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On Tarski's Type Theorems for L-Fuzzy Isotone and L-Fuzzy Relatively Isotone Maps on L-Complete Propelattices

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Abstract—Recently a new type of very general relational structures, the so called (L-)complete propelattices, was introduced. These significantly generalize complete lattices and completely lattice L-ordered sets, because they do not assume the technically very strong property of transitivity. For these structures also the main part of the original Tarski's fixed point theorem holds for (L-fuzzy) isotone maps, i.e., the part which concerns the existence of fixed points and the structure of their set. In this paper, fundamental properties of (L-)complete propelattices are recalled and the so called L-fuzzy relatively isotone maps are introduced. For these maps it is proved that they also have fixed points in L-complete propelattices, even if their set does not have to be of an awaited analogous structure of a complete propelattice.

Keywords—Fixed point, L-complete propelattice, L-fuzzy (relatively) isotone map, residuated lattice, transitivity.

I. Introductory Remarks

THE proofs of many different variants and generalizations of Tarski's fixed point theorem are substantially dependent on transitivity. In [1], there were introduced the (L-)*complete propelattices*, which do not assume transitivity in any its form, but still (L-*fuzzy*) *isotone maps* have fixed points on them and the set of all fixed points is a crisp complete propelattice. A cost for generality of these structures is the fact that analogous formulas expressing the least and the greatest fixed point, which are included in the original Tarski's theorem [2], do not hold for them. On the other hand, we show that also the so called (L-*fuzzy*) *relatively isotone maps*, which generalize (L-fuzzy) isotone maps and for the crisp case are presented in [3], have their fixed points, even if their structure does not need to be a complete propelattice.

The main reason for introducing of (L-)complete propelattices is the fact that a fuzzification of the proof of Tarski's theorem is for *strong transitivity*, see relation (47), more or less trivial, but for weak transitivity, see relation (45), during the "fuzzy transcription" some insurmountable difficulties arise (see the concluding section). So, a natural question is that if transitivity can be eliminated and what can be achieved by this in fuzzy setting. If we want to formulate an exact question and give a complete answer, first we have to recall the original theorem from [2].

Theorem 1 (Lattice-theoretical fixpoint theorem)**.** *Let*

(i) $\mathcal{A} = \langle A, \leq \rangle$ *be a complete lattice,*

(ii) f *be an increasing function on* A *to* A*,*

(iii) P *be the set of all fixpoints of* f*.*

then

$$\text{the set } P \text{ is not empty} \tag{1}$$

and

$$\text{the system } \langle P, \leq \rangle \text{ is a complete lattice}; \tag{2}$$

in particular we have

$$\bigvee P = \bigvee \{x \in X \mid f(x) \geq x\} \in P \tag{3}$$

and

$$\bigwedge P = \bigwedge \{x \in X \mid f(x) \leq x\} \in P. \tag{4}$$

The question is, which from the facts (1), (2), (3) and (4) and in what kind of form holds without the assumption of transitivity in fuzzy setting. The answer is somehow surprising, if we realize how irreplaceable (but maybe neglected) is the role played by transitivity in the original proof of Theorem 1. We show that even if we omit the fundamental assumption of transitivity in the structure $\mathcal{A} = \langle A, \leq \rangle$, still – in crisp and also in fuzzy setting – (1) and (2) keep holding and the structure of P is a complete propelattice. Only relations (3) and (4) do not need to hold.

II. Preliminaries and Basic Notions

Everywhere below we suppose that $L = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ is a complete *residuated lattice*, i.e., the algebra $\langle L, \wedge, \vee, 0, 1 \rangle = \langle L, \leq \rangle$ is a complete lattice. Properties of L are well known from literature and if we need some of them, we recall it at a relevant place. Now let us briefly summarize only the most needed notions of the theory of fuzzy sets [4], [5]. In theoretical parts let $X \neq \emptyset$ be an arbitrary but fixed *universe set*.

Every element $\Phi \in L^X$ is an L-*set*, i.e., an L-set is a map of the type $\Phi : X \rightarrow L$. Clearly $\emptyset, X \in L^X$, where $\emptyset(x) \equiv 0,\ X(x) \equiv 1$ $(x \in X)$. For a fixed $\alpha \in L$ α-*cut of* Φ is the crisp set ${}^{\alpha}\Phi = \{x \in X \mid \Phi(x) \geq \alpha\}$. Especially, the set $\mathrm{Ker}(\Phi) := {}^{1}\Phi = \{x \in X \mid \Phi(x) \geq 1\} = \{x \in X \mid \Phi(x) = 1\}$ is the *kernel* of Φ. If $\mathrm{Ker}(\Phi) \neq \emptyset$, then Φ is a *normal* L-set, otherwise it is *subnormal*.

Thank to completeness of $\langle L, \wedge, \vee, 0, 1 \rangle$ for an arbitrary indexed system $\mathcal{F} = \{\Phi_\lambda \in L^X \mid \lambda \in \Lambda\} \subseteq L^X$ the operations $\bigcap \mathcal{F} = \bigcap_{\lambda \in \Lambda} \Phi_\lambda$, $\bigcup \mathcal{F} = \bigcup_{\lambda \in \Lambda} \Phi_\lambda \in L^X$ can be introduced.

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Then for all $x \in X$ the following holds:

$$\left(\bigcap \mathcal{F}\right)(x) = \left(\bigcap_{\lambda\in\Lambda} \Phi_\lambda\right)(x) = \bigwedge_{\lambda\in\Lambda} \Phi_\lambda(x),$$

$$\left(\bigcup \mathcal{F}\right)(x) = \left(\bigcup_{\lambda\in\Lambda} \Phi_\lambda\right)(x) = \bigvee_{\lambda\in\Lambda} \Phi_\lambda(x).$$

The *inclusion* $\Phi \subseteq \Psi$ holds if and only if $\Phi(x) \leq \Psi(x)$ for all $x \in X$; the L-set Φ is an L-*subset* of Ψ. Very important in the next is the fact that with regard to completeness of L also the algebra $\langle L^X, \cap, \cup, \emptyset, X\rangle = \langle L^X, \subseteq\rangle$ is a complete lattice.

Every element $\triangle \in L^{X\times X}$ is a (*binary*) L-*relation*. For any $x, y \in X$ we write: $(x \triangle y) := \triangle(x, y)$. Often, we work with 1-cut ${}^1\triangle \subseteq X \times X$ of the L-relation $\triangle$. Then for $x, y \in X$ we write: $x\,{}^1\triangle\, y \Leftrightarrow (x \triangle y) = 1$. For fixed $x \in X$ and $\mathbf{0} < \alpha \in L$, $\{{}^\alpha/_x\}$ is such an L-set that $\{{}^\alpha/_x\}(x) = \alpha$ and $\{{}^\alpha/_x\}(y) = 0$ for all $y \neq x$. If the universe set $X = \{x_1, x_2, \ldots, x_n\}$ is finite, we write the L-set $A \in L^X$ in the form

$$A = \{{}^{\alpha_1}/_{x_1}, {}^{\alpha_2}/_{x_2}, \ldots, {}^{\alpha_n}/_{x_n}\},$$

where the terms with $\alpha_i = \mathbf{0}$ are omitted.

