of discrete variational splines. To formulate the problem, certain boundary conditions have been considered. After showing the existence and the uniqueness of the solution of such a problem, a convergence result of a discrete variational evolutionary spline is shown. A stress analysis of the occlusion of a human jaw with temporomandibular disorders by finite elements is carried out in FreeFem++ in order to prove the validity of the presented method.

***Keywords*—Approximation, evolutionary PDE, finite element method, temporomandibular disorders, variational spline.

I. Introduction

THE objective of this paper is intended to serve as a step to solve an evolutionary problem of a patient with temporomandibular disorders by the developing of a variational method. With regard to the approximation of curves and surfaces, we follow the methodology of [1] for variational approximation using methods from boundary problems and considering the finite element method. The mathematical model is derived using some facts supported by Dentistry, although it should be noted that there are not neither enough studies to date nor relevant information of the mathematical modeling of these issues, however it is possible to consider more than a hundred variables associated with them. This was the initial point for the research of this topic. With a first study in Multivariate Statistics to manipulate local groups of disorders, we complemented and extended the analysis carried out by other authors (see [2], [3]). A Factor Analysis was chosen for doing this task, obtaining 11 groups of patterns. After performing Factor Analysis to the problem of temporomandibular disorders from variables detected in a sample of patients, determining factors associated with different sets of variables required the implementation of a computer system based on the diagnosis of symptoms and signs for the determination of groups of relevant variables in the 11 factors found, but it was merely insufficient a classic system. Given the variable nature of examiners and examinees, it was difficult to obtain results without the integration of some degree of uncertainty, in consequence the intervention of an artificial intelligence system was necessary. That is why, in order to optimize the obtaining results, the original system was transformed into a fuzzy expert system for the diagnosis, applicable to any test subject which meets the exclusion criteria sample warned in [4]. This fuzzy expert system provided some indicators of membership of the progress of some of the factors of the disorder from which the test subject can be affected. Because of the lack of graphical or visual components of the developed classic systems from which patients does not look identified, it was necessary to build a prototype human jaw, using CAD, adaptable to any individual state in despite of the state of the disorder, in order to achieve a better diagnosis thereof. This objective required the coupling of some components as physical forces, material properties, and, in particular, the geometry of the curves used in its two-dimensional design for forming solid from numerous scans of a patient.

The structure of this study will be as follows: In the second section, we are going to study the Mathematical modeling of the occlusion process as an evolutionary problem and the determination of the variational formulation associated and will be studied into two parts. During the first part we will define the dental terminology that will be used in rest of the paper as temporomandibular disorder (TMD), outlining necessary topics for the understanding and the subsequent construction of the mathematical model of the occlusion. Throughout this section will be necessary to review some definitions like the proper of the system of data collection of signs and symptoms of a patient with temporomandibular disorder for the definition of the functions of compact support functions that have to do with the density of the forces needed in the model formulation. The second part, for the understanding the results of the process of occlusion some concepts like position, motion, deformation, displacement, stress and all physical concepts necessary for are defined. Furthermore, the assumptions of the occlusion model are described in the same way that the definition of the jaw, the boundary conditions and the forces involved in the process. Two problems of partial differential equations are formulated, one quasi-static and one dynamic, obtaining an evolutionary equation with contact conditions at the boundary, whose numerical solution is the main topic of the last section of the paper. In the following section we are going to design of an algorithm for numerical solution of variational problem formulated by finite differences in time and finite elements in space. We will present a set of theorems that will be demonstrated in Appendix in order to obtain convergency. The novelty of this model lies in the fact that we can chose a set of differential operators in order to transform an evolutionary problem in a variational problem from a constitutive law. The functional spaces and the variational formulation of the problem above mentioned are also studied. The result of existence and uniqueness of it and the convergence of the discrete solution to the exact solution of the original problem

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are performed. In the fourth section, after introducing the physical properties of the mandible in its constitutive version, the analysis of forces in the temporomandibular joint, and the explanation of each force used in the simulation we will show the results brought by the programming the previous algorithm in a suitable software to support the paper. Some simulations are made in order to construct, by variational methods with mesh, the mathematical model of occlusion of a patient with temporomandibular disorders, and the analysis of deformation and displacements to compare the obtained results with dental expected results by experts as can be seen in [5]. Finally, we will present the conclusions of the analysis focusing on the similarity of the numerical results that contrasts the assumptions made in this paper suggesting that its improvement under more realistic assumptions can serve for future research of chronic degenerative diseases in Dentistry, and in general in the field of Medicine.

II. Modeling the Occlusion Process

A. The Model

Let $\Omega \subset \mathbb{R}^3$ be the domain occupied by a jaw with boundary $\Gamma = \partial\Omega$, which by its own nature will be considered as the outer surface (see [6]). Let ϵ_{ij}, $i = 1, \ldots, 3$, $j = 1, \ldots, 3$ be the deformation. We assume the jaw is a homogeneous linear elastic isotropic solid so that the deformation is entirely elastic, hence body recovers its original shape upon removal of the occlusal force that causes deformation [7]. Due to the characteristics of the temporomandibular joint (TMJ) [8] and the rigidity of the temporal bone, we use an unilateral contact condition without friction in the upper part of left condyle. The displacement is restricted by the glenoid fossa and thus displacement in the tangential direction is produced without friction stress (see [9], [10]).

Let us consider the equilibrium of a body in space and a part of the boundary without including the support friction. If contact occurs then the reaction g is opposed to the outer normal to the body. If then we denote by $\sigma_n = \sum_{i=1}^{3} g_i n_i$, $u_n = \sum_{i=1}^{3} u_i n_i$, where u_n is the normal direction, and σ_n the normal stress, we have $\sigma_n \leq 0; u_n \leq 0; \sigma_n u_n = 0$ i.e. on frictionless contact problem (see [11]-[12]):

$$\begin{cases} \sigma_n \leq 0 \rightarrow u_n = 0, & \text{contact,} \\ u_n \leq 0 \rightarrow \sigma_n = 0, & \text{no contact.} \end{cases}$$

Therefore, we need to partition Γ into three sets:

- The surface area where occlusal contact may occur, denoted by Γ_C (see [10] for details about molars and condyles).
- The part of the boundary where there are not displacement u, denoted by Γ_D ($u = 0$ on Γ_D).
- The disjoint union of nine mandibular surfaces where different muscles can act (superficial masseter (SM), deep masseter (DM), medial pterygoid (MP), anterior temporalis (AT), middle temporalis (MT), posterior temporalis (PT), inferior pterygoid (IP), superior pterygoid (SP) and anterior digastric (AD)), each separately, denoted by Γ_M (see Table I).

Γ_C admits two parts: the upper side of the left condyle Γ_C^c and the upper side of the molar Γ_C^m (see [13], [14]). Γ_D will not be partitioned because in clenching the right condyle is always in contact with the maxillary. Then, Γ can be written as $\Gamma = (\Gamma_C^c \cup \Gamma_C^m) \cup \Gamma_D \cup \Gamma_M$.

For each point of the jaw we consider the displacement vector $u(x) = (u_i(x))$, where $u_i(x)$ denotes the displacement of x in the direction OX_i, for each $i = 1, \ldots, 3$.

We do not attempt to consider the weight of the jaw in this model because it does not have very influence on deformation (see [15]); hence, the components of the traction vector that we consider on Γ_M are g_i, $i = 1, \ldots, 3$ (see [16]).

Due to elastic properties of the bone we consider Lame-Hooke law

$$\sigma_{ij} = \sigma_{ij}(u) = \frac{E\nu}{(1+\nu)(1-2\nu)} \left(\sum_{k=1}^{3} \epsilon_{kk}(u) \right) \delta_{ij} + \frac{E}{1+\nu} \epsilon_{ij}(u), \tag{1}$$

for each $i = 1, \ldots, 3$, $j = 1, \ldots, 3$, where $\epsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$, ν is the Poisson's ratio, E is the Young's modulus and δ_{ij} is the Kronecker delta. We consider that the jaw has the same physical properties in all directions (isotropic), then (1) can be expressed compactly as

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix} = \frac{E}{1+\nu} \begin{pmatrix} \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \\ \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \\ \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & \\ & & & I_3 \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{23} \end{pmatrix}$$

with I_3 the identity matrix of order 3. Furthermore, since occlusion is a process where there are no substantial variation of forces in time it becomes steady, and thus there is no inertial effects related to acceleration. These properties define a quasi-static problem given by $-\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} = f_i$, with σ_{ij} is the stress tensor, and f_i are the components of the body forces, for each $i = 1, \ldots, 3$.

In the case of considering inertia effects, f_i must be written in the form $f_i = -\rho \frac{\partial^2 u_i}{\partial t^2}$, for each $i = 1, \ldots, 3$, where ρ is the mass density of jaw bone [17].

Imposing boundary conditions on Γ_M, we have

$$\sum_{j=1}^{3} \sigma_{ij} \cdot n_j = g_i, \quad i = 1, \ldots, 3,$$

where n_j is the outer vector with director cosines as components (see Table I). Note that only non-positive values of normal stress are allowed on Γ_C. We suppose that the frictional force is negligible, so it can be described by the equations

$$\sigma_{\tau i} = \sum_{j=1}^{3} \sigma_{ij} n_j - \sigma_n n_i = 0, \quad i = 1, \ldots, 3,$$

where $\sigma_{\tau i}$ is the shear stress.

