

# $(\lambda, \mu)$ -fuzzy Subrings and $(\lambda, \mu)$ -fuzzy Quotient Subrings with Operators

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**Abstract**—In this paper, we extend the fuzzy subrings with operators to the  $(\lambda, \mu)$ -fuzzy subrings with operators. And the concepts of the  $(\lambda, \mu)$ -fuzzy subring with operators and  $(\lambda, \mu)$ -fuzzy quotient ring with operators are given, while their elementary properties are discussed.

**Keywords**—Fuzzy subring with operators,  $(\lambda, \mu)$ -fuzzy subring with operators,  $(\lambda, \mu)$ -fuzzy quotient ring with operators.

## I. INTRODUCTION

SINCE the concept of the fuzzy set appeared, many scholars have applied it to the ring and obtained many fuzzy theories about the ring. In 1982, Liu [1] first raised the fuzzy subring. After that, [2] and [3] discussed fuzzy quotient ring. Reference [4] proposed the notion of fuzzy subrings and fuzzy quotient ring with operators. Reference [5] defined  $(\lambda, \mu)$ -fuzzy subrings. Besides, [6] gave  $(\lambda, \mu)$ -intuitionistic fuzzy subgroups with operators. In this paper, we further develop the fuzzy ring theory and give the definition of  $(\lambda, \mu)$ -fuzzy subring with operators and  $(\lambda, \mu)$ -fuzzy quotient ring with operators, while some elementary properties are discussed.

## II. PRELIMINARIES

In this paper, we always assume  $0 \leq \lambda < \mu \leq 1$ .

**Definition 1.** [1] Let  $A$  be a fuzzy subset of ring  $R$ . Then  $A$  is called a fuzzy subring of  $R$  if for all  $x, y \in R$ ,

1.  $A(x-y) \geq A(x) \wedge A(y)$ ;
2.  $A(xy) \geq A(x) \wedge A(y)$ .

**Definition 2.** [4] Let  $A$  be a fuzzy subring of  $M$ -ring  $R$ . Then  $A$  is called a  $M$ -fuzzy subring of  $R$  if for all  $x, y \in R, m \in M, A(mx) \geq A(x)$ .

**Definition 3.** [5] Let  $A$  be a fuzzy subset of ring  $R$ . Then  $A$  is called a  $(\lambda, \mu)$ -fuzzy subring of  $R$  if for all  $x, y \in R$ ,

1.  $A(x+y) \vee \lambda \geq (A(x) \wedge A(y)) \wedge \mu$ ;
2.  $A(-x) \vee \lambda \geq A(x) \wedge \mu$ ;
3.  $A(xy) \vee \lambda \geq (A(x) \wedge A(y)) \wedge \mu$ .

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**Definition 4.** [7] A subring  $R$  of  $M$ -ring is said to be an  $M$ -subring if for all  $\lambda \in M, a, b \in R$ ,

1.  $\lambda(a+b) = \lambda a + \lambda b$ ;
2.  $\lambda(ab) = (\lambda a)b$ .

**Definition 5.** [7] Let  $f: R \rightarrow R'$  be a homomorphism of  $M$ -rings. Then  $f$  is called a  $M$ -homomorphism if for all  $x \in R, m \in M, f(mx) = mf(x)$ .

**Proposition 1.** [5] Let  $A$  be a fuzzy subset of  $R$ . Then  $A$  is a  $(\lambda, \mu)$ -fuzzy subring of  $R$  iff for all  $x, y \in R$ ,

1.  $A(x-y) \vee \lambda \geq (A(x) \wedge A(y)) \wedge \mu$ ;
2.  $A(xy) \vee \lambda \geq (A(x) \wedge A(y)) \wedge \mu$ .

**Proposition 2.** [4] Let  $S$  be a nonempty subset of  $M$ -ring  $R$ . If  $I_S$  is the characteristic function then  $S$  is an  $M$ -subring of  $R$  iff  $I_S$  is an  $M$ -fuzzy subring of  $R$ .

**Proposition 3.** [5] Let  $A$  be a fuzzy subset of  $R$ . Then  $A$  is a  $(\lambda, \mu)$ -fuzzy subring of  $R$  iff for every  $\alpha \in (\lambda, \mu], A_\alpha$  is a subring of  $R$  when  $A_\alpha \neq \emptyset$ .

**Proposition 4.** [8] Let  $f: R \rightarrow R'$  be a homomorphism of  $M$ -rings,  $A$  be a fuzzy subring of  $R$ , and  $A'$  be a fuzzy subring of  $R'$ . Then the following statements hold:

1.  $f(A)$  is a fuzzy subring of  $R'$ ;
2.  $f^{-1}(A')$  is a fuzzy subring of  $R$ .