A crisp set $\emptyset \neq A \subseteq L$ is usually identified with the relevant L-set $A \equiv \bigcup_{x\in A} \{{}^{\mathbf{1}}/_x\}$. In other words, a crisp set is identified with its membership function.

The following notion can be often used for testing of "suitability" of fuzzificated notions (see Remark 1) or for introducing of the properties which have no analogies in the crisp case [6], [1].

Definition 1 ([6])**.** An element $\mathcal{N}(L) \in L$ (if it exists in L) is a *neutral of* L if $\mathcal{N}(L) = \mathcal{N}(L) \to 0$. The set of all neutrals is denoted $\mathbf{N}(L)$.

If for some $x_0 \in X$ we have $\Phi(x_0) = \mathcal{N}(L)$, this fact is interpreted like we have no information (positive nor negative) about the statement *"x_0 is an element of Φ"*. In L the so called *adjunction property* keeps holding, i.e., for arbitrary $a, b, c \in L$:

$$a \otimes b \leq c \Leftrightarrow a \leq b \to c.$$

From here, we get the following simple chain of implications, which derives some useful property of neutrals:

$$\mathcal{N}(L) \to 0 \leq \mathcal{N}(L) \to 0 \Rightarrow (\mathcal{N}(L) \to 0) \otimes \mathcal{N}(L) \leq 0 \Rightarrow$$
$$\Rightarrow \mathcal{N}(L) \otimes \mathcal{N}(L) \leq 0, \quad \text{i.e.,} \quad \mathcal{N}(L) \otimes \mathcal{N}(L) = 0. \tag{5}$$

Finally, we recall two in literature common but substantial definitions, which introduce the most important general notions.

Definition 2 ([4], [7], [8])**.** An L-relation $\approx \in L^{X\times X}$ is an L-*equality* if for any $x, y, z \in X$ the following four conditions are fulfilled:

$$(x \approx x) = 1\,(\textit{reflexivity});$$
$$(x \approx y) = (y \approx x)\,(\textit{symmetry});$$
$$(x \approx y) \otimes (y \approx z) \leq (x \approx z)\,(\textit{transitivity});$$
$$(x \approx y) = 1 \Rightarrow x = y\,(\textit{crisp antisymmetry}). \tag{6}$$

The following definition apparently presents the simplest possible variant of a singleton.

Definition 3 ([8])**.** A normal L-set $\Phi \in L^X$ is an *SC-singleton (at the point x_0)* if there exists $x_0 \in X$ such that $\mathrm{Ker}(\Phi) = \{x_0\}$. An arbitrary SC-singleton at the point x_0 we denote by a unique symbol $\mathcal{S}[x_0]$.

III. "Crisp" Motivation and Fundamental Notions

The notions presented here are not only of a motivational character but they are fundamental for an extension of our results into fuzzy setting. The basic idea is very simple: We introduce all the notions, which concern posets or lattices and which are not instantly dependent on the property of transitivity, without this assumption and find out what happens. In the next definitions we do not have to assume that $A \neq \emptyset$ (the case $A = \emptyset$ is naturally implicitly included in all the definitions).

Definition 4 ([1])**.** Let $\triangle \subseteq X \times X$ be a reflexive and antisymmetric relation, $A \subseteq X$ and $a \in X$, then we say that[1]

- a is a *lower propebound of* A if $\forall x \in A : a \triangle x$;
- a is an *upper propebound of* A if $\forall x \in A : x \triangle a$;
- a is a *propeleast element* of A if a is a lower propebound of A and $a \in A$; we denote $a = p\min(A)$;
- a is a *propegreatest element of* A if a is an upper propebound of A and $a \in A$; we denote $a = p\max(A)$.

The relation $\triangle$ is a *propeorder on* X and the pair $X = \langle\langle X, =\rangle, \triangle\rangle$ is called a *propeordered set*.

It is clear from antisymmetry that the elements $p\min(A)$ and $p\max(A)$, if they exist, are unique, and that is why we presented their notations directly in the definition. The same naturally holds also for the elements $p\inf(A)$ and $p\sup(A)$, which are introduced in the following definition.

Definition 5 ([1])**.** Let $X = \langle\langle X, =\rangle, \triangle\rangle$ be a propeordered set and let $A \subseteq X$. The *lower propecone of* A is the set $\mathcal{L}(A)$ of all lower propebounds of A. The *upper propecone of* A is the set $\mathcal{U}(A)$ of all upper propebounds of A. If there exists $a = p\max(\mathcal{L}(A))$, then a is the *propeinfimum of* A and we denote $a = p\inf(A)$. If there exists $a = p\min(\mathcal{U}(A))$, then a is the *propesupremum of* A and we denote $a = p\sup(A)$.

Apparently, if $p\inf(A)$ and $p\sup(A)$ exist, the following identities hold and they are independent on transitivity, because only definitional relations, antisymmetry and reflexivity are sufficient:

$$\{p\inf(A)\} = \mathcal{L}(A) \cap \mathcal{U}(\mathcal{L}(A)). \tag{7}$$
$$\{p\sup(A)\} = \mathcal{U}(A) \cap \mathcal{L}(\mathcal{U}(A)). \tag{8}$$

The next two definitions play a fundamental role even for our results in fuzzy setting. The first one generalizes in a crucial way the notion of a complete lattice.

Definition 6 ([1])**.** A propeordered set $X = \langle\langle X, =\rangle, \triangle\rangle$ is a *complete propelattice* if for every set $A \subseteq X$ there exist

[1]Let us remark that the prefix "prope" means "near", "close to" in Latin.

its $p\inf(A)$ as well as $p\sup(A)$. Especially, we denote $\perp := p\min(X) = p\inf(X)$ and $\top := p\max(X) = p\sup(X)$.

With respect to (7) and (8) clearly $\top = p\inf(\emptyset)$ and $\perp = p\sup(\emptyset)$, still independently on transitivity.

Definition 7 ([1])**.** Let $X = \langle\langle X, =\rangle, \triangle\rangle$ be a propeordered set. A map $f : X \to X$ is *isotone on* X iff

$$\forall x, y \in X : x \triangle y \Rightarrow f(x) \triangle f(y);$$

the map f is *relatively isotone on* X iff

$$\forall x, y \in X : f(x) \triangle y \ \&\ x \triangle y \ \&\ x \triangle f(y) \Rightarrow f(x) \triangle f(y).^2$$

Obviously, every isotone map is relatively isotone, but the opposite does not hold. The examples presented below show that in our case relative isotony is a substantial generalization of isotony. We denote the *set of all fixed points* of a map $f : X \to X$ as $\mathrm{Fix}(f) = \{x \in X \mid x = f(x)\}$.

The following statement shows that a great part of Tarski's fixed point theorem (more precisely (1) and (2)) keeps holding even for complete propelattices. Let us mention that propelattices are more general than some nontransitive structures like for example *weakly associative lattices* [9], [1]. The following Theorem 2 is only an immediate special case of Theorems 3 and 4 for classical logical environment represented by the two-element Boolean algebra $L = 2$, which are presented further.

