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Normalizing Logarithms of Realized Volatility in an ARFIMA Model

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Abstract**—**Modelling realized volatility with high-frequency returns is popular as it is an unbiased and efficient estimator of return volatility. A computationally simple model is fitting the logarithms of the realized volatilities with a fractionally integrated long-memory Gaussian process. The Gaussianity assumption simplifies the parameter estimation using the Whittle approximation. Nonetheless, this assumption may not be met in the finite samples and there may be a need to normalize the financial series. Based on the empirical indices S&P500 and DAX, this paper examines the performance of the linear volatility model pre-treated with normalization compared to its existing counterpart. The empirical results show that by including normalization as a pre-treatment procedure, the forecast performance outperforms the existing model in terms of statistical and economic evaluations.

Keywords**—**Long-memory, Gaussian process, Whittle estimator, normalization, volatility, value-at-risk.

I. Introduction

The availability of high-frequency data on financial assets has motivated research related to return volatility. Amongst the proxies of volatility, the realized volatility (RV) is popular as it is simple, yet an efficient estimator of return volatility [1]–[3]. RV is a sum of squares of the high-frequency returns over a desired estimation or forecast horizon. Andersen et al. [4] testified that although the distributions of the RV are right-skewed, the distributions of the logarithms of RV are approximately Gaussian, and the long-run dynamics of these quantities are well approximated by a fractionally-integrated long memory process. Subsequently, based on the simple long-memory Gaussian model for the logarithmic daily RVs, Andersen et al. [5] reported that the volatilities can be forecast with great accuracy, and their results are promising for practical modelling and forecasting of the large covariance matrices relevant in asset pricing and the related financial risk management applications.

With the assumption of Gaussian process, it is well-known that Whittle approximation offers an easier way to minimize the frequency domain approximation to the time domain Gaussian negative log likelihood [6]. The approximation is computationally fast due to the Fast Fourier Transform (FFT) [7]. Apart from the Gaussian case, the consistency of the Whittle estimator was proven for a general class of ergodic sequences [8]. It then becomes a popular methodology to obtain the approximate maximum-likelihood estimates in the long-memory time series analysis [9], [10].

This paper intends to examine the performance of the fractionally-integrated linear long memory process in modelling the log RV as proposed by Andersen et al. [5] henceforth referred to as ABDL. This is illustrated using the high-frequency returns of S&P500 and DAX. It is noted that the logarithms of the daily RV do not follow a Gaussian distribution, and hence, the performance of the parameter estimation assuming a Gaussian process is of interest of this paper. To meet the Gaussian assumption, a parsimonious normalization transformation is proposed before fitting the series to the linear long-memory model. The normalization method is adopted from Bivona et al. [11], of which the empirical cumulative probabilities are fitted to the Gaussian cumulative distribution function. The performance of the proposed model is compared to the ABDL model with several loss functions used in the literature and the superior predictive ability (SPA) developed by Hansen [12]. Besides the statistical evaluation, literatures show that an accurately predicted volatility is materialized into an accurate Value-at-Risk (VaR) forecast [13]–[15]. This motivates us to evaluate the performance of the models via the economic evaluation in the context of the forecast in VaR.

In the remainder of this paper, we proceed as follows. Section II describes the fractionally-integrated linear long memory Gaussian process, whereby the proposed normalization method as a data-pre-treatment procedure is detailed. Section III presents the empirical illustrations with the methodology used for statistical and economic forecast evaluations, and Section IV concludes.

II. Fractionally-Integrated Linear Long Memory Gaussian Process

An autoregressive fractionally integrated moving average ARFIMA(p,d,q) for process $y_t, t \in \mathbb{N}$ is given by:

$$\Phi(L)(1-L)^d y_t = \Theta(L)\varepsilon_t \tag{1}$$

where L denotes the backshift operator, ε_t are i.i.d. with zero mean and finite variance σ_ε^2, $\Phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j$, and $\Theta(z) = \sum_{j=0}^{q} \theta_j z^j$ are polynomials with no common roots and all roots lie outside the unit circle. The process is stationary if $d \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. The series $(1-z)^d$ can be expanded as $\sum_{j=0}^{\infty} a_j z^j$, where $a_j = \binom{d}{j}(-1)^j = \frac{\Gamma(d+1)}{\Gamma(j+1)\Gamma(d-j+1)}(-1)^j$. To estimate the $(p+q+2)$-dimensional parameter vector $\vartheta(\sigma_\varepsilon^2, d, \phi, \theta)$ with $\phi \in \Phi \subseteq \mathbb{R}^p$ and $\theta \in \Theta \subseteq \mathbb{R}^q$, it is convenient to assume that y_t is a zero mean Gaussian process so that the Gaussian