## III. $(\lambda, \mu)$ -FUZZY SUBRING WITH OPERATORS

**Definition 6.** Let  $A$  be a fuzzy subring of  $M$ -ring  $R$ . Then  $A$  is called a fuzzy subring with thresholds  $(\lambda, \mu)$  of operators or a  $(\lambda, \mu)$ -fuzzy subring with operators a  $(\lambda, \mu)$ - $M$ -fuzzy subring of  $R$  if for all  $x \in R, m \in M, A(mx) \vee \lambda \geq A(x) \wedge \mu$ , and denoted by a  $(\lambda, \mu)$ - $M$ -fuzzy subring of  $R$ .

**Proposition 5.** Let  $S$  be a nonempty subset of  $M$ -ring  $R$ . If  $I_S$  is the characteristic function then  $S$  is an  $M$ -subring of  $R$  iff  $I_S$  is an  $(\lambda, \mu)$ - $M$ -fuzzy subring of  $R$ .

**Proof.** According to Proposition 2,  $I_S$  is an  $M$ -fuzzy subring of  $R$  when  $S$  is an  $M$ -subring of  $R$ . For all  $x \in R, m \in M$ , let  $mx \in S$ , then

$$I_s(mx) \vee \lambda = 1 \vee \lambda = 1 \geq I_s(x) \wedge \mu = 1 \wedge \mu = \mu.$$

Also  $x \notin S$  when  $mx \notin S$ , and hence

$$I_s(mx) \vee \lambda = 0 \vee \lambda = \lambda \geq I_s(x) \wedge \mu = 0 \wedge \mu = 0.$$

Thus,  $I_s$  is an  $(\lambda, \mu)$ - $M$ -fuzzy subring of  $R$ . Conversely, it is can be obtained from Proposition 2.

**Proposition 6.** Let  $A$  be a  $(\lambda, \mu)$ - $M$ -fuzzy subring of  $M$ -ring  $R$ . Then the following statements hold:

1.  $A(m(xy)) \vee \lambda \geq A(mx) \wedge A(my) \wedge \mu$ ;
2.  $A(m(-x)) \vee \lambda \geq A(x) \wedge \mu$ .

**Proof.** (1) For all  $x, y \in R$ ,  $m \in M$ , we have

$$A(m(xy)) \vee \lambda = A((mx)(my)) \vee \lambda \geq A(mx) \wedge A(my) \wedge \mu.$$

Thus,  $A(m(xy)) \vee \lambda \geq A(mx) \wedge A(my) \wedge \mu$ .

(2) For all  $x \in R$ ,  $m \in M$ , we have

$$\begin{aligned} A(m(-x)) \vee \lambda &= A(-(mx)) \vee \lambda = (A(-mx) \vee \lambda) \vee \lambda \\ &\geq (A(mx) \wedge \mu) \vee \lambda = (A(mx) \vee \lambda) \wedge \mu \\ &\geq A(x) \wedge \mu \wedge \mu = A(x) \wedge \mu. \end{aligned}$$

Thus,  $A(m(-x)) \vee \lambda \geq A(x) \wedge \mu$ .

**Proposition 7.** Let both  $A$  and  $B$  are  $(\lambda, \mu)$ - $M$ -fuzzy subring of  $M$ -ring  $R$ . Then  $A \cap B$  is a  $(\lambda, \mu)$ - $M$ -fuzzy subring of  $R$ .

**Proof.** For all  $x \in R$ ,  $m \in M$ , we have

$$A(mx) \vee \lambda \geq A(x) \wedge \mu;$$

$$B(mx) \vee \lambda \geq B(x) \wedge \mu.$$

Then

$$\begin{aligned} (A \cap B)(mx) \vee \lambda &= (A(mx) \wedge B(mx)) \vee \lambda = (A(mx) \vee \lambda) \wedge (B(mx) \vee \lambda) \\ &\geq (A(x) \wedge \mu) \wedge (B(x) \wedge \mu) = (A(x) \wedge B(x)) \wedge \mu \\ &= (A \cap B)(x) \wedge \mu. \end{aligned}$$

Thus,  $A \cap B$  is a  $(\lambda, \mu)$ - $M$ -fuzzy subring of  $R$ .

**Proposition 8.** Let  $A$  be a  $(\lambda, \mu)$ - $M$ -fuzzy subring of  $M$ -ring  $R$ . Then  $A$  is a  $(\lambda, \mu)$ - $M$ -fuzzy subring of  $R$  iff for every  $\alpha \in (\lambda, \mu]$ ,  $A_\alpha$  is a  $M$ -subring of  $R$  when  $A_\alpha \neq \emptyset$ .