Theorem 2. *Let* $X = \langle\langle X, =\rangle, \triangle\rangle$ *be a complete propelattice, then for every relatively isotone map* $f : X \to X$ *on X the set* $\mathrm{Fix}(f)$ *is nonempty. Moreover, if* f *is isotone on* X*, then the system*

$$F = \langle\langle \mathrm{Fix}(f), =\rangle, \triangle \cap (\mathrm{Fix}(f) \times \mathrm{Fix}(f))\rangle$$

is a complete propelattice.

It is easy to find that the least cardinality of a complete propelattice, which is not automatically transitive and hence is not a complete lattice, is $|X| = 5$. The situation from Theorem 2 can be partially demonstrated in a rather simple example for $|X| = 6$.

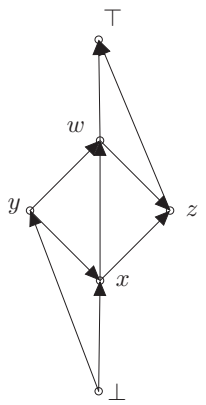

Fig. 1 Diagram for Example 1

[2]Here & denotes the classical logical conjunction.

Example 1. Let $X = \{\perp, x, y, w, z, \top\}$ and let the complete propelattice $X = \langle\langle X, =\rangle, \triangle\rangle$ have the "Hasse diagram" from Fig. 1. Obviously, $\triangle$ is not a transitive relation since $y \triangle x$, $x \triangle z$ and $y \triangle w$, $w \triangle z$, but $(y, z) \notin \triangle \subseteq X \times X$.

Now let us define the map $f : X \to X$ in this way: $f(\perp) = x$, $f(x) = w$, $f(y) = x$, $f(w) = w$, $f(z) = z$ and $f(\top) = z$. It is easy to verify that $X = \langle\langle X, =\rangle, \triangle\rangle$ is really a complete propelattice, the map f is isotone on X and $\mathrm{Fix}(f) = \{w, z\} \neq \emptyset$. Furthermore, $p\min(\mathrm{Fix}(f)) = w$ and $p\max(\mathrm{Fix}(f)) = z$. Now if we define the sets $M^\wedge$, $M^\vee$ in the following way, we have

$$M^\wedge := \{x \in X \mid f(x) \triangle x\} = \{w, z, \top\} \tag{9}$$

$$M^\vee := \{x \in X \mid x \triangle f(x)\} = \{\perp, x, y, w, z\} \tag{10}$$

Moreover, $p\inf(M^\wedge) = w$ and $p\sup(M^\vee) = \top$ and simultaneously (*by coincidence*)

$$p\inf(M^\wedge) = w = p\min(\mathrm{Fix}(f)); \tag{11}$$

but because $z = p\max(\mathrm{Fix}(f))$, $\top = p\sup(M^\vee)$, only

$$p\max(\mathrm{Fix}(f)) \triangle p\sup(M^\vee) \tag{12}$$

holds (*and identity does not hold*).

The rest of Theorem 2 demonstrates the next example.

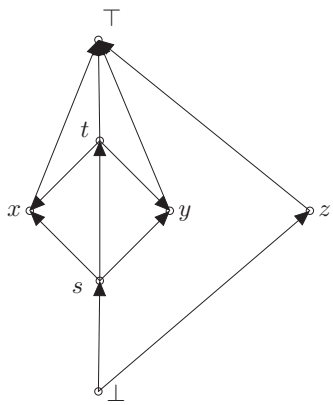

Fig. 2 Diagram for Example 2

Example 2. Let us have $X = \{\perp, s, t, x, y, z, \top\}$ and let $X = \langle\langle X, =\rangle, \triangle\rangle$ have the "Hasse diagram" given by Fig. 2. Obviously, X is a complete propelattice. The relation $\triangle \subseteq X \times X$ is not transitive, because $t \triangle x$, $x \triangle s$ and $t \triangle y$, $y \triangle s$, but $(s, t) \in \triangle$ (and hence with respect to antisymmetry $(t, s) \notin \triangle$). Now let $f : X \to X$ be such a map that $f(\{\perp, s, t, x, y, \top\}) = \{y\}$ and $f(z) = z$. The map $f : X \to X$ is not isotone, since $(\perp, z) \in \triangle$ but $(f(\perp), f(z)) = (y, z) \notin \triangle$. Nevertheless, f is trivially relatively isotone. Here $\mathrm{Fix}(f) = \{y, z\} \neq \emptyset$, but $\mathrm{Fix}(f)$ is not a "complete subpropelattice", because y and z are incomparable.

The following almost obvious statement formulates an analogy which holds clearly also for complete lattices.

Lemma 1. *Let $X = \langle\langle X, =\rangle, \triangle\rangle$ be a complete propelattice and let $A \subseteq B \subseteq X$ be arbitrary sets. Then the following relations hold:*

$$p\inf(B) \triangle p\inf(A) \quad \textit{and} \quad p\sup(A) \triangle p\sup(B). \tag{13}$$

Proof: Let us indicate the proof for the first from relations (13), the second one can be shown analogously. Because X is a complete propelattice, all elements in (13) exist according to Definition 6. From Definitions 4 and 5 we have the following. If $A = \emptyset$, then

$$\top = p\sup(X) = p\max(X) = p\max(\mathcal{L}(\emptyset)) = p\inf(\emptyset),$$

hence

$$p\inf(B) \triangle \top, \text{ that is } p\inf(B) \triangle p\inf(\emptyset).$$

If $A \neq \emptyset$, we gain the next simple chain of implications:

$$A \subseteq B \Rightarrow \mathcal{L}(B) \subseteq \mathcal{L}(A) \Rightarrow$$

$$\Rightarrow p\max(\mathcal{L}(B)) \triangle p\max(\mathcal{L}(A)) \Rightarrow p\inf(B) \triangle p\inf(A).$$

The proof is finished. ■

Reflexivity of the relation $\triangle \subseteq X \times X$ implies with regard to equalities (9) and (10) that $\mathrm{Fix}(f) \subseteq M^{\wedge}$ and $\mathrm{Fix}(f) \subseteq M^{\vee}$. From (13) the following relations immediately follow:

$$\begin{aligned} &p\max(\mathrm{Fix}(f)) \triangle p\sup(M^{\vee}),\\ &\quad p\inf(M^{\wedge}) \triangle p\min(\mathrm{Fix}(f));\\ &p\max(\mathrm{Fix}(f)) \triangle p\sup(M^{\wedge}),\\ &\quad p\inf(M^{\vee}) \triangle p\min(\mathrm{Fix}(f)). \end{aligned} \tag{14}$$

Relations (12) and (14) give a complete answer to the third part of the question from the introduction. Without transitivity equalities in (3) and (4) generally do not hold, but thank to completeness all relations (14) hold. At the same time the first two relations in (14) directly correspond to (3) and (4). The fact that $L = 2$ is now the Boolean algebra with the support $\{\mathbf{0}, \mathbf{1}\}$ is not important here, since an analogy of these relations holds generally for an arbitrary complete L (see relations (50)).

IV. Fundamental Notions in Fuzzy Setting and Basic Facts

We extend the notions presented in the previous section into fuzzy setting in a standard way. Let us mention that $\approx \in L^{X\times X}$ is everywhere below a fixed L-equality (see Definition 2).