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maximum likelihood estimate might have optimal asymptotic statistical properties. In particular, it is easy to minimize the frequency domain approximation to the time domain Gaussian negative log likelihood, called the Whittle estimator, given by (2):

$$\mathcal{L}_n(\vartheta) = \sum_{j=1}^{\left[\frac{n-1}{2}\right]} \left\{ \log\left(f_\vartheta(\omega_j)\right) + \frac{I_j}{f_\vartheta(\omega_j)} \right\} \quad (2)$$

where $f_\vartheta(\omega_j) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{|\Theta(-i\omega_j)|^2}{|\Phi(-i\omega_j)|^2 |1-\exp(-i\omega_j)|^{2d}}$ is the spectral density of y_t, $I_j = \frac{1}{2\pi n}\left|\sum_{t=1}^{n} y_t \exp(-i\omega_j)\right|^2$ is the periodogram of y_t at the j^{th}Fourier frequency $\omega_j = \frac{2\pi j}{n}$, and $[\cdot]$ is the integer part of $(\cdot)$. Although the Whittle estimates ϑ_0 are asymptotically efficient when y_t is Gaussian [16], [17], Hannan [8] showed that the limit distribution of the Whittle estimates is unchanged by many departures from Gaussianity.

It is interesting to know if the Gaussianity assumption is critical in the finite samples. To examine this, let us compare the performance of the ARFIMA models on the demeaned log RV series (ABDL) and the normalized demeaned log RV series (ABDLn). The normalization procedure adopted in this paper follows the procedure of Bivona et al. [11] that matches the empirical cumulative probabilities $P(y_t)$ to the Gaussian cumulative distribution $D(z_t)$, taking the mean $\bar{y} = E(y_t)$ and the variance $\sigma_y^2 = E\left(y_t - E(y_t)\right)^2$ as:

$$D(z_t) = \frac{1}{\sigma_y\sqrt{2\pi}} \int_{-\infty}^{z_t} \exp\left(\frac{(x-\bar{y})^2}{2\sigma_y^2}\right) dx = P(y_t) \quad (3)$$

Meanwhile, to prepare for the back-transformation in the process of forecasting, the empirical distribution is fitted with Weibull distribution. This is justified as Weibull distribution is flexible to assume the characteristics of many types of distributions [11], [18]–[20]. The probability density function is given in (4):

$$f(y_t^*) = \frac{b}{a}\left(\frac{y_t^*}{a}\right)^{b-1} \exp\left(-\left(\frac{y_t^*}{a}\right)^b\right), y_t^* \geq 0 \quad (4)$$

where a and b are the scale and shape parameters to be estimated using maximum likelihood estimates given the values in the series $\{y_t^*\}$. To ensure that $y_t^* \geq 0$, the empirical data $\{y_t\}$ are adjusted such that $y_t^* = y_t - y_m + 0.1$, where y_m is the minimum of $\{y_t\}$. As a Gaussian process, the series $\{z_t\}$ is modelled with (1), and the parameters are then estimated with the Whittle estimator r given by (2). The model can be used to obtain the one-step ahead forecast $\hat{z}_{n+1}$. However, this quantity has to be back-transformed in order to be of any utility. This involves two steps; namely, (i) matching the Gaussian cumulative distribution function $D(\hat{z}_{n+1})$ to the Weibull cumulative distribution function $F_w(\hat{y}_{n+1}^*) = 1 - \exp\left(-\left(\frac{\hat{y}_{n+1}^*}{a}\right)^b\right)$ such that $D(\hat{z}_{n+1}) = F_w(\hat{y}_{n+1}^*)$, and (ii) obtain $\hat{y}_{n+1} = \hat{y}_{n+1}^* + y_m - 0.1$.