TABLE I
MAGNITUDES OF MUSCLE LOADS USED IN FE MODEL OF UNILATERAL RIGHT MOLAR CLENCHING TYPE (RMOL) [10]

	NODES		FACTOR			MAGNITUDES-RMOL (for composing fuzzy f)						DIRECTION COSINES (for composing fuzzy g)					
	R	L	R	L	[N]	R			L			R			L		
			SCALE		W	X	Y	Z	X	Y	Z	X	Y	Z	X	Y	Z
SM	59	59	0.72	0.60	190.4	-0.48	2.05	0.97	0.40	1.71	0.81	-0.20	0.88	0.41	0.20	0.88	0.41
DM	38	38	0.72	0.60	81.6	-0.84	1.17	-0.55	0.70	0.97	-0.46	-0.54	0.75	-0.35	0.54	0.75	-0.35
MP	44	43	0.84	0.60	174.8	1.62	2.63	1.24	-1.18	1.92	0.90	0.48	0.79	0.37	-0.48	0.79	0.37
AT	43	40	0.73	0.58	158.0	-0.40	2.65	0.11	0.34	2.26	0.10	-0.14	0.98	0.04	0.14	0.98	0.04
MT	18	18	2.97	-1.77	0.78	0.66	0.67	95.6	-0.77	2.93	-1.75	-0.22	0.83	-0.50	0.22	0.83	-0.50
PT	15	15	0.59	0.39	75.6	-0.62	1.41	-2.54	0.41	0.93	-1.68	-0.20	0.47	-0.85	0.20	0.47	-0.85
IP	5	5	0.30	0.65	66.9	2.52	-0.69	3.03	-5.47	-1.51	6.58	0.63	-0.17	0.75	-0.63	-0.17	0.75
SP	4	4	0	0	28.7	0	0	0	0	0	0	0.76	0.07	0.64	-0.76	0.07	0.64
AD	8	8	0	0	40.0	0	0	0	0	0	0	-0.24	-0.23	-0.94	0.24	-0.23	-0.94

Then we have the problem:

$$\begin{aligned} -\sum_{j=1}^{3}\frac{\partial\sigma_{ij}(u)}{\partial x_j} &= f_i \text{ in } \Omega, \quad i=1,\dots,3,\\ u &= 0 \text{ on } \Gamma_D,\\ \sum_{j=1}^{3}\sigma_{ij}(u)n_j &= g_i \text{ on } \Gamma_M, \quad i=1,\dots,3, \end{aligned}$$

$$\begin{cases} u_n &= \sum_{i=1}^{3} u_i n_i \le 0, \quad \sigma_n = \sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}n_in_j \le 0,\\ u_n\sigma_n &= 0,\\ \sigma_{\tau i} &= \sum_{j=1}^{3}\sigma_{ij}n_j - \sigma_n n_i = 0, \quad i=1,\dots,3, \text{ on } \Gamma_C. \end{cases}$$

If we consider the inertial effect of acceleration and the condition that the outer normal vector at a point on the surface is constant with respect to time and varies only according to its position during a short interval of time $[0,T]$, $T>0$, then the problem (2)-(II-A) can be stated as follows.

B. The Problem

Find $u:\overline{\Omega}\times[0,T]\to\mathbb{R}^3$, with $u(x,t)=(u_i(x,t))$, $i=1,\dots,3$, such that:

$$\begin{aligned} \sum_{j=1}^{3}\frac{\partial\sigma_{ij}}{\partial x_j}(u(x,t)) &= \rho\frac{\partial^2 u_i}{\partial t^2}(x,t) \text{ in } \Omega\times[0,T],\\ u(x,t) &= 0, \qquad \text{on } \Gamma_D\times[0,T],\\ \sum_{j=1}^{3}\sigma_{ij}(u(x,t))n_j(x) &= g_i(x,t), \text{ on } \Gamma_M\times[0,T], \end{aligned}$$

$$\begin{cases} u_n(x,t)\le 0, \sigma_n(x,t)\le 0, u_n(x,t)\sigma_n(x,t)=0,\\ \sigma_\tau(x,t)=0, \quad \text{on } \Gamma_C\times[0,T],\\ u(x,0)=0, \quad \dfrac{\partial u}{\partial t}(x,0)=0, \quad \text{in } \Omega. \end{cases}$$

For each $i=1,\dots,3$, we have to solve the set of partial differential equations:

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{3}\frac{\partial\sigma_{ij}}{\partial x_j}(u).$$

From (1) we have

$$\sigma_{ij} = \frac{E\nu}{(1+\nu)(1-2\nu)}\left(\sum_{k=1}^{3}\epsilon_{kk}(u)\right)\delta_{ij} + \frac{E}{1+\nu}(\epsilon_{ij}),$$

and by substituting we obtain

$$\begin{aligned} \sigma_{ij} =& \frac{E\nu}{(1+\nu)(1-2\nu)}\left(\sum_{k=1}^{3}\frac{1}{2}\left(\frac{\partial u_k}{\partial x_k}+\frac{\partial u_k}{\partial x_k}\right)\right)\delta_{ij}+\\ & \frac{E}{1+\nu}\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right), \end{aligned} \tag{2}$$

if we denote $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$, $\mu = \dfrac{E}{2(1+\nu)}$, by using (2), we deduce that

$$\sigma_{ij} = \begin{cases} \lambda\left(\sum_{k=1}^{3}\dfrac{\partial u_k}{\partial x_k}\right) + 2\mu\left(\dfrac{\partial u_i}{\partial x_i}\right), \text{ if } i=j,\\ \mu\left(\dfrac{\partial u_j}{\partial x_i}+\dfrac{\partial u_i}{\partial x_j}\right), \text{ if } i\ne j. \end{cases}$$

Applying derivatives to σ_{ij} this implies

$$\frac{\partial\sigma_{ij}}{\partial x_j} = \begin{cases} \lambda\left(\sum_{j=1}^{3}\dfrac{\partial^2 u_j}{\partial x_i\partial x_j}\right) + 2\mu\left(\dfrac{\partial^2 u_i}{\partial x_i^2}\right), \text{ if } j=i,\\ \mu\left(\dfrac{\partial^2 u_j}{\partial x_i\partial x_j}+\dfrac{\partial^2 u_i}{\partial x_j^2}\right), \text{ if } j\ne i. \end{cases}$$

which means

$$\frac{\partial\sigma_{ij}}{\partial x_j} = \begin{cases} \lambda\left(\dfrac{\partial^2 u_i}{\partial x_i^2}\right) + \lambda\sum_{j\ne i}^{3}\dfrac{\partial^2 u_j}{\partial x_i\partial x_j} + \\ \mu\left(\dfrac{\partial^2 u_i}{\partial x_i^2}\right) + \mu\sum_{j=i}^{3}\dfrac{\partial^2 u_j}{\partial x_i^2}, \text{ if } j=i,\\ \mu\left(\dfrac{\partial^2 u_j}{\partial x_i\partial x_j}+\mu\dfrac{\partial^2 u_i}{\partial x_j^2}\right), \text{ if } j\ne i. \end{cases}$$

and consequently

$$\frac{\partial\sigma_{ij}}{\partial x_j} = \begin{cases} (\lambda+\mu)\dfrac{\partial^2 u_i}{\partial x_i^2} + \lambda\sum_{j\ne i}^{3}\dfrac{\partial^2 u_j}{\partial x_i\partial x_j} + \mu\left(\dfrac{\partial^2 u_i}{\partial x_i^2}\right),\\ \text{if } j=i,\\ \mu\left(\dfrac{\partial^2 u_j}{\partial x_i\partial x_j}+\mu\dfrac{\partial^2 u_i}{\partial x_j^2}\right), \text{ if } j\ne i. \end{cases}$$

By adding the derivatives $\dfrac{\partial \sigma_{ij}}{\partial x_j}$, for each $j = 1, \ldots, 3$ we have

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} =(\lambda+\mu)\frac{\partial^2 u_i}{\partial x_i^2} + \lambda \sum_{\substack{j=1\\ j\neq i}}^{3} \frac{\partial^2 u_j}{\partial x_i \partial x_j} + \mu \sum_{\substack{j=1\\ j\neq i}}^{3} \frac{\partial^2 u_j}{\partial x_i \partial x_j} + \mu \frac{\partial^2 u_i}{\partial x_i^2} + \mu \sum_{\substack{j=1\\ j\neq i}}^{3} \frac{\partial^2 u_i}{\partial x_j^2},$$

since

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} = (\lambda+\mu)\frac{\partial^2 u_i}{\partial x_i^2} + (\lambda+\mu) \sum_{\substack{j=1\\ j\neq i}}^{3} \frac{\partial^2 u_j}{\partial x_i \partial x_j} + \mu \left(\sum_{\substack{j=1\\ j\neq i}}^{3} \frac{\partial^2 u_j}{\partial x_j^2} + \frac{\partial^2 u_i}{\partial x_i^2} \right),$$

we obtain

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} = (\lambda+\mu) \left(\sum_{\substack{j=1\\ j\neq i}}^{3} \frac{\partial^2 u_j}{\partial x_i \partial x_j} + \frac{\partial^2 u_i}{\partial x_i^2} \right) + \mu \sum_{j=1}^{3} \frac{\partial^2 u_i}{\partial x_j^2},$$

or equivalently

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} = (\lambda+\mu) \sum_{j=1}^{3} \frac{\partial^2 u_j}{\partial x_i \partial x_j} + \mu \sum_{j=1}^{3} \frac{\partial^2 u_i}{\partial x_j^2},$$

or

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} = (\lambda+\mu) \left(\frac{\partial}{\partial x_i} \left(\sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j} \right) \right) + \mu \Delta u_i, \qquad (3)$$

it turns out from the divergence of a vector that

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} = (\lambda+\mu) \left(\frac{\partial}{\partial x_i} div u \right) + \mu \Delta u_i .$$

Thus, for each $i = 1, \ldots, 3$ the system becomes

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda+\mu) \frac{\partial}{\partial x_i}(div u) + \mu \Delta u_i,$$

or in vector form

$$\rho u_{tt} = (\lambda+\mu)\nabla div u + \mu \Delta u,$$

which is also the solution of the variational problem:

Find $u : \Omega \to \mathbb{R}^3$, with $u(0) = 0$, $\dfrac{\partial u}{\partial t}(0) = 0$ in Ω and $u(t) = 0$ in $[0, T]$, on Γ_D, such that for each $v : \Omega \to \mathbb{R}^3$ with $v = 0$ on Γ_D we have

$$\rho \int_\Omega u_{tt}(x,t)\cdot v + \lambda \int_\Omega (\nabla\cdot u(x,t))(\nabla \cdot v(x)) + 2\mu \int_\Omega \partial_x u(x,t) : \partial_x v(x) + \int_{\Gamma_M} \alpha_g (u_{\alpha_g} - u)\cdot v = \int_\Omega 0 \cdot v.$$

III. Proposed Methodology

A. The Proposed Method: Hananel's Method

To solve the previous problem we follow two steps, first we split (3) obtaining

$$\begin{cases} \rho \dfrac{\partial^2 u_1}{\partial t^2} = \mu(\dfrac{\partial^2 u_1}{\partial x_1^2} + \dfrac{\partial^2 u_1}{\partial x_2^2} + \dfrac{\partial^2 u_1}{\partial x_3^2}) + (\lambda+\mu)\left(\dfrac{\partial^2 u_1}{\partial x_1^2} + \dfrac{\partial^2 u_2}{\partial x_1 \partial x_1} + \dfrac{\partial^2 u_3}{\partial x_1 \partial x_3} \right) \\ \rho \dfrac{\partial^2 u_2}{\partial t^2} = \mu(\dfrac{\partial^2 u_2}{\partial x_1^2} + \dfrac{\partial^2 u_2}{\partial x_2^2} + \dfrac{\partial^2 u_2}{\partial x_3^2}) + (\lambda+\mu)\left(\dfrac{\partial^2 u_1}{\partial x_1 \partial x_2} + \dfrac{\partial^2 u_2}{\partial x_2^2} + \dfrac{\partial^2 u_3}{\partial x_2 \partial x_3} \right) \\ \rho \dfrac{\partial^2 u_3}{\partial t^2} = \mu(\dfrac{\partial^2 u_3}{\partial x_1^2} + \dfrac{\partial^2 u_3}{\partial x_2^2} + \dfrac{\partial^2 u_3}{\partial x_3^2}) + (\lambda+\mu)\left(\dfrac{\partial^2 u_1}{\partial x_1 \partial x_3} + \dfrac{\partial^2 u_2}{\partial x_2 \partial x_3} + \dfrac{\partial^2 u_3}{\partial x_3^2} \right) \end{cases} \qquad (4)$$

and second, we shall consider this system as constructed with a double differential operator in such way that:

$$u = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \\ u_3(x,t) \end{pmatrix} \overset{\partial_x^{(1,0,0)}}{\longrightarrow} \begin{pmatrix} \dfrac{\partial u_1}{\partial x_1}(x,t) \\ \dfrac{\partial u_2}{\partial x_1}(x,t) \\ \dfrac{\partial u_3}{\partial x_1}(x,t) \end{pmatrix} \overset{\partial_x^{(1,0,0)}}{\longrightarrow} \begin{pmatrix} \dfrac{\partial^2 u_1}{\partial x_1^2}(x,t) \\ \dfrac{\partial^2 u_2}{\partial x_1^2}(x,t) \\ \dfrac{\partial^2 u_3}{\partial x_1^2}(x,t) \end{pmatrix},$$

and by multiplying it by a matrix of the form

$$a_{(1,0,0),(1,0,0)} = \begin{pmatrix} \mu + (\mu+\lambda) & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

we obtain

$$\begin{pmatrix} (\mu + (\lambda+\mu))\dfrac{\partial^2 u_1}{\partial x_1^2}(x,t) \\ \mu \dfrac{\partial^2 u_2}{\partial x_1^2}(x,t) \\ \mu \dfrac{\partial^2 u_3}{\partial x_1^2}(x,t) \end{pmatrix}$$

which can be defined as

$$a_{(1,0,0),(1,0,0)} \partial_x^{(1,0,0)}(\partial_x^{(1,0,0)} u(x,t)). \qquad (5)$$

Similarly,

$$u = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \\ u_3(x,t) \end{pmatrix} \overset{\partial_x^{(0,1,0)}}{\longrightarrow} \begin{pmatrix} \dfrac{\partial u_1}{\partial x_2}(x,t) \\ \dfrac{\partial u_2}{\partial x_2}(x,t) \\ \dfrac{\partial u_3}{\partial x_2}(x,t) \end{pmatrix} \overset{\partial_x^{(0,1,0)}}{\longrightarrow} \begin{pmatrix} \dfrac{\partial^2 u_1}{\partial x_2^2}(x,t) \\ \dfrac{\partial^2 u_2}{\partial x_2^2}(x,t) \\ \dfrac{\partial^2 u_3}{\partial x_2^2}(x,t) \end{pmatrix},$$

and by multiplying it by a matrix of the form

$$a_{(0,1,0),(0,1,0)} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu + (\mu+\lambda) & 0 \\ 0 & 0 & \mu \end{pmatrix},$$

we obtain

$$\begin{pmatrix} \mu \dfrac{\partial^2 u_1}{\partial x_2^2}(x,t) \\ (\mu + (\lambda+\mu)) \dfrac{\partial^2 u_2}{\partial x_2^2}(x,t) \\ \mu \dfrac{\partial^2 u_3}{\partial x_3^2}(x,t) \end{pmatrix}$$

which can be defined as

$$a_{(0,1,0),(0,1,0)} \partial_x^{(0,1,0)} (\partial_x^{(0,1,0)} u(x,t)). \tag{6}$$

Similarly,

$$u = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \\ u_3(x,t) \end{pmatrix} \stackrel{\partial_x^{(0,0,1)}}{\longrightarrow}$$

$$\begin{pmatrix} \dfrac{\partial u_1}{\partial x_3}(x,t) \\ \dfrac{\partial u_2}{\partial x_3}(x,t) \\ \dfrac{\partial u_3}{\partial x_3}(x,t) \end{pmatrix} \stackrel{\partial_x^{(0,0,1)}}{\longrightarrow} \begin{pmatrix} \dfrac{\partial^2 u_1}{\partial x_3^2}(x,t) \\ \dfrac{\partial^2 u_2}{\partial x_3^2}(x,t) \\ \dfrac{\partial^2 u_3}{\partial x_3^2}(x,t) \end{pmatrix},$$

and by multiplying it by a matrix of the form

$$a_{(0,0,1),(0,0,1)} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu + (\mu+\lambda) \end{pmatrix}$$

we obtain

$$\begin{pmatrix} (\mu + (\lambda+\mu)) \dfrac{\partial^2 u_1}{\partial x_3^2}(x,t) \\ \mu \dfrac{\partial^2 u_2}{\partial x_3^2}(x,t) \\ (\mu + (\lambda+\mu)) \dfrac{\partial^2 u_3}{\partial x_3^2}(x,t) \end{pmatrix},$$

which can be defined as

$$a_{(0,0,1),(0,0,1)} \partial_x^{(0,0,1)} (\partial_x^{(0,0,1)} u(x,t)). \tag{7}$$

Overall process is similar to that used for finding (7). For more details on obtaining this equation and the rest of the proof, see Appendix (A).

If we denote by $H^1(\Omega)$ the Sobolev space of order 1 of continuous classes of functions $u \in L^2(\Omega)$, with weak derivatives $\partial^i u$, of order 1, $|i| \leq 1$, for any $i = (i_1, i_2, i_3) \in \mathbb{N}^3$, $|i| = i_1 + i_2 + i_3$ and $\partial_x^i u(x) = \dfrac{\partial^{|i|} u}{\partial x_1^{i_1} \partial x_2^{i_2} \partial x_3^{i_3}}$, for any $x = (x_1, x_2, x_3) \in \Omega$.

Instead of writing the sum of the expressions: (5)-(7), (16)-(21) it is convenient to write

$$\sum_{|i|,|j|=1} a_{ij} \partial_x^j (\partial_x^i u(x,t))$$

which is equal to the following components of a vector

$$\begin{aligned} &\mu \left(\sum_{i=1}^{3} \frac{\partial^2 u_1}{\partial x_i^2} \right) + (\lambda+\mu) \left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_1} + \frac{\partial^2 u_3}{\partial x_1 \partial x_3} \right) \\ &\mu \left(\sum_{i=1}^{3} \frac{\partial^2 u_2}{\partial x_i^2} \right) + (\lambda+\mu) \left(\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_2 \partial x_3} \right) \\ &\mu \left(\sum_{i=1}^{3} \frac{\partial^2 u_3}{\partial x_i^2} \right) + (\lambda+\mu) \left(\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + \frac{\partial^2 u_2}{\partial x_2 \partial x_3} + \frac{\partial^2 u_3}{\partial x_3^2} \right) \end{aligned} \tag{8}$$

considering the vector

$$\begin{pmatrix} \rho \dfrac{\partial^2 u_1}{\partial t^2} \\ \rho \dfrac{\partial^2 u_2}{\partial t^2} \\ \rho \dfrac{\partial^2 u_3}{\partial t^2} \end{pmatrix} \tag{9}$$

and by comparing (8) and (9), we obtain (4) that can be written as

$$\rho u_{tt} = (\lambda+\mu) \nabla divu + \mu \Delta u,$$

which is the linear system consisting of equations

$$\partial_t (\rho \partial_t u(x,t)) = \rho \partial_{tt} u(x,t) = \sum_{|i|,|j|=1} a_{ij} \partial_x^j (\partial_x^i u(x,t)),$$