Definition 8 ([1])**.** Let an L-relation $\triangle \in L^{X\times X}$ be reflexive and antisymmetric with respect to (w.r.t.) $\approx$, i.e., for every $x, y \in X$:

$$(x \triangle x) = 1 \ (\textit{reflexivity}); \tag{15}$$

$$(x \triangle y) \otimes (y \triangle x) \leq (x \approx y) \ (\textit{antisymmetry}). \tag{16}$$

Then $\triangle$ is an L-*propeorder on X (w.r.t. $\approx$)*. The pair $X = \langle\langle X, \approx\rangle, \triangle\rangle$ is an L-*propeordered set*.

The following definitions extend the notions from Definition 5 into fuzzy setting. If nothing else is explicitly said, we suppose that $X = \langle\langle X, \approx\rangle, \triangle\rangle$ is a fixed L-propeordered set (as for example in the following two definitions).

Definition 9 ([1])**.** Let $\Phi \in L^X$, then $\mathcal{U}(\Phi) \in L^X$ is the *upper propecone of* Φ if

$$\mathcal{U}(\Phi)(x_0) := \bigwedge_{x\in X} (\Phi(x) \to (x \triangle x_0)), \quad \forall x_0 \in X; \tag{17}$$

$\mathcal{L}(\Phi) \in L^X$ is the *lower propecone of* Φ if

$$\mathcal{L}(\Phi)(x_0) := \bigwedge_{x\in X} (\Phi(x) \to (x_0 \triangle x)), \quad \forall x_0 \in X. \tag{18}$$

Special cases of (17) and (18) are for a fixed $x_0 \in X$ the following notions: the *upper propecone of* x_0 is the L-set $\mathcal{C}^{\vee}_{x_0} \in L^X$, for which

$$\mathcal{C}^{\vee}_{x_0} := (x_0 \triangle x), \quad \forall x \in X; \tag{19}$$

the *lower propecone of* x_0 is the L-set $\mathcal{C}^{\wedge}_{x_0} \in L^X$, for which

$$\mathcal{C}^{\wedge}_{x_0} := (x \triangle x_0), \quad \forall x \in X. \tag{20}$$

Relations (7) and (8) give in the following definition immediate extensions of notions propeinfimum and propesupremum from Definition 5 into fuzzy setting.

Definition 10 ([1])**.** Let $\Phi \in L^X$, then the *propeinfimum of* Φ is the L-set

$$p\inf(\Phi) := \mathcal{L}(\Phi) \cap \mathcal{U}(\mathcal{L}(\Phi)); \tag{21}$$

the *propesupremum of* Φ is the L-set

$$p\sup(\Phi) := \mathcal{U}(\Phi) \cap \mathcal{L}(\mathcal{U}(\Phi)). \tag{22}$$

The following two definitions, which generalize Definitions 6 and 7, play a fundamental role in our results. The first of them copies almost verbatim the definitions of the so called *completely lattice* L-*ordered sets* [4], [7], [8].

Definition 11 ([1])**.** An L-propeordered set $X = \langle\langle X, \approx\rangle, \triangle\rangle$ is called an L-*complete propelattice* if $p\inf(\Phi)$ and $p\sup(\Phi)$ are normal L-sets, i.e., $\mathrm{Ker}(p\inf(\Phi)) \neq \emptyset$ and $\mathrm{Ker}(p\sup(\Phi)) \neq \emptyset$, for every L-set $\Phi \in L^X$.

Also here we denote $\perp := p\inf(X)$ and $\top := p\sup(X)$, even if these "elements" are now in fact of course L-sets. The last definition in this section extends standardly into fuzzy setting the first part of Definition 7.

Definition 12 ([1])**.** Let $X = \langle\langle X, \approx\rangle, \triangle\rangle$ be an L-propeordered set. A map $f : X \to X$ is *L-fuzzy isotone on* X if the following condition holds:

$$\forall x, y \in X : (x \triangle y) \leq (f(x) \triangle f(y)). \tag{23}$$

The following auxiliary statement summarizes some simple properties of introduced notions and some necessary facts for a proof of Theorem 3 in the next section. General facts from these clearly have to hold also for "fuzzy" lattices, no matter how they are defined.

Lemma 2 ([1])**.** *Let $X = \langle\langle X, \approx\rangle, \triangle\rangle$ be an L-propeordered set. Then the following statements hold:*

1) *Let $x_0 \in X$ and let $\psi \subseteq \mathcal{C}^{\vee}_{x_0}$ be an arbitrary L-subset. Then $p\inf(\psi) \subseteq \mathcal{C}^{\vee}_{x_0}$. Analogously if $\phi \subseteq \mathcal{C}^{\wedge}_{x_0}$, then $p\sup(\phi) \subseteq \mathcal{C}^{\wedge}_{x_0}$.*
2) *Let $\emptyset \neq A \subseteq X$ be a crisp subset. Then*
$$\mathcal{L}(A) = \bigcap_{x\in A} \mathcal{C}^{\wedge}_x, \quad \mathcal{U}(A) = \bigcap_{x\in A} \mathcal{C}^{\vee}_x;$$
if $\psi \in L^X$ is an arbitrary L-set, then the next implications hold:
$$\psi \subseteq \mathcal{U}(A) \Rightarrow p\inf(\psi) \subseteq \mathcal{U}(A),$$
$$\psi \subseteq \mathcal{L}(A) \Rightarrow p\sup(\psi) \subseteq \mathcal{L}(A).$$
3) *Relations $\perp = p\inf(X) = \mathcal{L}(X)$ and $\top = p\sup(X) = \mathcal{U}(X)$ hold.*
4) *For arbitrary $x_0 \in X$ we have $\top \subseteq \mathcal{C}^{\vee}_{x_0}$ and $\perp \subseteq \mathcal{C}^{\wedge}_{x_0}$.*
5) *The following holds:*
$$p\inf(\emptyset) = \top \quad \text{and} \quad p\sup(\emptyset) = \perp. \tag{24}$$
6) *If $X = \langle\langle X, \approx\rangle, \triangle\rangle$ is even an L-complete propelattice and $\Theta, \Xi \in L^X$, then*
   (a) *there exist x_0, x_1 so that*
$$p\sup(\Theta) = \mathcal{S}[x_0] \quad \text{and} \quad p\inf(\Theta) = \mathcal{S}[x_1]; \tag{25}$$
   (b) *if $\Theta \subseteq \Xi$, $\mathcal{S}[x_0] = p\inf(\Xi)$, $\mathcal{S}[x_1] = p\inf(\Theta)$, $\mathcal{S}[x_2] = p\sup(\Theta)$ and $\mathcal{S}[x_3] = p\sup(\Xi)$, then the following relations hold:*
$$x_0 \, {}^{\mathbf{1}}\!\triangle \, x_1 \quad \text{and} \quad x_2 \, {}^{\mathbf{1}}\!\triangle \, x_3. \tag{26}$$

Let us mention that equalities (24) express generally subnormal L-sets, even if their formal form is the same as in the crisp case. But in the case, where X is an L-complete propelattice, these L-sets are according to (25) normal: *SC*-singletons.

V. Main Results for Fuzzy Setting

Statements without proofs are proved in [1]. Let us mention that all our proofs have a common core. Always we have a construction of the least L-set of specific properties in the complete lattice $\langle L^X, \cap, \cup, \emptyset, X\rangle = \langle L^X, \subseteq\rangle$ – in the same way as in the proof of the following Theorem 3, see further relations (28) and (30).