III. Empirical Illustration

In this section, we compare the RV forecast performance of ABDL and the normalized-ABDL (ABDLn) of which log RV is transformed following (3) and (4). In the first application, we consider the half hourly returns on the S&P500 indices spanning a period from 2/1/08 to 19/7/13. The half-hourly returns are computed as $r_t = \log(P_t) - \log(P_{t-1})$, where P_t is the asset price at the t^{th} half hourly observation. There are 13 returns per day, computed from 9:30 a.m. to 3:30 p.m. In the second application, we examine the same using DAX indices spanning a period from 2/1/08 to 9/5/13, with 18 returns per day from 3:00 a.m. to 11.30 p.m. For both indices, the RV for day-i is computed as $RV_i = \sum_{t=1+(i-1)*m}^{i*m} r_t^2$, where m is the number of intraday returns per day. The information regarding these data sets is detailed in Table I.

TABLE I
DESCRIPTIVE STATISTICS FOR THE FULL DATA SET, S&P 500 (2/1/08 – 19/7/13) AND DAX (2/1/08 – 9/5/13)

	S&P 500		DAX	
	r	RV	r	RV
Mean	$7.5238\ 10^{-06}$	$2.0345\ 10^{-04}$	$9.8656\ 10^{-07}$	$2.9055\ 10^{-04}$
Std dev	0.0040	$4.2067\ 10^{-04}$	0.0040	$5.9095\ 10^{-04}$
Skewness	-0.1165	6.0906	-0.3188	8.5049
Kurtosis	22.4317	54.1799	35.5520	101.6352
JB (p-value)	1	1	1	1
Q_{20} (p-value)	1	1	1	1

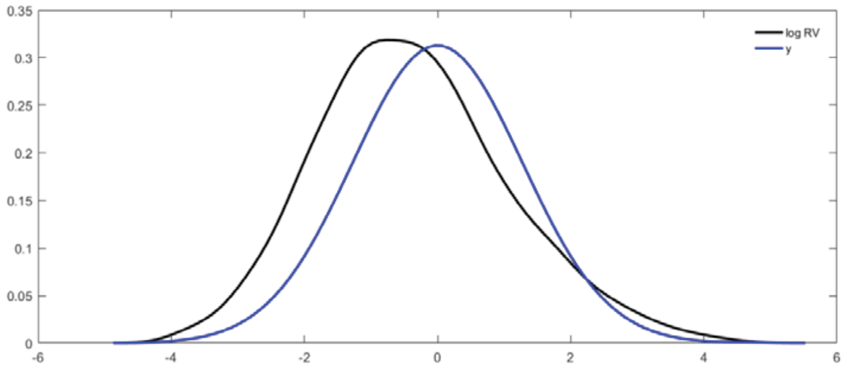
Note: JB is the Jarque-Bera statistic and Q_{20} is the 20th order of Ljung-Box test.

It is noted that both data sets are not normally distributed in their returns as well as RVs. Besides, these series portray strong autocorrelations, indicating possible existence of long memory. We compare the performance of ABDL and ABDLn based on the daily RV forecasts. To project a fair comparison between the two indices, the performances of these models are judged based on 300 out-of-sample forecasts for both S&P 500 and DAX.

To examine if the logarithms of daily RV (denoted as log RV) are normally distributed, we first compute the logarithms of $RV_i, i = 1, \cdots \left[\frac{n}{m}\right]$, where n = sample size and m = number of intraday returns per day. In this empirical illustration, we take a sample of 18196 S&P500 and 24145 DAX half hourly indices. These log RVs are then demeaned with the respective sample means and they are compared to the normalized counterparts $\{z_i\}$ that follow the normalization procedure in Section II. The comparison is shown in Fig. 1. It is clear that the log RVs of both indices do not follow Gaussian distribution, and this motivates the examination with ABDLn.