therefore there exists a variational problem

$$L_t u(x,t) + L_x u(x,t) = 0, \quad x \in \Omega, t \in (0,T),$$

where $L_x : \Omega \to L^2(\Omega \times (0,T)) \times H^2((0,T))$ is a differential operator given by

$$L_x u(x,t) = -\sum_{|i|,|j|=1} a_{ij} \partial_x^j (\partial_x^i u(x,t)), \quad x \in \Omega, t \in (0,T),$$

where $a_{ij} \in C^{|j|}(\Omega)$ and $a_{ij} = a_{ji}$ for all $|i|, |j| \leq 1$, and ∂_x^i is a partial derivative of order $|i|$ with respect to x, similarly, $L_t : H^1(\Omega \times (0,T)) \to \Omega \times L^2((0,T))$ is an differential operator given by

$$L_t u(x,t) = \partial_t (\rho \partial_t u(x,t)) = \rho \partial_{tt} u(x,t), \quad x \in \Omega, t \in (0,T),$$

with $\rho \in \mathbb{R}$, which is a problem with the same boundary conditions of Problem (II-B), in consequence to solve this problem becomes to solve the Problem:

$$\begin{cases} L_t u(x,t) + L_x u(x,t) = 0, \quad x \in \Omega, \quad t \in (0,T), \\ u(x,0) = 0, \quad x \in \Omega, \\ \dfrac{\partial u}{\partial t}(x,0) = 0, \quad x \in \Omega, \\ u(x,t) = g(x,t), \quad x \in \Gamma, \quad t \in (0,T). \end{cases} \tag{10}$$

To solve the problem we follow two steps, first we solve the problem by discretizing with respect to time, i.e. the variable t, this means we take an uniform partition $\{t_0 = 0 < t_1 < \cdots < T_R = T\}$ of $[0,T]$, and second, for each $i = 1, \ldots, R$, we treat to find an unique solution by discretizing the position, depending on i, i.e. the variable x, this means we take a partition of Ω.

B. Discretizing in Time

Now, we consider an uniform partition $\{t_0 = 0 < t_1 < \cdots < T_R = T\}$ of $[0,T]$ of diameter $s = t_i - t_{i-1}$ for $i = 1,\ldots,R$. Let $i \in \{1,\ldots,R\}$ fixed, we are going to approximate the derivatives of the functions u by finite difference in each t_i. So, we have

$$u_t(x,t_i) \approx \frac{u(x,t_i+s)-u(x,t_i)}{s}.$$

Then, for each $i = 1,\ldots,R-1$, and each $x \in \Omega$ the differential operator L_t becomes

$$L_t u(x,t_i) = \frac{\rho\dfrac{u(x,t_i+2s)-u(x,t_i+s)}{s} - \rho\dfrac{u(x,t_i+s)-u(x,t_i)}{s}}{s},$$

which is equivalent to consider

$$L_t u(x,t_i) = \rho\frac{u(x,t_i+2s)-2u(x,t_i+s)+u(x,t_i)}{s^2}$$

Hence, to solve the problem (10) becomes to solve i variational problems, for $i,\ldots,R-1$,

$$\begin{cases} L_t u(x,t_i) + L_x u(x,t_i) = 0, \quad x\in\Omega, \\ u(x,0) = 0, \quad x\in\Omega, \\ u_t(x,0) = 0, \quad x\in\Omega, \\ u(x,t_i) = g(x,t_i), \quad x\in\Gamma. \end{cases}$$

C. Discretizing in Space

Now, for each $i = 1,\ldots,R-1$, let us consider the differential operator defined in $H^2(\Omega\times(0,T))$ by $(L_x^i u)(x) = L_x u(x,t_i)$ and the differential operator defined in $H^1(\Omega\times(0,T))$ by $(L_t^i u)(x) = L_t u(x,t_i)$. Moreover, we define

$$(u,v)_{x,i} = \sum_{|i|,|j|\le n} (a_{ij}(\cdot)\partial_x^i u(\cdot,t_i), \partial_x^j v(\cdot,t_i))_{0,\Omega} + \rho(\partial_t u(t_i,\cdot), \partial_t v(t_i,\cdot))_{0,\Omega}$$

and we assume that

$$\sum_{|i|,|j|=0} (\xi^i)^t a_{ij}\xi^j \ge 0, \quad \forall x\in\Omega, \tag{11}$$

and that there exists $\nu > 0$ such that

$$\sum_{|i|,|j|=1} (\xi^i)^t a_{ij}\xi^j \ge \nu\,\langle\xi\rangle_2^2, \quad \forall x\in\Omega, \tag{12}$$

for all $\xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3$, where $\xi^i = \xi_1^{i_1}\xi_2^{i_2}$, for any $i = (i_1,i_2,i_3) \in \mathbb{N}^3$.

Due to (12), the differential operator L_x^i is said to be strongly elliptic on Ω.

It can be shown that according to the hypotheses (11)-(12) the bilinear form $(\cdot,\cdot)_{x,i}$ defines a semi-inner product on $H^1(\Omega\times(0,T))$ whose associated semi-norm is denoted by $|u|_{x,i} = (u,u)_{x,i}^{\frac{1}{2}}$.

Suppose are given:

- an ordered set $A^r = \{a_1,\ldots,a_m\}$ of $m = m(r) \ge 0$ distinct points of Ω;
- an ordered set $B^N = \{b_1,\ldots,b_N\}$ of $N \in \mathbb{N}^*$ distinct points of Γ, none of which is a geometric vertex of $\overline{\Omega}$;
- a data vector $\beta = (\beta_1,\ldots,\beta_m) \in \mathbb{R}^m$;
- a finite dimensional space X_h made up over a partition T_h of Ω verifying that the length of $T_h \le h$ and $h \to 0$ as $dim(X_h) \to +\infty$.
- For each non-negative h, X_h will be a finite element space.

If Ω is not poligonal, we approximate Ω by Ω_h for each $h \in H$, in such way that $\lim_{h\to 0} \Omega\setminus\Omega_h = 0$ and we construct T_h over Ω_h.

We define the **operator** $\rho : H^1(\Omega) \to \mathbb{R}^m$, given by $\rho(v) = (v(a_j))_{1\le j\le m}$, the convex set $H_i^{Nh} = \{v \in X_h : v(b_j) = g(b_j,t_i), j = 1,\ldots,N\}$, the vectorial subspace $H_0^{Nh} = \{v \in X_h : v(b_j) = 0, j = 1,\ldots,N\}$ and $\tau v = (v(b_j))_{j=1,\ldots,N}$. Then, we suppose that

$$\ker\rho \cap \mathbb{P}_0(\Omega) = \{0\}.$$

Definition 1. *We say that σ_h^i is a discrete variational evolutionary PDE spline associated with L_x^i, B^N, A^r, β y $\epsilon > 0$, if σ_h^i is a solution of the problem*

$$\begin{cases} \sigma_h^i \in H_i^{Nh}, \\ \forall v \in H_i^{Nh}, \quad J_i(\sigma_h^i) \le J_i(v), \end{cases} \tag{13}$$

where J_i is the functional defined on $H^1(\Omega)$ by

$$J_i(v) = \langle\rho(v)-\beta\rangle_m^2 + \epsilon(|v|_{x,i}^2),$$

where ϵ is a non-negative real number.

The next result shows the uniqueness of Problem (13).

Theorem 1. *Problem (13) admits a unique solution which is also the unique solution of the variational problem: find $\sigma_h^i \in H_i^{Nh}$ such that*

$$\forall v \in H_0^{Nh}, \quad \left\langle\rho\sigma_h^i, \rho v\right\rangle_m + \epsilon(\sigma_h^i, v)_{x,i} = \langle\beta,\rho v\rangle_m.$$

See Appendix (B) for proof.

Theorem 2. *There exists a unique $(\sigma_h^i,\lambda) \in H_i^{Nh}\times\mathbb{R}^N$ such that*

$$\left\langle\rho\sigma_h^i, \rho v\right\rangle_m + \epsilon(\sigma_h^i, v)_{x,i} + \langle\tau v,\lambda\rangle_N = \langle\beta,\rho v\rangle_m, \tag{14}$$

for all $v \in H_i^{Nh}$, where σ_h^i is the unique solution of Problem (13).

See Appendix (C) for proof.

D. Discrete Solution in Space

We are now going to obtain for each $i = 1,\ldots,R$, the expression of the discrete variational spline σ_h^i.

We set h and we consider a partition T_h of rectangles of Ω, such that the points of B^N are knots of T_h. We number the basis functions of X_h by $\{w_1,\ldots,w_l\}$. We can then express σ_h^i as the linear combination $\sigma_h^i(x) = \sum_{j=1}^{I}\gamma_j w_j(x)$, and if we calculate the unknown coefficients γ_j, we then have the

expression of σ_h^i.
By substituting in (14), we obtain, for all $v \in H_i^{Nh}$,

$$\sum_{j=1}^{I} \gamma_j \left(\langle \rho w_j, \rho v \rangle_m + \epsilon (w_j, v)_{x,i} \right) + \langle \lambda, \tau v \rangle_N = \langle \beta, \rho v \rangle_m ,$$

subject to the constraints $\tau \left(\sum_{j=1}^{I} \gamma_j w_j \right) = y$, which are equivalent to

$$\begin{cases} \sum_{j=1}^{I} \gamma_j \left(\langle \rho w_j, \rho w_k \rangle_m + \epsilon (w_j, w_k)_{x,i} \right) + \langle \lambda, \tau w_k \rangle_N = \\ \langle \beta, \rho w_k \rangle_m , \quad 1 \le k \le I, \\ \sum_{j=1}^{I} \gamma_j \left(w_j(b_k) \right) = y_k, \quad 1 \le k \le N, \end{cases}$$

that is, a linear system with $I+N$ equations and the unknowns

$$\{\gamma_1, \ldots, \gamma_l, \lambda_1, \ldots, \lambda_N\}.$$

Its matrix form is

$$\begin{pmatrix} C & D \\ D^T & 0 \end{pmatrix} \begin{pmatrix} \gamma \\ \lambda \end{pmatrix} = \begin{pmatrix} \hat{f} \\ y \end{pmatrix},$$

where $D = (d_{jk})_{1 \le j \le I, 1 \le k \le N}$, with $d_{jk} = w_j(b_k)$,
$C = (c_{jk})_{1 \le j,k \le I}$, with $c_{jk} = \langle \rho w_j, \rho w_k \rangle_m + \epsilon (w_j, w_k)_{x,i}$,
$\gamma = (\gamma_1, \ldots, \gamma_I)^T$, $\lambda = (\lambda_1, \ldots, \lambda_N)^T$, $y = (y_1, \ldots, y_N)^T$,
$\hat{f} = \left(\langle \beta, \rho w_1 \rangle_m , \ldots, \langle \beta, \rho w_I \rangle_m \right)^T$.