**Proof.** It is easy to know by Proposition 3  $A_\alpha$  is a subring of  $R$  when  $A_\alpha \neq \emptyset$  for every  $\alpha \in (\lambda, \mu]$  in case of  $A$  being an  $M$ -fuzzy subring of  $R$ . Also for all  $x \in A_\alpha$ ,  $m \in M$ , we have

$$A(x) \geq \alpha.$$

Then

$$A(mx) \geq A(x) \geq \alpha,$$

and hence  $mx \in A_\alpha$ . Thus,  $A_\alpha$  is a  $M$ -subring of  $R$ . Conversely, we get the information from Proposition 3 that  $A$  is a  $(\lambda, \mu)$ -fuzzy subring of  $R$  for every  $\alpha \in (\lambda, \mu]$  when  $A_\alpha \neq \emptyset$ . If there exists  $x_0 \in R$ ,  $m_0 \in M$  such that

$$A(m_0x_0) \vee \lambda < A(x_0) \wedge \mu$$

Let

$$\alpha = A(x_0) \wedge \mu,$$

then for  $\alpha \in (\lambda, \mu]$ ,

$$A(m_0x_0) < \alpha$$

and

$$x_0 \in A_\alpha.$$

But  $m_0x_0 \notin A_\alpha$ , so here emerges a contradiction. Hence

$$A(mx) \vee \lambda \geq A(x) \wedge \mu$$

always holds for any  $x \in R$ ,  $m \in M$ . Therefore,  $A$  is a  $(\lambda, \mu)$ - $M$ -fuzzy subring of  $R$ .

**Proposition 9.** Let  $f: R \rightarrow R'$  be a  $M$ -homomorphism of  $M$ -rings and  $A$  be a  $(\lambda, \mu)$ - $M$ -fuzzy subring of  $R$ . Then  $f(A)$  is a  $(\lambda, \mu)$ - $M$ -fuzzy subring of  $R'$ .

**Proof.** It is clear from Proposition 2.4 that  $f(A)$  is a fuzzy subring of  $R'$ .

For all  $y \in R$ ,  $m \in M$ , we have

$$\begin{aligned}
f(A)(my) \vee \lambda &= \sup \left\{ A(x) \mid x \in f^{-1}(my) \right\} \vee \lambda \\
&= \sup \left\{ A(x) \mid f(x) = my \right\} \vee \lambda \\
&\geq \sup \left\{ A(x') \mid f(mx') = my, mx' \in R \right\} \vee \lambda \\
&= \sup \left\{ A(x') \vee \lambda \mid f(mx') = my, mx' \in R \right\} \\
&\geq \sup \left\{ A(x') \wedge \mu \mid f(x') = y, x' \in R \right\} \\
&= \sup \left\{ A(x') \mid f(x') = y, x' \in R \right\} \wedge \mu \\
&= f(A)(y) \wedge \mu.
\end{aligned}$$

Thus,  $f(A)$  is a  $(\lambda, \mu) - M$  - fuzzy subring of  $R'$ .

**Proposition 10.** Let  $f: R \rightarrow R'$  be a  $M$  - homomorphism of  $M$  - rings and  $A'$  be a  $(\lambda, \mu) - M$  - fuzzy subring of  $R'$ .

Then  $f^{-1}(A')$  is a  $(\lambda, \mu) - M$  - fuzzy subring of  $R$ .

**Proof.** It is clear from Proposition 4 that  $f^{-1}(A')$  is a fuzzy subring of  $R$ .

For all  $x \in R$ ,  $m \in M$ , we have

$$\begin{aligned}
f^{-1}(A')(mx) \vee \lambda &= A'(f(mx)) \vee \lambda = A'(mf(x)) \vee \lambda \\
&\geq A'(f(x)) \wedge \mu = f^{-1}(A')(x) \wedge \mu.
\end{aligned}$$

Thus,  $f^{-1}(A')$  is a  $(\lambda, \mu) - M$  - fuzzy subring of  $R$ .

#### IV. $(\lambda, \mu) -$ FUZZY QUOTIENT RING WITH OPERATORS

Let  $B$  be a  $(\lambda, \mu) -$  fuzzy ideal of ring  $R$ . For all  $a, b \in R$ , we define a fuzzy set  $a + B$  of  $R$  as:

$$(a + B)(x) = (B(x - a) \vee \lambda) \wedge \mu, \forall x \in R.$$

Let  $R/B = \{r+B \mid r \in R\}$ . For all  $r_1, r_2 \in R$ , we define them on  $R/B$  as:

$$\begin{aligned}
(r_1 + B) + (r_2 + B) &= (r_1 + r_2) + B; \\
(r_1 + B) \cdot (r_2 + B) &= r_1 r_2 + B.
\end{aligned}$$

Reference [2] proved that  $(R/B; +, \cdot)$  is a ring.

**Proposition 11.** Let  $R$  be a  $M$  - ring and  $B$  be a  $(\lambda, \mu) -$  fuzzy ideal of  $R$ . For any  $R + B \in R/B$ ,  $m \in M$ , we define  $m(r + B) = mr + B$ . Then  $(R/B; +, \cdot)$  is a  $M$  - ring.