Let us take ABDLn model to forecast day ahead RV of S&P500 as an example. Based on the computed series $RV_i = \sum_{t=1+(i-1)*13}^{i*13} r_t^2, i = 1, \cdots, 1399$, we set the estimation window $n_d = 742$. Based on the autocorrelation plot in Fig. 2, both the series $\{log\, \mathrm{RV_i}\}$ *and* $\{\mathrm{z_i}\}$ depict slow decaying autocorrelation functions indicating the existence of long memory. As such, we keep the approach of ABDL that fits the 742 log RV (as well as z, the normalized log RV) with an

ARFIMA$(1, d, 0)$ model. With the Whittle estimator, the parameters of ABDL model are estimated as $\phi = -0.3036, \sigma_\varepsilon = 0.7796$, and $d = 0.5503$, whilst the parameters of ABDLn model are estimated as $\phi = -0.0006, \sigma_\varepsilon = 0.826$, and $d = 0.3896$. Based on these fitted ARFIMA models, the predicted log RV is then exponentiated, and $\widehat{RV}_{743}$ is estimated as 2.4 10^{-05} by ABDL and 2.39 10^{-05} by ABDLn. To proceed to the subsequent forecast, we rotate the estimation window forward by a day, that is, $\{RV_i\}_{i=2}^{742} \cup \widehat{RV}_{743}$, and the procedures to estimate the parameters of ARFIMA$(1, d, 0)$ and forecast day ahead RV are repeated for both ABDL and ABDLn, respectively.



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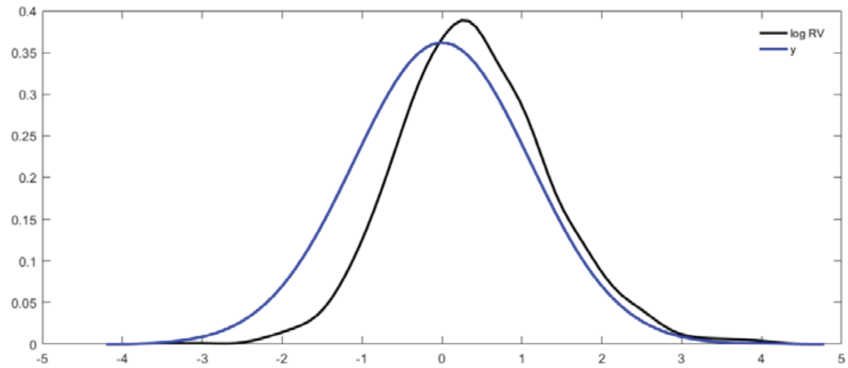

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Fig. 1 Density plot of log RV and the normalized log RV for (a) S&P500 and (b) DAX

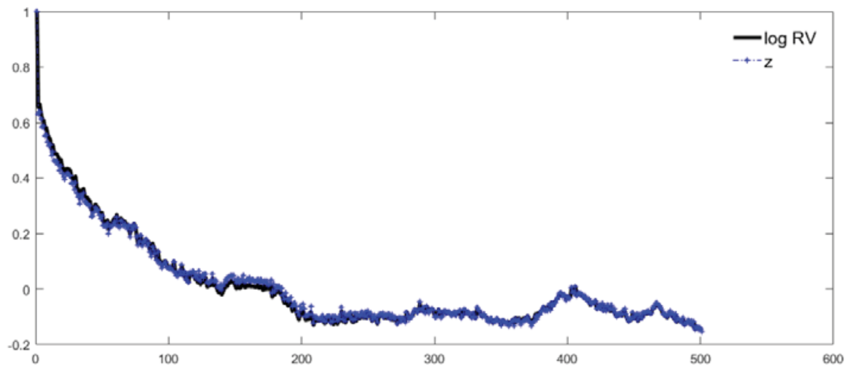
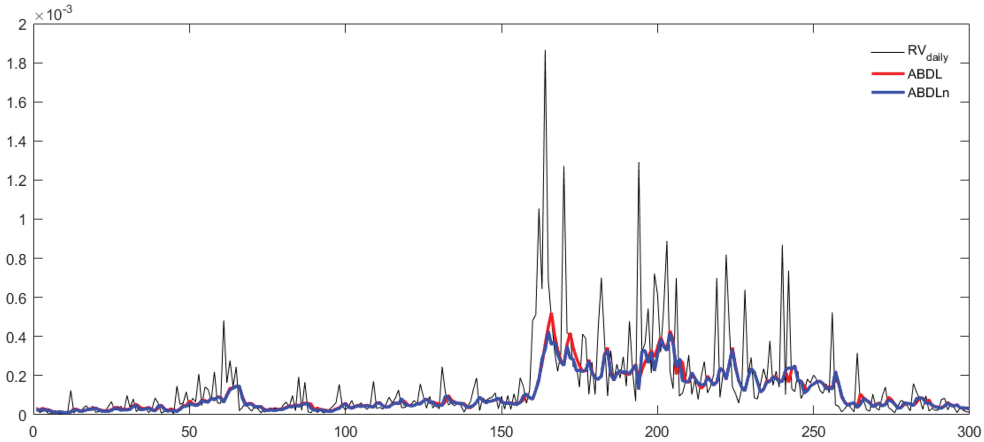