If we call $A = (w_k(a_j))_{1 \le j \le m, 1 \le k \le I}$, $\hat{f} = A^T \beta$, and, $R = ((w_j, w_k)_{x,i})_{1 \le j,k \le I}$, then $C = A^T A + \epsilon R$.

We have obtained a discrete-time solution, for each value t_i, as a spline function of a finite dimensional space. Therefore, the overall solution is a set of surfaces in $\mathbb{R}^3$, one for each discrete value of time.

E. Convergence

Suppose that

$$\sup_{x \in \Omega} \min_{a \in A^r} \langle x - a \rangle_2 = o\left(\frac{1}{r} \right), \qquad \text{as } r \to +\infty. \qquad (15)$$

Let $\Delta = \frac{(n+2)!}{n!2!}$ be the dimension of $\mathbb{P}_n$, with $\mathbb{P}_n$ designs the space of polynomial functions of total degree n defined in Ω. Then we have the following useful result.

Theorem 3. *Suppose that $\Omega \subset \mathbb{R}^3$ is an open set with Lipschitz-continuous boundary. Let $\Delta_0 = \{a_{01}, \ldots, a_{0\Delta}\}$ be a $\mathbb{P}_{n-1}$-unisolvent set of points of $\overline{\Omega}$, with $n \ge 1$. Then, there exists $\eta > 0$ such that, if T_n denotes the set of Δ-uplas $G = \{a_1, \ldots, a_\Delta\}$ of points of $\overline{\Omega}$ that verify the condition*

$$\forall j = 1, \ldots, \Delta, \quad \langle a_j - a_{0j} \rangle_2 \le \eta,$$

the application

$$[[v]]_n^{\Delta_0} = \left(\sum_{j=1}^{\Delta} |v(a_j)|^2 + |v|_{x,i}^2 \right)^{\frac{1}{2}},$$

defined for all $\Delta_0 \in T_\eta$ is a norm over $H^n(\Omega)$ uniformly equivalent over T_η to the usual norm $\| \cdot \|_n$.

See Proposition 2.1 of [18] for proof.

Corollary 1. *Suppose that the hypothesis for convergence (15) holds and $n > 1$. Then, there exists $\eta > 0$ and, for all $r \in \mathbb{N}$, a subset A_0^r of A^r such that, for all $r \ge \eta$, the application $[[\cdot]]_r^0$, defined by*

$$[[v]]_r^0 = \left(\sum_{a \in A_0^r} |v(a)|^2 + |v|_{x,i}^2 \right)^{\frac{1}{2}},$$

is a norm of $H^n(\Omega)$ uniformly equivalent with respect to r, to the norm $\| \cdot \|_n$.

See Appendix (D) for proof.

Theorem 4. *We suppose that (15) holds and*

$$\epsilon = o(r^p), \qquad \text{as } r \to +\infty.$$

Then, one has

$$\lim_{r \to +\infty} \| \sigma_h^i - u_i \|_n = 0.$$

See Appendix (E) for proof.

IV. Numerical Results and Discussion

A. Considerations in the Numerical Simulation

For the numerical simulation we considered an arbitrary individual which after the clinical analysis by system H resulted with a TMD (see [3]).

The level design of the mandibular model [19] was conducted by acquiring CT scans and its subsequent reconstruction using curves and surfaces as can be seen in [20]-[21] (Fig. 1). We considered a subject with 14 teeth instead of 16, because the third molars do not usually sprout in some people and many times these teeth do unstable the analysis of temporomandibular disorders.

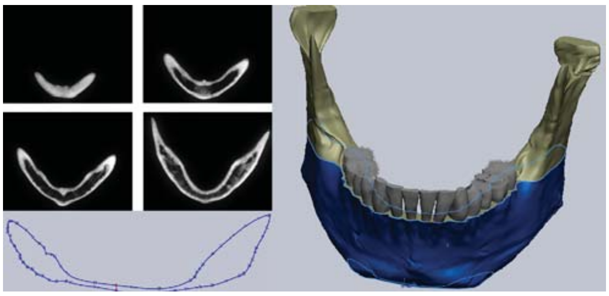

Fig. 1 Acquisition of CT scans and mandibular reconstruction

The system H processed the registration information and determined through the membership function of a fuzzy set related to the eleven factors of the Factorial Analysis applied to the database sample and having a dental detailed interpretation (see [3]). For modeling the biomechanics of the jaw: the involved muscles, the physical properties and the fuzzy components over the defined forces on muscles we used MATLAB. To present the necessary conditions for the

TABLE II
MANDIBULAR SURFACES AND ASSOCIATED TMD FACTORS

NR	Mandibular section	Lat	Pos	Affected factor(s)
2	Mandibular body			$F_{3,11}$
3	Ramus	L		F_2
4	Condyle	L		$F_{1-4,6-7,10}$
5	Apophysis	L		F_9
6	Ramus	R		F_2
8	Condyle	R		$F_{1-4,6-7,10}$
10	Apophysis	R		F_9
11	Central incisor	R	Lw	$F_{5,9}$
12	Central incisor	L	Lw	$F_{5,9}$
13	Lateral incisor	L	Lw	$F_{5,9}$
15	Lateral incisor	R	Lw	$F_{5,9}$
16	Canine	R	Lw	$F_{2,9}$
17	Canine	L	Lw	$F_{2,9}$
22	First premolar	R	Lw	F_9
24	First premolar	L	Lw	F_9
25	Second premolar	L	Lw	F_9
27	Second premolar	R	Lw	F_9
29	First molar	L	Lw	F_9
30	Mesial lingual cusp (*)	L	Lw	F_9
33	Vestibular cusp (*)	L	Lw	F_9
34	Mesial buccal cusp (*)	L	Lw	F_9
35	Disto vestibular cusp (*)	L	Lw	F_9
36	Disto lingual cusp (*)	L	Lw	F_9
37	First molar	R	Lw	F_9
38	Second molar	L	Lw	F_{8-9}
40	Second molar	R	Lw	F_{8-9}

implementation of the variational formulation of the problem and its resolution by the Finite Method Element, we did a connection between MATLAB and FreeFem++.

The original model developed included 102821 vertices, 2352 edges, 554752 tetrahedra and 40368 triangles, however it should be noted that for the numerical simulation of this work and its further analysis only 2095 vertices, 588 edges, 2523 triangles and 8668 tetrahedra were used instead.

Consequently, factors numbered as 3, 4, 5, 7, 8, 9 from the eleven of the performed Factorial Analysis as shown in [2], were able to provide numerical patterns according to their degree of membership in order to generate of forces f, and, g, needed for implementing the dynamic model in FreeFem++. To simulate the impact it was assumed a density of strength in the interval $[0, T]$, exponentially growing, g given by

$$g(x,t) = \begin{cases} \mu_j e^{2\left(\sum_{\substack{i=1\\ i\neq j}}^{11} \mu_i\right)(t-t_0)} g_0, & \text{if } 0 \leq t \leq t_0 \\ 0, & \text{if } t > t_0 \end{cases}$$

which is a function of compact support where $g_0 \in \mathbb{R}^3$ is the maximum intensity of the impact at the instant $t_0 > 0$, where $j \in \{j : \mu_j = \max M\}$, with $M = \{\mu_i : i \in \{1, \ldots, 11\}\}$, such that $\mu_i \in [0, 1]$ is the degree of membership of their associated factor, where for each x, we assigned a fuzzy set X related to each of the boundary zones according to its affected factor (see Table II where NR is the cardinality, Lat is laterality: left and right, Pos is position: upper and lower, and with * as first molar).

We consider for the simulation $\|g_0\| = 1000\text{N/cm}^2$, with $t_0 = 1$ and for the calculus, $T = 1$ and $\Delta t = 0.01$.

B. Stress and Deformation Analysis: Interpretation of Results

To facilitate the visualization of the results we use FreeFem++ where colorbar values in the figures correspond to the intensity of the incidence of the stresses and displacements on jaw movement of a patient with TMD. This analysis makes it clear that every deformation of the jaw is not serious because it is difficult to fracture by a simple clenching even in patients with TMD (Fig. 2).

Fig. 2 Jaw movement of a patient with TMD. (Overlapping)

Fig. 3 shows the jaw-closing movements during chewing where we can appreciate the small displacement of the mandibular condyle (blue color) and the greatest displacement of the anterior mandibular ramus and coronoid apophysis. Thus, patients with TMD tend to experience pain, limited movement, or asymmetric jaw and temporomandibular joint sounds concentrated on the chin which can be seen in the color bar of the plot (red color). Some incisors might be affected in particularly bad cases (see [22]). Fig. 6 shows the stress distribution during free opening and closing, and during chewing of a patient with TMD. Only the area near the Spix's spine, in the condyle branch can suffer stress as well as affecting the muscles and the TMD (see [23]).