A *relatively isotone map* [3] is a generalization of isotone maps on posets. Its analogy can be introduced also for L-propeordered sets. The following definition generalizes the second part of Definition 7.

Definition 13. Let $X = \langle\langle X, \approx\rangle, \triangle\rangle$ be an L-propeordered set. A map $f : X \to X$ is L-*fuzzy relatively isotone on* X, if for every $x, y \in X$ the following condition holds:
$$(f(x) \triangle y) \wedge (x \triangle y) \wedge (x \triangle f(y)) \leq (f(x) \triangle f(y)). \tag{27}$$

Clearly the next implication holds:
$$(x \triangle y) \leq (f(x) \triangle f(y)) \Rightarrow$$
$$\Rightarrow (f(x) \triangle y) \wedge (x \triangle y) \wedge (x \triangle f(y)) \leq (f(x) \triangle f(y)).$$

Relation (23) implies (27) and hence every L-fuzzy isotone map is at the same time L-fuzzy relatively isotone, but again the opposite does not generally hold.

Remark 1. Obviously, it makes a not very substantial sense to substitute the expression $(f(x) \triangle y) \otimes (x \triangle y) \otimes (x \triangle f(y))$ for the left side of inequality (27), because according to (5) if two of its terms are equal to some (fixed) neutral $\mathcal{N}(L) \in \mathbf{N}(L)$, then the third term has no influence on the value of the expression, no matter what is its value like. One can observe that in the case of such a formulation of (27) Theorem 3 would not hold.

The following theorem is partially proved in [1]. It gives a positive answer onto the first part of the question in the introduction, i.e., that (1) holds. Its proof presents quite general idea from the theory of ordinal numbers, which universally pervades proofs of all the theorems in [1].

Theorem 3. (i) *Let $X = \langle\langle X, \approx\rangle, \triangle\rangle$ be an L-complete propelattice and $f : X \to X$ be L-fuzzy relatively isotone map on X. Then $\mathrm{Fix}(f) \neq \emptyset$.*
(ii) *Moreover, if $f : X \to X$ is L-fuzzy isotone on X, then the set $\mathrm{Fix}(f)$ contains the propeleast and the propegreatest element with respect to the propeorder ${}^{\mathbf{1}}\!\triangle \subseteq X \times X$, i.e., $p\min(\mathrm{Fix}(f))$ and $p\max(\mathrm{Fix}(f))$ exist.*

Proof: (i) Let us show that $x_0 \in \mathrm{Fix}(f)$ exists, i.e., that $\mathrm{Fix}(f) \neq \emptyset$, if $f : X \to X$ is L-fuzzy relatively isotone on X.

Let $\mathcal{F} = \{\Phi_\lambda \in L^X \,|\, \lambda \in \Lambda\} \subseteq L^X$ be the indexed system of all L-sets such that for every $\Phi_\lambda \in L^X$ the following two conditions hold:

(a) $\Phi_\lambda \subseteq \Phi_\lambda \circ f$, i.e., $\Phi_\lambda(x) \leq \Phi_\lambda(f(x))$ for every $x \in X$;
(b) the implication $\phi \subseteq \Phi_\lambda \Rightarrow p\sup(\phi) \subseteq \Phi_\lambda$ holds for every $\phi \in L^X$.

The system $\mathcal{F}$ is nonempty since $X \in \mathcal{F}$: clearly $X(x) = X(f(x)) = \mathbf{1}$ for every $x \in X$. Also $p\sup(\phi) \subseteq X$ for every $\phi \in L^X$. Further let us define the L-set
$$\Phi := \bigcap \mathcal{F} = \bigcap_{\lambda\in\Lambda} \Phi_\lambda \neq \emptyset. \tag{28}$$

With respect to (b) for $\phi = \emptyset$ we have according to (24) $p\sup(\emptyset) = \perp \subseteq \Phi_\lambda$ for all $\lambda \in \Lambda$. Hence according to (25) $\emptyset \neq \mathcal{S}[x_0] = \perp \subseteq \Phi$ for some $x_0 \in X$, after which $\Phi \neq \emptyset$.

First let us show that the L-set Φ meets with both the conditions (a) and (b), i.e., that Φ is the least element of $\mathcal{F}$ with respect to the order in the complete lattice $\langle L^X, \subseteq\rangle = \langle L^X, \cap, \cup, \emptyset, X\rangle$. For arbitrary $\lambda \in \Lambda$ and $x \in X$ we have
$$\Phi(x) = \left(\bigcap_{\lambda\in\Lambda} \Phi_\lambda\right)(x) = \bigwedge_{\lambda\in\Lambda} \Phi_\lambda(x) \leq \Phi_\lambda(x) \leq \Phi_\lambda(f(x));$$
hence
$$\Phi(x) \leq \bigwedge_{\lambda\in\Lambda} \Phi_\lambda(f(x)) = \left(\bigcap_{\lambda\in\Lambda} \Phi_\lambda\right)(f(x)) = \Phi(f(x)), \tag{29}$$
so $\Phi \subseteq \Phi \circ f$ and the condition (a) holds for Φ.

The condition (b) can be shown in an even simpler way. Let $\phi \subseteq \Phi$ be an arbitrary L-subset. Then according to (28),

$\phi \subseteq \Phi_\lambda$ for every $\lambda \in \Lambda$ and hence $p\sup(\phi) \subseteq \Phi_\lambda$, that is $p\sup(\phi) \subseteq \bigcap_{\lambda\in\Lambda} \Phi_\lambda = \Phi$. Hence the condition (b) holds for Φ too. All in all, we have in $\langle L^X, \subseteq\rangle$ that

$$\Phi = \inf(\mathcal{F}) = \min(\mathcal{F}) \in \mathcal{F}. \tag{30}$$

As $\Phi \subseteq \Phi \in \mathcal{F}$, we have according to (b) $p\sup(\Phi) \subseteq \Phi$. According to Lemma 2 (6) (a), relation (25), there exists $x_0 \in X$ such that $p\sup(\Phi) = \mathcal{S}[x_0]$. Then $\Phi(x_0) = \mathbf{1}$ and therefore according to (a) also $\Phi(f(x_0)) = \mathbf{1}$. Furthermore, with respect to (22)

$$\mathcal{S}[x_0] = p\sup(\Phi) = \mathcal{U}(\Phi) \cap \mathcal{L}(\mathcal{U}(\Phi)). \tag{31}$$

Herefrom

$$\mathcal{U}(\Phi)(x_0) = \bigwedge_{x\in X} (\Phi(x) \to (x \triangle x_0)) = \mathbf{1},$$

i.e., $\Phi(x) \to (x \triangle x_0) = \mathbf{1}$ for every $x \in X$. In every residuated lattice L for every $a, b \in L$ the following equivalence holds: $a \to b = \mathbf{1} \Leftrightarrow a \leq b$. From here we have this important inequality:

$$\Phi(x) \leq (x \triangle x_0), \quad \forall x \in X. \tag{32}$$

Especially, for the choice $x = f(x_0)$ in (32) we have

$$\mathbf{1} = \Phi(f(x_0)) \leq (f(x_0) \triangle x_0) \Rightarrow (f(x_0) \triangle x_0) = \mathbf{1}. \tag{33}$$