Fig. 2 Autocorrelation of the training series $\{log\ RV_i\}$ and $\{z_i\}$ for $\widehat{RV}_{743}$ of S&P500



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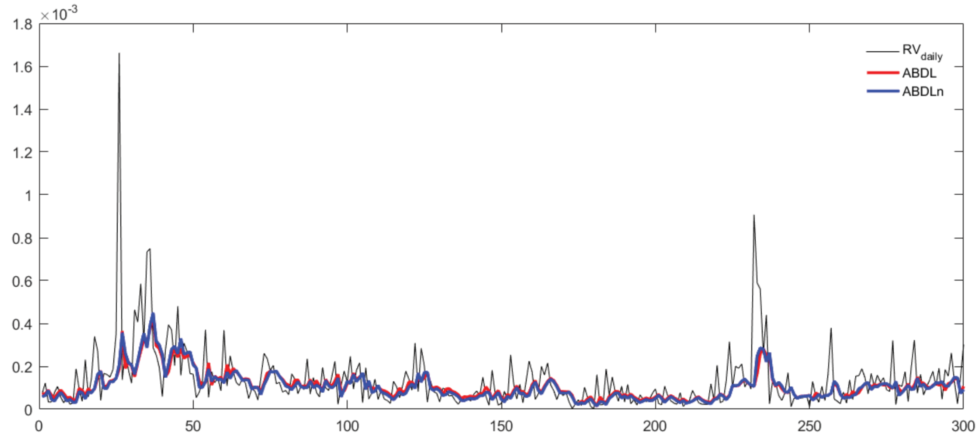


(b)

Fig. 3 Out-of-sample forecasts of daily RV for (a) S&P500 and (b) DAX

TABLE II
THE AVERAGES OF PARAMETER ESTIMATES OF ARFIMA $(1, d, 0)$ ABDL AND ABDLN OVER 300 REPLICATES FOR S&P500 AND DAX

	S&P500		DAX	
	ABDL	ABDLn	ABDL	ABDLn
ϕ	-0.2827	-0.2426	-0.1682	-0.1797
σ_ε	0.8081	0.8291	0.7115	0.7164
d	0.5247	0.5018	0.4790	0.4806

TABLE III
FORECASTING PERFORMANCE OF ABDL AND ABDLN FOR S&P500 AND DAX

	S&P500		DAX	
	ABDL	ABDLn	ABDL	ABDLn
MSE	$3.539\ 10^{-08}$	$3.588\ 10^{-08}$	$1.868\ 10^{-08}$	$1.835\ 10^{-08}$
MAD	$8.8140\ 10^{-05}$	$8.7130\ 10^{-05}$	$7.390\ 10^{-05}$	$7.361\ 10^{-05}$
MAPD	0.8033	0.7534	0.8773	0.8249

The logarithms of RV of DAX depict similar characteristic as log RV of S&P500. As such, we keep the same ARFIMA$(1, d, 0)$ to fit the series of log RV as well as the normalized log RV. The averages of the parameter estimates over 300 replicates for both indices are shown in Table II. It can be seen that the normalization of log RV does not produce a very different result, but it mitigates the characteristic of non-stationarity in modelling the volatility in S&P500. The out-of-sample forecasts of the daily RV for S&P500 and DAX are presented in Fig. 3. The forecast performance of these volatility models is then examined with the commonly used loss functions; namely, (i) the mean squared error (MSE), (ii) the mean absolute deviation (MAD), and (iii) the mean absolute percentage deviation (MAPD). The results of these forecasting models for S&P 500 and DAX are summarized in Table III. The best model in the respective performance measure is set forth in bold. We observe that ABDLn is consistently marked as a better model by the loss functions, except for S&P500, whereby the MSE of ABDLn is 1.384% higher than the result of ABDL. As a whole, ABDLn performs better than ABDL in forecasting RV for these indices.