Fig. 8 shows that the most affected area corresponds to the left condyle, therefore is concluded that there is a relationship between TMD and occlusal factors. There can also be corroborating evidence related to a certain source of system H; therefore, we should be interested not only in the Fricton Index in order to obtain more complete results.

Fig. 4 shows that both condyles are affected with high levels of heterogeneity. These results were very similar to those obtained by Korioth [24] from stress distribution along the condyle which becomes more intense, especially during occlusion (see Fig. 5).

In the verification process the calculation results obtained were compared with those from the research of Korioth

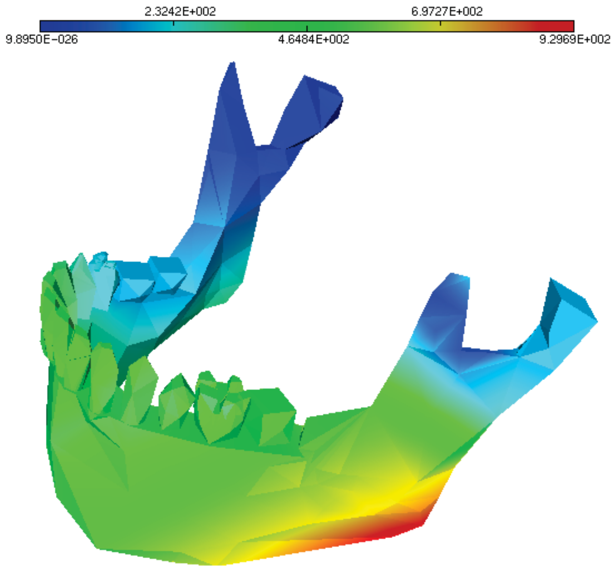

Fig. 3 Analysis of Displacements

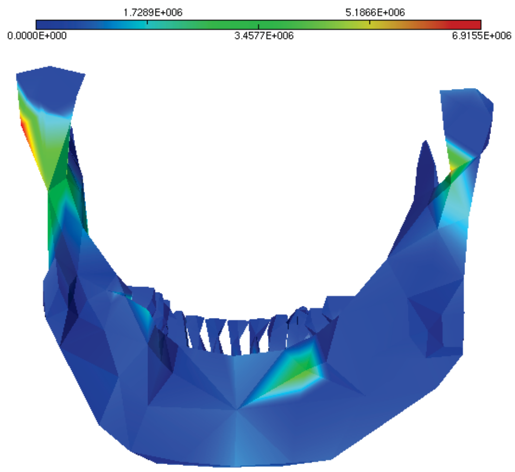

Fig. 4 Condylar affected branches

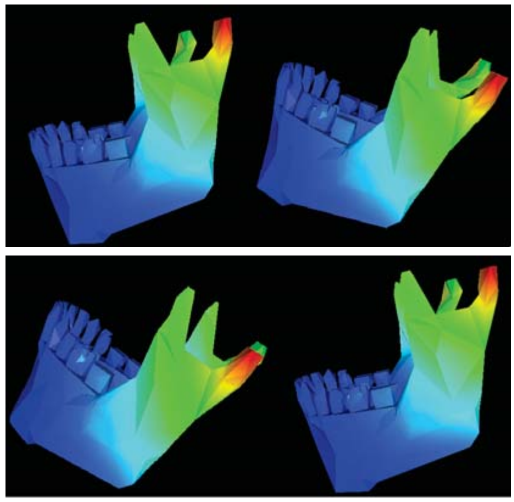

Fig. 5 Variation of stress in the condyles in different time instants

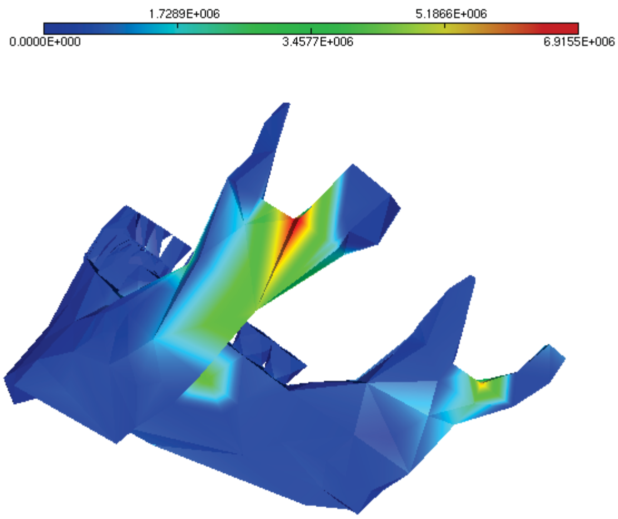

Fig. 6 Affected zones in the area of the molars, left and right condyles

(see [24]). The calculations and measured results were compared for validation. The comparisons were made using conditions where contact occurs as well as loading patterns (10^8 Pa) and jaw movements during occlusion (see [25]). The simulation results of the maximum value of Von Mises stress on the opposite condyle occlusion are shown when we applied a load of 51.51×10^6 Pa at ICP. When we applied a lingual and distal load, the maximum value of Von Mises on the opposite condyle increases (72.14×10^6 Pa and 69.57×10^6 Pa respectively).

Some occlusal conditions related to tiny eruption of the third molar or second molar buccal crossbite has committed to publish clinical study reports rather than by simulation to prevent erroneous data [26].

Our simulation results show that the highest stress occurs at the condyle (69.16×10^6 Pa), similar to those obtained by Korioth (see [27]) whose value was 69.57×10^6 Pa (see Figs. 6-8).

This study also emphasizes the presence of a different variable that would involve the incisors, which should be isolated for further analysis that according to the existing dental literature is not considered and may also contribute to the development of TMD as shown in Fig. 8.

An acceptable compatibility of the results proves that the model can be applied with a TMD patient in practice.

V. Conclusion

In this paper, we have been developed a different method for the approximation of surfaces as well as the resolution of a boundary value problems for the detection of temporomandibular disorders. We conclude that our results

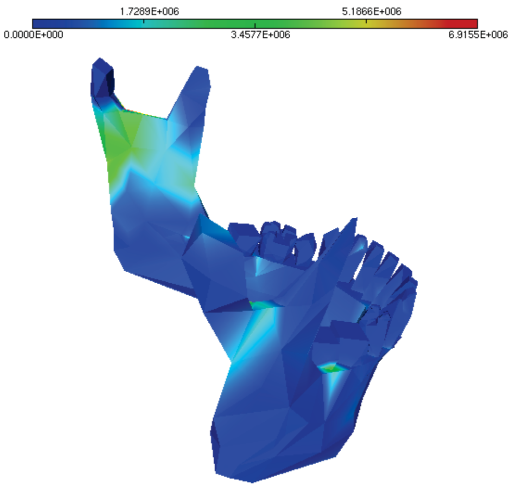

Fig. 7 Condylar branches affected with heterogeneous values

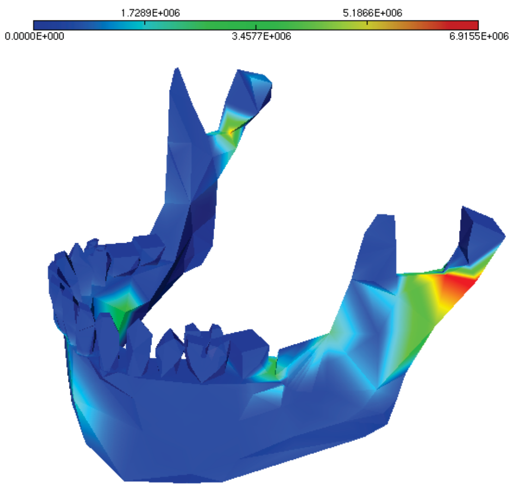

Fig. 8 Incisors as variables not considered in TMD

match with those expected by experts in previous studies. Hence, we consider the presented variational method as a valid tool to solving many PDEs.

Appendix

A. Proof of the Method (Final Part)

:

$$u \overset{\partial_x^{(1,0,0)}}{\longrightarrow} \begin{pmatrix} \frac{\partial u_1}{\partial x_1}(x,t) \\ \frac{\partial u_2}{\partial x_1}(x,t) \\ \frac{\partial u_3}{\partial x_1}(x,t) \end{pmatrix} \overset{\partial_x^{(0,1,0)}}{\longrightarrow} \begin{pmatrix} \frac{\partial^2 u_1}{\partial x_1 \partial x_2}(x,t) \\ \frac{\partial^2 u_2}{\partial x_1 \partial x_2}(x,t) \\ \frac{\partial^2 u_3}{\partial x_1 \partial x_2}(x,t) \end{pmatrix},$$

with a factor

$$a_{(1,0,0),(0,1,0)} = \begin{pmatrix} 0 & \frac{1}{2}(\lambda+\mu) & 0 \\ \frac{1}{2}(\lambda+\mu) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the result

$$\begin{pmatrix} \frac{1}{2}(\lambda+\mu)\frac{\partial^2 u_2}{\partial x_1 \partial x_2}(x,t) \\ \frac{1}{2}(\lambda+\mu)\frac{\partial^2 u_1}{\partial x_1 \partial x_2}(x,t) \\ 0 \end{pmatrix},$$

and denoted by

$$a_{(1,0,0),(0,1,0)}\partial_x^{(0,1,0)}(\partial_x^{(1,0,0)}u(x,t)). \tag{16}$$