According to (16) and (6), it is now sufficient to prove that $(x_0 \triangle f(x_0)) = \mathbf{1}$. Let us define the L-set $\Phi^\wedge \in L^X$ in the following way:

$$\Phi^\wedge := \Phi \cap \mathcal{C}^\wedge_{f(x_0)}. \tag{34}$$

We show that $\Phi^\wedge = \Phi$. Obviously $\Phi^\wedge \subseteq \Phi$. The reverse inclusion $\Phi^\wedge \supseteq \Phi$ has to be shown. If we show that $\Phi^\wedge$ satisfies both the conditions (a) and (b), then this inclusion has to hold with respect to (30), i.e., to minimality of Φ in $\mathcal{F}$. According to (34) with respect to (29) for every $x \in X$ we have:

$$\begin{aligned}\Phi^\wedge(x) = (\Phi \cap \mathcal{C}^\wedge_{f(x_0)})(x) = \Phi(x) \wedge \mathcal{C}^\wedge_{f(x_0)}(x) \leq \\ \leq \Phi(x) \leq \Phi(f(x)).\end{aligned} \tag{35}$$

Further, with respect to (34), (29) and (32) we have

$$\Phi^\wedge(x) \leq \Phi(x) \leq \Phi(f(x)) \leq (f(x) \triangle x_0) \tag{36}$$

and

$$\Phi^\wedge(x) \leq \Phi(x) \leq (x \triangle x_0). \tag{37}$$

According to (34) and (20) we finally have:

$$\Phi^\wedge(x) \leq \mathcal{C}^\wedge_{f(x_0)}(x) = (x \triangle f(x_0)). \tag{38}$$

Because $f : X \to X$ is L-fuzzy relatively isotone on X, according to (27) then with respect to (36), (37) and (38) we obtain:

$$\begin{aligned}\Phi^\wedge(x) \leq (f(x) \triangle x_0) \wedge (x \triangle x_0) \wedge (x \triangle f(x_0)) \leq \\ \leq (f(x) \triangle f(x_0)) = \mathcal{C}^\wedge_{f(x_0)}(f(x)).\end{aligned} \tag{39}$$

Relations (35) and (39) imply that for every $x \in X$ according to (34)

$$\begin{aligned}\Phi^\wedge(x) = \Phi(x) \wedge \mathcal{C}^\wedge_{f(x_0)}(x) \leq \Phi(f(x)) \wedge \mathcal{C}^\wedge_{f(x_0)}(f(x)) = \\ = (\Phi \cap \mathcal{C}^\wedge_{f(x_0)})(f(x)) = \Phi^\wedge(f(x)),\end{aligned}$$

that is $\Phi^\wedge \subseteq \Phi^\wedge \circ f$, i.e., $\Phi^\wedge$ satisfies the condition (a).

Finally let us prove for $\Phi^\wedge$ also the condition (b). Let $\phi \in L^X$ be an arbitrary L-set such that $\phi \subseteq \Phi^\wedge$. Since $\phi \subseteq \Phi^\wedge = \Phi \cap \mathcal{C}^\wedge_{f(x_0)} \subseteq \Phi$, then according to the condition (b) we have for Φ:

$$p\sup(\phi) \subseteq \Phi. \tag{40}$$

Because also $\phi \subseteq \mathcal{C}^\wedge_{f(x_0)}$, then with respect to Lemma 2 (1) we have:

$$p\sup(\phi) \subseteq \mathcal{C}^\wedge_{f(x_0)}. \tag{41}$$

From (40) and (41) we finally obtain:

$$p\sup(\phi) \subseteq \Phi \cap \mathcal{C}^\wedge_{f(x_0)} = \Phi^\wedge. \tag{42}$$

Hence according to (42) the L-set $\Phi^\wedge \in L^X$ satisfies also the condition (b). With respect to minimality of Φ in $\mathcal{F}$ the inclusion $\Phi^\wedge \supseteq \Phi$ then has to hold and hence the next equality is valid:

$$\Phi^\wedge = \Phi. \tag{43}$$

With respect to (31), (43) and (34) we have

$$\mathcal{S}[x_0] = p\sup(\Phi) = p\sup(\Phi^\wedge) \subseteq \Phi^\wedge = \Phi \cap \mathcal{C}^\wedge_{f(x_0)} \subseteq \mathcal{C}^\wedge_{f(x_0)},$$

hence

$$\begin{aligned}\mathbf{1} = \mathcal{S}[x_0](x_0) \leq \mathcal{C}^\wedge_{f(x_0)}(x_0) = (x_0 \triangle f(x_0)) \Rightarrow \\ \Rightarrow (x_0 \triangle f(x_0)) = \mathbf{1}.\end{aligned} \tag{44}$$

From (16), (6), (33) and (44) we finally obtain

$$\begin{aligned}\mathbf{1} = (f(x_0) \triangle x_0) \otimes (x_0 \triangle f(x_0)) \leq (f(x_0) \approx x_0) = \mathbf{1} \Rightarrow \\ \Rightarrow f(x_0) = x_0,\end{aligned}$$

that is $x_0 \in \mathrm{Fix}(f)$, i.e., $\mathrm{Fix}(f) \neq \emptyset$. The first part of the proof is finished.

(ii) Now let $f : X \to X$ be L-fuzzy isotone on X and let $x_0 \in \mathrm{Fix}(f)$ be the same as in the part (i). We prove that $x_0 = p\min(\mathrm{Fix}(f)) = p\inf(\mathrm{Fix}(f))$ in the propeordered set $\langle\langle \mathrm{Fix}(f), =\rangle, {}^{\mathbf{1}}\triangle \cap (\mathrm{Fix}(f) \times \mathrm{Fix}(f))\rangle$. Suppose (even if it is not absolutely necessary) that $|\mathrm{Fix}(f)| \geq 2$ and $x^* \in \mathrm{Fix}(f)$ be such fixed point of f that $x^* \neq x_0$. We have to show that $(x_0 \triangle x^*) = \mathbf{1}$, that is $x_0 \, {}^{\mathbf{1}}\triangle \, x^*$. For x^* we have $f(x^*) = x^*$ and because $f : X \to X$ is L-fuzzy isotone on X, for every $x \in X$ according to (20) and (23) we have

$$\begin{aligned}\mathcal{C}^\wedge_{x^*}(x) = (x \triangle x^*) \leq (f(x) \triangle f(x^*)) = \\ = (f(x) \triangle x^*) = \mathcal{C}^\wedge_{x^*}(f(x)),\end{aligned}$$

hence $\mathcal{C}^\wedge_{x^*} \subseteq \mathcal{C}^\wedge_{x^*} \circ f$ and $\mathcal{C}^\wedge_{x^*}$ satisfies the condition (a). The condition (b) is fulfilled according to Lemma 2 (1). Now by an analogous argumentation as above we have that $\Phi = \Phi \cap \mathcal{C}^\wedge_{x^*}$. Herefrom, according to (31)

$$\mathcal{S}[x_0] = p\sup(\Phi) = p\sup(\Phi \cap \mathcal{C}^\wedge_{x^*}) \subseteq \Phi \cap \mathcal{C}^\wedge_{x^*} \subseteq \mathcal{C}^\wedge_{x^*}.$$

Finally, this implies that

$$\mathbf{1} = \mathcal{S}[x_0](x_0) \leq \mathcal{C}^\wedge_{x^*}(x_0) = (x_0 \triangle x^*) \Rightarrow (x_0 \triangle x^*) = \mathbf{1},$$

hence $x_0 \, {}^{1}\triangle \, x^*$. Because $x_0 \in \mathrm{Fix}(f)$, according to Definition 4 with respect to the propeorder ${}^{\mathbf{1}}\triangle \subseteq X \times X$ then $x_0 = p\min \mathrm{Fix}(f)$. This proves the second part.