A. Economic Evaluation of the Forecast RVs

Besides the statistical evaluation, we examine the economic appraisal of the forecast RVs in this study. VaR has been widely used as a measurement of the market risk of financial assets. It is a quantile forecast, of which $\widehat{VaR}^{\alpha}$ is the α^{th} quantile of the conditional returns, which can be written in.

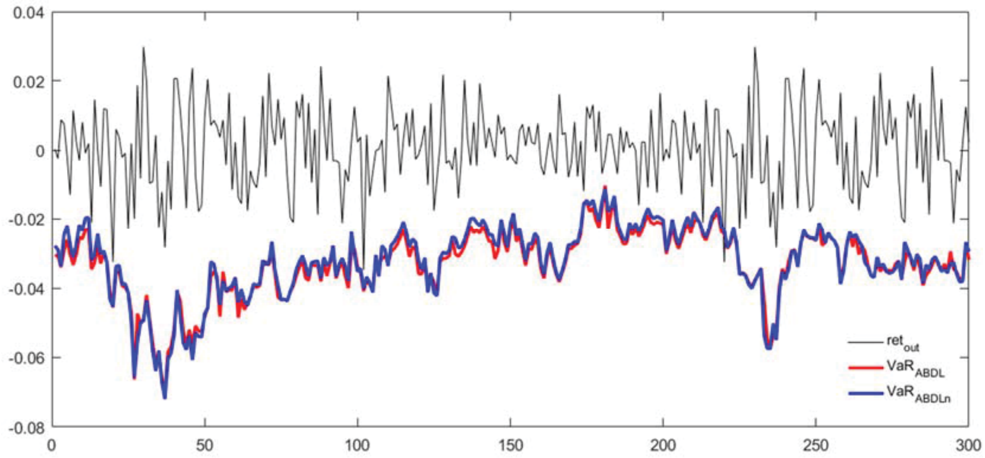
$$\widehat{VaR}^{\alpha}_{t_d+1,j} = \hat{\mu}_{t_d+1,j} + \hat{\sigma}_{t_d+1,j} F_{\varrho}^{-1}(\alpha),\ t_d = \left[\frac{n}{m}\right], \cdots, 299 + \left[\frac{n}{m}\right] \quad (5)$$

where $\hat{\mu}_{t_d+1,j}$ and $\hat{\sigma}_{t_d+1,j}$ are the j^{th} model's day ahead conditional mean and conditional volatility forecasts respectively, and F_{ϱ}^{-1} is the inverse cumulative distribution function of the innovations, $\varrho_{t_d} = \frac{r_{t_d} - \mu_{t_d}}{\sigma_{t_d}}$. From (5), the day ahead VaR is predicted by replacing the quantity $\hat{\sigma}_{t_d+1,j}$ with the square root of the volatility forecasts obtained from the ABDL and ABDLn models. In line with the characteristics of financial series, the α^{th} quantile of the ϱ_{t_d} process is estimated based on a skewed student distribution. With these results, we compute the forecasts $\widehat{VaR}^{.01}$and $\widehat{VaR}^{.05}$ for both S&P500 and DAX. The forecast results for DAX are illustrated in Fig. 4. It can be seen that the VaR forecasts following the volatilities predicted by ABDLn are in general closer to the daily returns.