the same with

$$u \overset{\partial_x^{(1,0,0)}}{\longrightarrow} \begin{pmatrix} \frac{\partial u_1}{\partial x_1}(x,t) \\ \frac{\partial u_2}{\partial x_1}(x,t) \\ \frac{\partial u_3}{\partial x_1}(x,t) \end{pmatrix} \overset{\partial_x^{(0,0,1)}}{\longrightarrow} \begin{pmatrix} \frac{\partial^2 u_1}{\partial x_1 \partial x_3}(x,t) \\ \frac{\partial^2 u_2}{\partial x_1 \partial x_3}(x,t) \\ \frac{\partial^2 u_3}{\partial x_1 \partial x_3}(x,t) \end{pmatrix},$$

with

$$a_{(1,0,0),(0,0,1)} = \begin{pmatrix} 0 & 0 & \frac{1}{2}(\lambda+\mu) \\ 0 & 0 & 0 \\ \frac{1}{2}(\lambda+\mu) & 0 & 0 \end{pmatrix}$$

having

$$\begin{pmatrix} \frac{1}{2}(\lambda+\mu)\frac{\partial^2 u_2}{\partial x_1 \partial x_2}(x,t) \\ 0 \\ \frac{1}{2}(\lambda+\mu)\frac{\partial^2 u_1}{\partial x_1 \partial x_2}(x,t) \end{pmatrix}$$

and denoted by

$$a_{(1,0,0),(0,0,1)}\partial_x^{(0,0,1)}(\partial_x^{(1,0,0)}u(x,t)). \tag{17}$$

and the same for

$$u \overset{\partial_x^{(0,1,0)}}{\longrightarrow} \begin{pmatrix} \frac{\partial u_1}{\partial x_2}(x,t) \\ \frac{\partial u_2}{\partial x_2}(x,t) \\ \frac{\partial u_3}{\partial x_2}(x,t) \end{pmatrix} \overset{\partial_x^{(0,0,1)}}{\longrightarrow} \begin{pmatrix} \frac{\partial^2 u_1}{\partial x_2 \partial x_3}(x,t) \\ \frac{\partial^2 u_2}{\partial x_2 \partial x_3}(x,t) \\ \frac{\partial^2 u_3}{\partial x_2 \partial x_3}(x,t) \end{pmatrix},$$

with

$$a_{(0,1,0),(0,0,1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(\lambda+\mu) \\ 0 & \frac{1}{2}(\lambda+\mu) & 0 \end{pmatrix},$$

obtaining

$$\begin{pmatrix} 0 \\ \frac{1}{2}(\lambda+\mu)\frac{\partial^2 u_3}{\partial x_2 \partial x_3}(x,t) \\ \frac{1}{2}(\lambda+\mu)\frac{\partial^2 u_2}{\partial x_2 \partial x_3}(x,t) \end{pmatrix}$$

and denoted by

$$a_{(0,1,0),(0,0,1)}\partial_x^{(0,0,1)}(\partial_x^{(0,1,0)}u(x,t)). \quad (18)$$

Hence, the following results are shown:

$$u \overset{\partial_x^{(0,1,0)}}{\longrightarrow} \begin{pmatrix} \frac{\partial u_1}{\partial x_2}(x,t) \\ \frac{\partial u_2}{\partial x_2}(x,t) \\ \frac{\partial u_3}{\partial x_2}(x,t) \end{pmatrix} \overset{\partial_x^{(1,0,0)}}{\longrightarrow} \begin{pmatrix} \frac{\partial^2 u_1}{\partial x_1 \partial x_2}(x,t) \\ \frac{\partial^2 u_2}{\partial x_1 \partial x_2}(x,t) \\ \frac{\partial^2 u_3}{\partial x_1 \partial x_2}(x,t) \end{pmatrix},$$

consequently by multiplying the last vector for a matrix of the form

$$a_{(0,1,0),(1,0,0)} = \begin{pmatrix} 0 & \frac{1}{2}(\lambda+\mu) & 0 \\ \frac{1}{2}(\lambda+\mu) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we have

$$\begin{pmatrix} \frac{1}{2}(\lambda+\mu)\frac{\partial^2 u_2}{\partial x_1 \partial x_2}(x,t) \\ \frac{1}{2}(\lambda+\mu)\frac{\partial^2 u_1}{\partial x_1 \partial x_2}(x,t) \\ 0 \end{pmatrix}$$

which can be denoted by

$$a_{(0,1,0),(1,0,0)}\partial_x^{(1,0,0)}(\partial_x^{(0,1,0)}u(x,t)). \quad (19)$$

The process is complemented by

$$u \overset{\partial_x^{(0,0,1)}}{\longrightarrow} \begin{pmatrix} \frac{\partial u_1}{\partial x_3}(x,t) \\ \frac{\partial u_2}{\partial x_3}(x,t) \\ \frac{\partial u_3}{\partial x_3}(x,t) \end{pmatrix} \overset{\partial_x^{(1,0,0)}}{\longrightarrow} \begin{pmatrix} \frac{\partial^2 u_1}{\partial x_1 \partial x_3}(x,t) \\ \frac{\partial^2 u_2}{\partial x_1 \partial x_3}(x,t) \\ \frac{\partial^2 u_3}{\partial x_1 \partial x_3}(x,t) \end{pmatrix},$$

and by multiplying the vector for a matrix of the form

$$a_{(0,0,1),(1,0,0)} = \begin{pmatrix} 0 & 0 & \frac{1}{2}(\lambda+\mu) \\ 0 & 0 & 0 \\ \frac{1}{2}(\lambda+\mu) & 0 & 0 \end{pmatrix}$$

we have

$$\begin{pmatrix} \frac{1}{2}(\lambda+\mu)\frac{\partial^2 u_2}{\partial x_1 \partial x_2}(x,t) \\ 0 \\ \frac{1}{2}(\lambda+\mu)\frac{\partial^2 u_1}{\partial x_1 \partial x_2}(x,t) \end{pmatrix}$$

which can be denoted by

$$a_{(0,0,1),(1,0,0)}\partial_x^{(1,0,0)}(\partial_x^{(0,0,1)}u(x,t)). \quad (20)$$

Finally,

$$u \overset{\partial_x^{(0,0,1)}}{\longrightarrow} \begin{pmatrix} \frac{\partial u_1}{\partial x_3}(x,t) \\ \frac{\partial u_2}{\partial x_3}(x,t) \\ \frac{\partial u_3}{\partial x_3}(x,t) \end{pmatrix} \overset{\partial_x^{(0,1,0)}}{\longrightarrow} \begin{pmatrix} \frac{\partial^2 u_1}{\partial x_2 \partial x_3}(x,t) \\ \frac{\partial^2 u_2}{\partial x_2 \partial x_3}(x,t) \\ \frac{\partial^2 u_3}{\partial x_2 \partial x_3}(x,t) \end{pmatrix},$$

and by multiplying the vector for a matrix of the form

$$a_{(0,0,1),(0,1,0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(\lambda+\mu) \\ 0 & \frac{1}{2}(\lambda+\mu) & 0 \end{pmatrix}$$

we have

$$\begin{pmatrix} 0 \\ \frac{1}{2}(\lambda+\mu)\frac{\partial^2 u_3}{\partial x_2 \partial x_3}(x,t) \\ \frac{1}{2}(\lambda+\mu)\frac{\partial^2 u_2}{\partial x_2 \partial x_3}(x,t) \end{pmatrix}$$

which can be denoted by

$$a_{(0,0,1),(0,1,0)}\partial_x^{(0,1,0)}(\partial_x^{(0,0,1)}u(x,t)). \quad (21)$$

■

B. Proof of Theorem 1

: We consider the application $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ given by

$$a(u,v) = 2(\langle \rho u, \rho v \rangle_m + \epsilon(u,v)_{x,i}).$$

The form $a(\cdot,\cdot)$ is bilinear and symmetric in $H^1(\Omega)$. From (11) and (12) we have that a is coercive [28] and its continuity is deduced from the continuity of ρ and $(\cdot,\cdot)_{x,i}$. Let $\varphi(v) = 2(\langle \beta, \rho v \rangle)$, which is clearly linear and continuous in $H^1(\Omega)$. We conclude that there exists a unique $\sigma \in H_i^{Nh}$ such that $a(\sigma, w-\sigma) \geq \varphi(w-\sigma)$, for all $w \in H_i^{Nh}$, which implies that $a(\sigma,v) \geq \varphi(v)$ for all $v \in H_0^{Nh}$. As H_0^{Nh} is a vectorial subspace, then if $v \in H_0^{Nh}$ hence $-v \in H_0^{Nh}$, and it follows that $a(\sigma,-v) \geq \varphi(-v)$, for any $v \in H_0^{Nh}$. We obtain that $a(\sigma,v) = \varphi(v)$ for any $v \in H_0^{Nh}$. Furthermore, σ is the minimum in H_i^{Nh} of the functional $\Phi(v) = \frac{1}{2}a(v,v) - \varphi(v)$, which is the minimum of J_i, since $\Phi(v) = J_i(v) - \langle \beta \rangle_m^2$. Hence we conclude the result. ■

C. Proof of Theorem 2

: Let us denote by $\{w_1, \ldots, w_N\}$ as the basis functions of X_h associated with the degree of freedom $\{v(b_j)\}_{j=1,\ldots,N}$. For each $v \in H_i^{Nh}$, let $w = v - \sum_{j=1}^{Nh} v(b_j)w_j$. Then, $w \in X_h$ and for each $k = 1,\ldots,N$, $\phi_k(w) = \phi_k(v) - \sum_{j=1}^{N} v(b_j)w_j(b_k) = 0$, so $\tau w = 0$ and consequently, $w \in H_0^{Nh}$.