An argumentation, which leads to the proof of the fact that $\mathrm{Fix}(f) \neq \emptyset$ and of the potential existence of the propegreatest element of $\mathrm{Fix}(f)$ with respect to ${}^{1}\triangle \subseteq X \times X$, is dual. Here we introduce the system of all L-sets $\mathcal{P} = \{\Psi_\lambda \in L^X | \lambda \in \Lambda\} \subseteq L^X$ such that they satisfy the next two conditions:

(aa) $\Psi_\lambda \subseteq \Psi_\lambda \circ f$, i.e., $\Psi_\lambda(x) \leq \Psi_\lambda(f(x))$ for every $x \in X$;
(bb) the implication $\psi \subseteq \Psi_\lambda \Rightarrow p\inf(\psi) \subseteq \Psi_\lambda$ holds for every $\psi \in L^X$.

Further, we introduce the L-set $\emptyset \neq \Psi := \bigcap \mathcal{P} = \bigcap_{\lambda \in \Lambda} \Psi_\lambda \in L^X$. If $f : X \to X$ is L-fuzzy relatively isotone on X and $\mathcal{S}[x_1] = p\inf(\Psi)$, then we can symmetrically show that $x_1 \in \mathrm{Fix}(f)$, and if it is L-fuzzy isotone, then also $x_1 = p\max(\mathrm{Fix}(f))$. ■

Let us notice that for an (L-fuzzy) relatively isotone map its $p\min(\mathrm{Fix}(f))$ nor $p\max(\mathrm{Fix}(f))$ does not need to exist and hence $\mathrm{Fix}(f)$ does not need to have a structure of a complete propelattice. Obviously, if $L = 2$ is the Boolean algebra with the support $\{\mathbf{0}, \mathbf{1}\}$, then the first part of Theorem 2 follows directly from Theorem 3(i).

Remark 2. The proof of Theorem 3 shows that while in $X = \langle\langle X, \approx\rangle, \triangle\rangle$ transitivity is eliminable, for $L = \langle L, \wedge, \vee, \mathbf{0}, \mathbf{1}\rangle = \langle L, \leq\rangle$ transitivity remains fundamental, see for example relations (29), (35), (36), (37), (39) and others. Transitivity of L is irreplaceable also for validity of Lemma 2 and all the other statements. The question of transitivity for $L = 2$ (as for example in Theorem 2) is pointless, but it is important to emphasize that the Boolean algebra 2 is transitive trivially.

The last result of this section gives a positive answer to the second part of the question given in the introduction: Even for L-complete propelattices an analogy of the fact (2) keeps holding. Nevertheless, let us point out that the theorem holds only for L-fuzzy isotone maps, for L-fuzzy relatively isotone maps no similar statement holds. Of course, this theorem immediately implies the second part of Theorem 2 for the case $L = 2$.

Theorem 4 ([1]). *Let $X = \langle\langle X, \approx\rangle, \triangle\rangle$ be an L-complete propelattice and $f : X \to X$ be L-fuzzy isotone on X. Then for an arbitrary subset $P \subseteq \mathrm{Fix}(f)$ there exist its $p\inf(P)$ and $p\sup(P)$ in $\mathrm{Fix}(f)$ with respect to the propeorder ${}^{1}\triangle \cap (\mathrm{Fix}(f) \times \mathrm{Fix}(f))$, i.e.,*

$$F_f = \langle\langle \mathrm{Fix}(f), =\rangle, {}^{1}\triangle \cap (\mathrm{Fix}(f) \times \mathrm{Fix}(f))\rangle$$

is a complete propelattice.

In connection to Theorem 4 it is important to realize that for an L-fuzzy isotone map $f : X \to X$ the set of fixed points $\mathrm{Fix}(f)$ creates a crisp complete propelattice. This fact obviously relates (but loosely) to similar results in [7] and this is also the reason why we could not omit the third part of the paper.

VI. Concluding Remarks

It is obvious that our results have to be fully compatible with Theorem 1 for logical setting represented by $L = 2$. Theorem 2 shows that (1) and (2) hold in this case for complete propelattices. Moreover, if the L-relation $\triangle \in L^{X \times X}$ is transitive, also (3) and (4) must hold. But there exist some counterexamples that *weak transitivity* in the form

$$(x \triangle y) \otimes (y \triangle z) \leq (x \triangle z), \quad \forall x, y, z \in X, \tag{45}$$

is not sufficient for equalities (3) and (4). Nevertheless, the following theorem holds.

Theorem 5 ([1]). *Let $X = \langle\langle X, \approx\rangle, \triangle\rangle$ be an L-complete propelattice, $f : X \to X$ be an L-isotone map on X and L-sets $\Omega, \Phi \in L^X$ be defined for an arbitrary $x \in X$ as follows:*

$$\Omega(x) := (f(x) \triangle x) \quad \textit{and} \quad \Phi(x) := (x \triangle f(x)). \tag{46}$$

Let an L-relation $\triangle \in L^{X \times X}$ be strongly transitive, i.e., for every $x, y, z \in X$:

$$(x \triangle y) \wedge (y \triangle z) \leq (x \triangle z). \tag{47}$$

If $\mathcal{S}[x_0] = \inf(\Omega)$, $\mathcal{S}[x_1] = \sup(\Phi)$, then

$$x_0 = \min(\mathrm{Fix}(f)) \quad \textit{and} \quad x_1 = \max(\mathrm{Fix}(f)). \tag{48}$$

For the case $L = 2$, relations (45) and (47) are equivalent and relations (48) clearly transfer into (3) and (4). Theorem 2 together with this fact hence give the complete version of Theorem 1 – Quomodo videmus circulum plane inclusum iri! (prof. Petr Vopěnka[3]).

Remark 3. (a) Let us point out, that the statement of Theorem 5 does not hold for L-fuzzy relatively isotone maps, even if the L-propeorder $\triangle \in L^{X \times X}$ is strongly transitive. Simple counterexamples exist even in the crisp case [3].

(b) In every residuated lattice L for all $a, b \in L$ obviously $a \otimes b \leq a \wedge b$ holds. Hence every strongly transitive relation is weakly transitive too.

The situation from Theorem 3 (and partially from Theorem 5) is demonstrated in the following example.