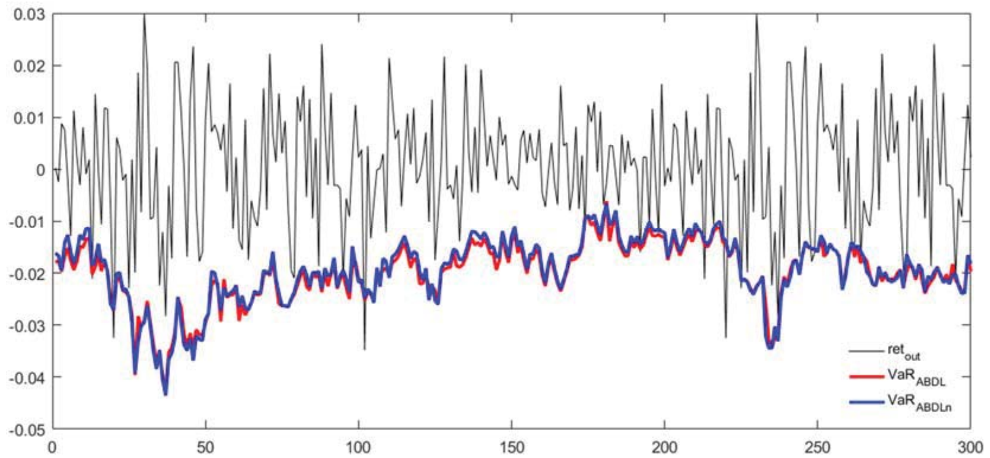
The forecasts of VaR are further evaluated in terms of capital efficiency. We examine this aspect using FABL firm's loss function by Abad et al. [21] given in (6):

$$FABL_{t_d+1,j} = \begin{cases} \left(\widehat{VaR}^{\alpha}_{t_d+1,j} - r_{t_d+1}\right)^2, & if\ r_{t_d+1} < \widehat{VaR}^{\alpha}_{t_d+1,j} \\ -c\left(r_{t_d+1} - \widehat{VaR}^{\alpha}_{t_d+1,j}\right), & if\ r_{t_d+1} \geq \widehat{VaR}^{\alpha}_{t_d+1,j} \end{cases} \quad (6)$$

where c is the firm's cost of capital.



(a)



(b)

Fig. 4 DAX daily returns and the respective VaR forecasts for (a) 1% (b) 5% long positions

The results are confirmed with the superior predictive ability (SPA) test by Hansen [12]. This examines the null hypothesis that the benchmark model is not inferior to its competing models. In our case here, we have only a benchmark and a competing model with 300 out-of-sample forecasts. The test statistic is deduced from the loss function differential $d_i = L_{i,0} - L_{i,c}$, $\mathrm{i} = 1, \cdots, 300$ where $L_{i,0}$ and $L_{i,c}$ are the loss variables (see (6)) of the benchmark and the competing model at observation i respectively. Under the assumption of the null hypothesis and that $d_{i,c}$ is stationary, we expect that on average, the loss variable of the benchmark model is not bigger than the competing model, that is, H_0: $\mu_c = \mathrm{E}(d_{i,c}) \leq 0$. The test statistic is given below.

$$T_{n_f}^{SPA} = \max\left[\frac{\sqrt{300}\mu_c}{\widehat{\omega}_c}, 0\right] \tag{7}$$

where $\widehat{\omega}_c$ is a consistent estimator of $\omega_c = var(\sqrt{300}\mu_c)$. The test statistic p-values are then estimated using stationary bootstrap of Politis and Romano [22]. Based on 2000 bootstraps, the performances of the VaR forecasts due to the volatilities from ABDL and ABDLn models are shown in Table IV. Interestingly, the volatilities from ABDLn model are consistently contributing to a better predictive ability of VaRs across the pre-set quantile (1% and 5%) as well as the stock indices.

TABLE IV
ECONOMIC EVALUATIONS OF VAR RESULTS FOR S&P 500 AND DAX INDICES

	S&P500		DAX	
	FABL	SPA (p-value)	FABL	SPA (p-value)
1% VaR				
ABDL	0.0014	0	0.0017	0
ABDLn	0.0014	0.1565	0.0016	0.1535
5% VaR				
ABDL	0.00070	0	0.001	0
ABDLn	0.00068	0.098	0.001	0.1865

IV. CONCLUSION

Modelling the logarithmic daily RVs with fractionally-integrated long memory Gaussian process is commonly practiced in the literature. Nonetheless, the assumption of Gaussianity in finite samples is uncertain. This paper examines the characteristic of the distribution of log RV in finite samples using the indices S&P500 and DAX. We noted that the series are not Gaussian, and a parsimonious normalization procedure is proposed. The performances of ABDL and ABDLn are compared in terms of statistical as well as economic evaluations. The empirical results show that the volatilities predicted by ABDLn outperform the counterpart from ABDL. This shows that pre-treating the financial series with normalization is indeed beneficial in the case where Gaussianity is assumed in the process of parameter estimation.

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