Let σ_h^i be the solution (13). Then, by Theorem 1, we have $\sigma_h^i \in H_i^{Nh}$ and

$$\left\langle \rho\sigma_h^i, \rho w\right\rangle_m + \epsilon(\sigma_h, w)_{x,i} = \langle \beta, \rho w\rangle_m\,,$$

by substituting and by linearity, we obtain

$$\langle \beta, \rho v\rangle_m = \left\langle \rho\sigma_h^i, \rho v\right\rangle_m + \epsilon(\sigma_h^i, v)_{x,i} + \sum_{j=1}^{N}\left(\left\langle \beta - \rho\sigma_h^i, \rho w_j\right\rangle_m - \epsilon(\sigma_h^i, w_j)_{x,i}\right) v(b_j).$$

If we denote $\lambda = \left(\left\langle \beta - \rho\sigma_h^i, \rho w_j\right\rangle_m - \epsilon(\sigma_h^i, w_j)_{x,i}\right)_{j=1,\dots,N}$, then we conclude that

$$\left\langle \rho\sigma_h^i, \rho v\right\rangle_m + \epsilon(\sigma_h^i, v)_{x,i} + \langle \lambda, \tau v\rangle_N = \langle \beta, \rho v\rangle_m\,,$$

and (14) is verified. Now, we suppose that there exists $\lambda, \overline{\lambda} \in \mathbb{R}^N$ such that (σ_h^i, λ) and $(\sigma_h^i, \overline{\lambda})$ verify (14). Then

$$\begin{aligned}\left\langle \rho\sigma_h^i, \rho v\right\rangle_m + \epsilon(\sigma_h^i, v)_{x,i} + \langle \lambda, \tau v\rangle_N &= \langle \beta, \rho v\rangle_m\,,\\ \left\langle \rho\sigma_h^i, \rho v\right\rangle_m + \epsilon(\sigma_h^i, v)_{x,i} + \left\langle \overline{\lambda}, \tau v\right\rangle_N &= \langle \beta, \rho v\rangle_m\,,\end{aligned}$$

and, by substracting, we have $\left\langle \lambda - \overline{\lambda}, \tau v\right\rangle_N = 0$, $\forall v \in H_i^{Nh}$, from which we derive $\lambda = \overline{\lambda}$ and, hence, the uniqueness of (σ_h^i, λ). ■

D. Proof of Theorem 1

: Let $\Delta_0 = \{a_{01}, \dots, a_{0\Delta}\}$ be anyone $\mathbb{P}_{n-1}$-unisolvent subset of $\overline{\Omega}$. From (15) for all $r \in \mathbb{N}$ an all $j = 1, \dots, \Delta$, there exists $a_j^r \in A^r$ verifying

$$\left\langle a_j - a_j^r\right\rangle_2 \le \frac{1}{r}.$$

Let $A_0^r = \{a_1^r, \dots, a_\Delta^r\}$. Then, it is sufficient to apply Proposition 3, taking into account that for all $r \ge \eta$, $A_0^r \in T_\eta$, written $[[\cdot]]_r^0$ instead of $[[\cdot]]_n^{A_0^r}$.

We suppose that $\epsilon = \epsilon(r)$. Let $u_i = u(x, t_i)$, $x \in \Omega$ the displacement function in time $t = t_i$. Clearly $u_i \in H_i^{Nh}$ and we denote by σ_h^i the evolutionary discrete variational spline associated to $L_x^i, B^N, A^r, \rho g$ y ϵ. ■

E. Proof of Theorem 4

: We know that $J_i(\sigma_h^i) \le J_i(u_i)$. This implies that

$$\left\langle \rho(\sigma_h^i - u_i)\right\rangle_m^2 + \epsilon(|\sigma_h^i|_{x,i}^2) \le \epsilon(|u_i|_{x,i}^2). \tag{22}$$

Thus, we obtain

$$|\sigma_h^i|_{x,i}^2 \le |u_i|_{x,i}^2. \tag{23}$$

Let $\tilde{J}$ be the functional defined above and let $\tilde{\sigma}$ be the minimum of $\tilde{J}$ in H_i^{Nh}. Since $\tilde{\sigma} - u_i \in H_0$, we have that

$$(\tilde{\sigma}, \sigma_h^i - u_i)_{x,i} = 0. \tag{24}$$

By adding $(\tilde{\sigma}, \tilde{\sigma})_{x,i}$ in both terms of (23) and by using (24), we deduce that

$$|\sigma_h^i|_{x,i}^2 + (\tilde{\sigma}, \tilde{\sigma})_{x,i} \le |u_i|_{x,i}^2 + 2(\tilde{\sigma}, \sigma_h^i - u_i)_{x,i} + (\tilde{\sigma}, \tilde{\sigma})_{x,i}$$

and we obtain $|\sigma_h^i - \tilde{\sigma}|_{x,i}^2 \le |u_i - \tilde{\sigma}|_{x,i}^2$. Hence, we conclude that

$$|\sigma_h^i|_{x,i} \le 2|u_i - \tilde{\sigma}|_{x,i} + 2|\tilde{\sigma}|_{x,i}. \tag{25}$$

In the same way we obtain from (22)

$$\left\langle \rho(\sigma_h^i - u_i)\right\rangle_m^2 \le \epsilon(|u_i|_{x,i}^2)$$

or equivalently

$$\left\langle \rho(\sigma_h^i - u_i)\right\rangle_m^2 \le \epsilon(|v|_{x,i}^2)$$

and again using (24) we obtain

$$\left\langle \rho(\sigma_h^i - u_i)\right\rangle_m^2 \le \epsilon(|u_i|_{x,i}^2 + 2(\tilde{\sigma}, \sigma_h^i - u_i)_{x,i})$$

or

$$\left\langle \rho(\sigma_h^i - u_i)\right\rangle_m^2 \le \epsilon(|u_i|_{x,i}^2 - 2(\tilde{\sigma}, u_i)_{x,i} + 2(\tilde{\sigma}, \sigma_h^i)_{x,i}). \tag{26}$$

We know that

$$2(\tilde{\sigma}, \sigma_h^i)_{x,i} \le |\tilde{\sigma}|_{x,i}^2 + |\sigma_h^i|_{x,i}^2.$$

Then, by (26) we deduce

$$\left\langle \rho(\sigma_h^i - u_i)\right\rangle_m^2 \le \epsilon(|u_i|_{x,i}^2 - 2(\tilde{\sigma}, u_i)_{x,i} + |\tilde{\sigma}|_{x,i}^2 + |\sigma_h^i|_{x,i}^2)$$

but the last term in the right side of the inequality is bounded by (25) and u_i and $\tilde{\sigma}$ are fixed functions. This implies

$$\left\langle \rho(\sigma_h^i - u_i)\right\rangle_m^2 = O(\epsilon). \tag{27}$$

Let $\Delta_0 = \{a_{01}, \dots, a_{0\Delta}\}$ a $\mathbb{P}_{n-1}$-unisolvent subset of points of Ω and let η be the constant of Proposition 3. Obviously, there exists $\eta' \in (0, \eta]$ such that

$$\forall j = 1, \dots, \Delta, \quad \overline{B}(a_{0j}, \eta') \subset \overline{\Omega}.$$

From (15), we have that there exists $C > 0$ such that

$$\forall r \in \mathbb{N}, r > \frac{C}{\eta'}, \quad \forall j = 1, \dots, \Delta,$$

$$\overline{B}\left(a_{0j}, \eta' - \frac{C}{r}\right) \subset \bigcup_{a \in A^r \cap \overline{B}(t_{0j}, \eta')} \overline{B}\left(a, \frac{C}{r}\right).$$

If $\mathcal{N}_j = card(A^r \cap \overline{B}(a_{0j}, \eta'))$, it follows $\forall r \in \mathbb{N}$, $r > \dfrac{C}{\eta'}, \forall j = 1, \dots, \Delta$, $\exists C_1 > 0$, $\left(\eta' - \dfrac{C}{r}\right)^p \le \dfrac{C_1 \mathcal{N}_j}{r^2}$ and consequently, for any $r_0 > \dfrac{C}{\eta'}$ we have

$$\forall r \in \mathbb{N}, \forall j = 1, \dots, \Delta, \exists C_2 > 0, \quad \mathcal{N}_j \ge C_2 \left(\eta' - \frac{C}{r_0}\right)^{2r^2}. \tag{28}$$

Meanwhile, from (15) and (27) we deduce that $\forall j = 1, \dots, \Delta$

$$\sum_{a \in A^r \cap \overline{B}(t_{0j}, \eta')} |(\sigma_h^i - u_i)(a)|^2 = o(r^2), r \to +\infty. \tag{29}$$

If $a_j^r \in A^r \cap \overline{B}(a_{0j}, \eta')$ such that

$$|(\sigma_h^i - u_i)(a_j^r)| = \min_{a \in A^r \cap \overline{B}(t_{0j}, \eta')} |(\sigma_h^i - u_i)(a)|, \quad r \to +\infty$$

we deduce from (28) and (29) that

$$\forall j = 1, \dots, \Delta, \quad |(\sigma_h^i - u_i)(a)| = o(1), r \to +\infty. \tag{30}$$

Now, we denote by Δ^r the set $\{a_1^r, \dots, a_\Delta^r\}$. By applying Proposition 3 to $\Delta = \Delta^r$ for r sufficiently close to infinity, it turns out from (25), (30) and Corollary 1 that

$$\exists C > 0, \exists \gamma > 0, \forall r \in \mathbb{N}, \quad r \ge \gamma, \quad \|\sigma_h^i\|_n \le C,$$

which means that the family (σ_h^i) is bounded in $H^n(\Omega)$. Therefore, there exists a subsequence $(\sigma_{\epsilon_l}^{r_l})_{l\in\mathbb{N}}$ extracted from this family, with $\epsilon = \epsilon(r_l)$, and an element u_i^* of $H^n(\Omega)$ such that

$$u_i^* = \lim_{l\to+\infty} \sigma_{\epsilon_l}^{r_l} \quad \text{weakly in } H^n(\Omega).$$

■

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