Example 3. Let $L = \langle L, \wedge, \vee, \otimes, \to, \mathbf{0}, \mathbf{1}\rangle$ be such that $L = \{\mathbf{0}, N, \mathbf{1}\}$, where $\mathbf{0} < N < \mathbf{1}$ and N is the only neutral of L, i.e., $N \in \{\mathcal{N}(L)\} = \mathbf{N}(L)$. The operations $\otimes, \to$ have all the required properties. Obviously, L is a complete residuated lattice. In place of the L-equality we use the L-equality $\approx_N \in L^{X \times X}$ which is defined in the following way: $(w \approx_N w) = \mathbf{1}, (w \approx_N z) = N$ for $w \neq z$, where $w, z \in X$.

Let $X = \{\bot, x, y, \top\}$. The diagram in Fig. 3 indicates a "skeleton" of the L-complete propelattice $X = \langle\langle X, \approx_N\rangle, \triangle\rangle$ (hardly we can talk about a "Hasse diagram" and only hardly the whole L-relation $\triangle \in L^{X \times X}$ can be simply displayed).

The L-propeorder $\triangle \in L^{X \times X}$ is given in this way: $(w \triangle \top) = \mathbf{1}$, $(\bot \triangle w) = \mathbf{1}$, $(w \triangle w) = \mathbf{1}$, $(\top \triangle w) = N$, $(w \triangle \bot) = N$ for $w \in X$ and $(x \triangle y) = N = (y \triangle x)$. The L-equality $\approx_N$ obviously fulfills with respect to (5) all the conditions of Definition 2. And also $\triangle$ is antisymmetric

[3]The famous founder of the alternative set theory and alternative mathematics in general.

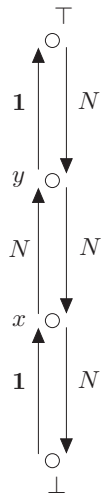

Fig. 3 Diagram for Example 3

w.r.t. $\approx_N$, i.e., (16) in Definition 8 holds. The propeorder $\triangle$ is obviously strongly transitive (compare to the text preceding Example 1, because here $|X| = 4$) and hence also weakly transitive. For example, for $\bot, x \in X$ we have:

$$(\bot \triangle x) \wedge (x \triangle y) = \mathbf{1} \wedge N = N \leq \mathbf{1} = (\bot \triangle y),$$
$$(x \triangle y) \wedge (y \triangle \bot) = N \wedge N = N \leq N = (x \triangle \bot).$$

It is easy to verify that X is an L-complete propelattice. For example the following holds:

$$A = \{{}^{\mathbf{1}}/_x, {}^{\mathbf{1}}/_y\} \Rightarrow p\inf(A) = \mathcal{S}[\bot],$$
$$A = \{{}^{N}/_x, {}^{\mathbf{1}}/_y\} \Rightarrow p\inf(A) = \mathcal{S}[y],$$
$$A = \{{}^{\mathbf{1}}/_x, {}^{\mathbf{1}}/_y\} \Rightarrow p\sup(A) = \mathcal{S}[\top],$$
$$A = \{{}^{\mathbf{1}}/_x, {}^{N}/_y\} \Rightarrow p\sup(A) = \mathcal{S}[x],$$
$$A = \{{}^{N}/_x, {}^{N}/_y\} \Rightarrow p\inf(A) = \mathcal{S}[\top],$$
$$p\inf(X) = \mathcal{S}[\bot] \quad \text{and} \quad p\sup(\emptyset) = \mathcal{S}[\bot].$$

Finally let us define the map $f : X \to X$ as follows: $f(\bot) = \bot, f(x) = y, f(y) = \bot, f(\top) = \bot$. One can easily verify that f is L-fuzzy relatively isotone; for example for $x, \top \in X$ we have:

$$\begin{aligned}&(f(x) \triangle \top) \wedge (x \triangle \top) \wedge (x \triangle f(\top)) = \\ &= (y \triangle \top) \wedge (x \triangle \top) \wedge (x \triangle \bot) = \\ &= \mathbf{1} \wedge \mathbf{1} \wedge N = N \leq (f(x) \triangle f(\top)) = (y \triangle \bot) = N. \quad (49)\end{aligned}$$

The other cases can be verified in an even simpler way. Inequality (49) also shows that even if the map f is L-fuzzy relatively isotone, it is not L-fuzzy isotone. However, at the same time $\mathrm{Fix}(f) = \{\bot\} \neq \emptyset$.

We briefly show in which sense an analogy of relations (14) holds. With respect to reflexivity (15) of the relation $\triangle \in L^{X \times X}$ the following inclusions hold for L-sets (46): $\mathrm{Fix}(f) \subseteq \Omega$ and $\mathrm{Fix}(f) \subseteq \Phi$. Let $\mathcal{S}[x_2] = p\inf(\Omega)$ and $\mathcal{S}[x_3] = p\sup(\Phi)$, then according to Lemma 2(6)(b), relation (26), the following relations hold:

$$p\max(\mathrm{Fix})f)) \,{}^{\mathbf{1}}\!\triangle\, x_3 \quad \text{and} \quad x_2 \,{}^{\mathbf{1}}\!\triangle\, p\min(\mathrm{Fix}(f)). \quad (50)$$

These in fuzzy setting correspond to the first two relations in (14), which directly correspond to relations (3) and (4); other analogies of relations (14) would be similar. And all three parts of the question from the introduction are fully answered.

In the end we show, where lies the problem of the fuzzification of Tarski's theorem in connection with transitivity and validity of relations (3) and (4). In relation (47) there occurs the operation meet $\wedge : L \times L \to L$. The fact, that this operation is idempotent, i.e., $a \wedge a = a$ for every $a \in L$ (and it is known that this is the only operation of appropriate requirements with this property), is fundamental in the proof of Theorem 5. Contrarily, the operation multiplication $\otimes : L \times L \to L$, which occurs in (45), is not idempotent (except for the trivial case $L = 2$, where $\otimes = \wedge$). Generally only the inequality $a \otimes a \leq a$ holds for every $a \in L$. This "trouble" cannot be in principal overcome, as the existing counterexamples show.

References

[1] F. Včelař, Z. Pátíková, *On fuzzification of Tarskis fixed point theorem without transitivity* (submitted to Fuzzy Sets and Systems).
[2] A. Tarski, *A lattice-theoretical fixpoint theorem and its applications*, Pacific Journal of Mathematics 5 (1955), 285–309.
[3] J. Klimeš, *Fixed point characterization of completeness on lattices for relatively isotone mappings*, Archivum Mathematicum 20 (3) (1984), 125–132.
[4] R. Bělohlávek, *Fuzzy Relational Systems: Foundations and Principles*, Kluwer, New York, 2002.
[5] V. Novák, *Fuzzy Sets and Their Applications*, Adam-Hilger, Bristol, 1989.
[6] F. Včelař, *On an approach to the fuzzification of classical Arrow's aggregation problem*, Int. J. of General Systems 2 (1994), 139–154.
[7] R. Bělohlávek, *Lattice-type fuzzy order is uniquely given by its 1-cut: proof and consequences*, Fuzzy Sets and Systems 123 (2004), 447–458.
[8] P. Martinek, *On generalization of fuzzy concept lattices based on change of underlying fuzzy order*, CLA 2008, Proc. of the Sixth International Conference on Concept Lattices and Their Applications, Palacky University, Olomouc (2008), 207–215.
[9] E. Fried, G. Grätzer, *Partial and free weakly associative lattices*, Houston J. of Mathematics 2 (4) (1976), 501–512.