**Proof.** First we prove the existence of the definition  $m(r + B) = mr + B$ .

If  $r_1 + B = r_2 + B$ , then

$$B(r_1 - r_2) = B(r_2 - r_1) = B(0).$$

$$B(mr_1 - mr_2) = B(m(r_1 - r_2)) \geq B(r_1 - r_2) = B(0).$$

Hence,  $mr_1 + B \supset mr_2 + B$ . Similarly, we have

$$B(mr_2 - mr_1) = B(m(r_2 - r_1)) \geq B(r_2 - r_1) = B(0).$$

Hence,  $mr_2 + B \supset mr_1 + B$ . Therefore, we have

$$mr_2 + B = mr_1 + B.$$

Namely,

$$m(r_1 + B) = m(r_2 + B).$$

Thus, the above definition is reasonable.

On the one hand,

$$\begin{aligned}
m((r_1 + B) + (r_2 + B)) &= m((r_1 + r_2) + B) = m(r_1 + r_2) + B \\
&= mr_1 + mr_2 + B = (mr_1 + B) + (mr_2 + B) \\
&= m(r_1 + B) + m(r_2 + B).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
m((r_1 + B)(r_2 + B)) &= m(r_1 r_2 + B) = m(r_1 r_2) + B \\
&= (mr_1) r_2 + B = (mr_1 + B)(r_2 + B) \\
&= (m(r_1 + B))(r_2 + B) = r_1(mr_2) + B = (r_1 + B)(mr_2 + B) \\
&= (r_1 + B)(m(r_2 + B)).
\end{aligned}$$

Thus,  $R/B$  is a  $M$  - ring.

Let  $R$  be a  $M$  - ring,  $A$  be a  $(\lambda, \mu) - M$  - fuzzy subring of  $R$ ,  $B$  be a  $(\lambda, \mu) -$  fuzzy ideal of  $R$ , and  $A/B$  is a fuzzy set of  $R/B$ . Now for any  $r + B \in R/B$ , we define it as:

$$A/B: R/B \rightarrow [0,1] \text{ satisfying } A/B(r+B) = \sup_{x+B=r+B} A(x).$$

Reference [4] proved  $A/B$  is a  $M$  - fuzzy subring of  $R/B$ .

**Proposition 12.** The above fuzzy subset  $A/B$  is a  $(\lambda, \mu) - M$  - fuzzy subring of  $R/B$ .

**Proof.** Let  $A$  be a  $(\lambda, \mu) - M$  - fuzzy subring of  $R$ . Then  $A/B$  is an  $M$  - fuzzy subring of  $R/B$ . For any  $r + B \in R/B$ ,  $m \in M$ , we have

$$\begin{aligned}
A/B(m(r+B)) \vee \lambda &= A/B(mr+B) \vee \lambda = \sup_{x+B=mr+B} A(x) \vee \lambda \\
&\geq \sup_{my+B=mr+B} A(my) \vee \lambda \geq \sup_{y+B=r+B} A(my) \vee \lambda \\
&\geq \sup_{y+B=r+B} A(y) \wedge \mu = A/B(r+B) \wedge \mu.
\end{aligned}$$

Thus,  $A/B$  is a  $(\lambda, \mu) - M$  - fuzzy subring of  $R/B$ .

**Definition 7.** The  $(\lambda, \mu) - M$  - fuzzy subring  $A/B$  is called a  $(\lambda, \mu) -$  fuzzy quotient ring of  $A$  with operators with respect to  $B$ , denoted by the  $(\lambda, \mu) - M$  - fuzzy quotient ring of  $A$  with respect to  $B$ .

**Proposition 13.** Let  $R$  be a  $M$  - ring,  $A$  be a  $(\lambda, \mu) - M$  - fuzzy subring of  $R$ ,  $B$  be a  $M$  - fuzzy ideal of  $R$ , and

$$\begin{aligned}
f: R &\rightarrow R/B, \\
x &\rightarrow x+B.
\end{aligned}$$

Then  $f$  is a  $M$  - homomorphism from  $R$  to  $R/B$ , and

$$f(A) = A/B.$$

**Proof.** It is clear that  $f$  is a homomorphism from  $R$  to  $R/B$ .

For any  $x \in R$ ,  $m \in M$ , we have

$$f(mx) = mx + B = m(x+B) = m(f(x)).$$

And for any  $a+B \in R/B$ , we have

$$\begin{aligned}
f(A)(a+B) &= \sup_{f(x)=a+B} A(x) = \sup_{x+B=a+B} A(x) \\
&= A/B(a+B).
\end{aligned}$$

Thus,  $f$  is a  $M$  - homomorphism from  $R$  to  $R/B$ , and

$$f(A) = A/B.$$

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