International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:10, No:8, 2016

Quintic Spline Solution of Fourth-Order Parabolic Equations Arising in Beam Theory

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Abstract—We develop a method based on polynomial quintic spline for numerical solution of fourth-order non-homogeneous parabolic partial differential equation with variable coefficient. By using polynomial quintic spline in off-step points in space and finite difference in time directions, we obtained two three level implicit methods. Stability analysis of the presented method has been carried out. We solve four test problems numerically to validate the derived method. Numerical comparison with other methods shows the superiority of presented scheme.

Keywords—Fourth-order parabolic equation, variable coefficient, polynomial quintic spline, off-step points, stability analysis.

I. Introduction

We consider the one dimensional linear fourth-order parabolic equation with variable coefficient of the form:

$$\frac{\partial^2 u}{\partial t^2} + \mu(x)\frac{\partial^4 u}{\partial x^4} = f(x,t), \qquad L_0 \le x \le L_1, \quad t \ge 0, \quad (1)$$

subject to the initial conditions

$$\begin{aligned} &u(x,0) = \phi(x), \\ &u_t(x,0) = \psi(x), \qquad L_0 \le x \le L_1, \end{aligned} \quad (2)$$

with appropriate boundary conditions

$$\begin{aligned} &u(L_0,t) = g_0(t), \quad u(L_1,t) = g_1(t), \\ &u_{xx}(L_0,t) = p_0(t), \quad u_{xx}(L_1,t) = p_1(t), \quad t \ge 0, \end{aligned} \quad (3)$$

where $u = u(x,t)$ is transverse displacement of the beam, $\mu(x) > 0$ is the ratio of flexural rigidity of the beam to its mass per unit length [1], t and x are time and distance variables respectively, $f(x,t)$ is the dynamic driving force per unit mass, and $\phi(x)$, $\psi(x)$, $g_0(t)$, $g_1(t)$, $p_0(t)$ and $p_1(t)$ are continuous functions. This equation represents (in dimensionless terms) the problem of predicting the transient response of a uniform flexible beam clamped at both ends, whose displacements and velocities are initially known [2].

We shall assume that the initial and boundary conditions are given with sufficient smoothness to maintain the order of accuracy of the difference scheme and spline function.

Numerical methods for the solution of (1) have been carried out by many authors. References [3]-[10] used finite difference methods for the numerical solution of transverse vibrations. But we can not apply the above procedures when the bending moment is not prescribed at the end that is $x = L_0$ and L_1.

References [11]-[14], Collatz [8], Jain [3] have proposed both explicit and implicit methods successfully. Parametric quintic spline methods are given by [15], [16] using nodal points.

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All above authors considered the homogenous case of (1) with constant coefficient. The non-homogeneous problem with constant coefficient has been studied by [17] based on parametric quintic spline and by [18] based on sextic spline using nodal points.

Reference [1] developed a family of numerical methods, which are second order accurate in space and time, based on an exact recurrence relation but in a case (1) is homogeneous. The analytic solution of homogeneous case of (1) was obtained by [19] using Adomian decomposition method.

Our aim is to construct a direct numerical method for the solution of (1). In this study, we have derived three level methods of $\mathcal{O}(k^2 + h^2)$ and $\mathcal{O}(k^4 + h^2)$ based on polynomial quintic spline for the solution of fourth-order, nonhomogeneous, parabolic equation with variable coefficient governing the transverse vibrations of a homogeneous rod, using off-step points.

The spline function we present has the form

$$T_4 = Span\{1, x, x^2, x^3, x^4, x^5\}.$$

In Section II, derivation of the polynomial quintic spline and spline relations are given. In Section III, for discretization of (1) we present the formulation of our methods. We obtain truncation error of our scheme and show that by choosing different values of σ we can obtain various methods. In Section IV, stability analysis have been carried out. Finally in Section V, numerical evidence are included to demonstrate the practical usefulness of our methods and confirm their theoretical behavior.

II. Quintic Spline Functions

We give the set of grid points in the interval $[L_0, L_1]$

$$\begin{aligned} x_0 = L_0, \quad x_{l-\frac{1}{2}} = L_0 + (l - \tfrac{1}{2})h, \quad h = \tfrac{L_1 - L_0}{N}, \\ l = 1, 2, ..., N, \quad x_N = L_1. \end{aligned}$$

Definition 1. *A quintic spline function $S_l(x)$, interpolating to a function $u(x)$ on $[L_0, L_1]$ defined as:*

(1) In each subinterval $[x_l, x_{l+1}]$, $S_l(x)$ is a polynomial of degree at most five.

(2) The first-fourth derivatives of $S_l(x)$ are continuous on $[L_0, L_1]$.

(3) $S_l(x_{l-\frac{1}{2}}) = u(x_{l-\frac{1}{2}})$, $l = 1(1)N$.

The spline function $S_l(x)$ for $x \in [x_l, x_{l+1}]$ is defined by

$$S_l(x) = \sum_{j=0}^{5} a_l^{(j)} (x - x_l)^j, \quad l = 0, 1, 2, ..., N, \quad (4)$$

where $a_l^{(j)}$, $j = 0, 1, ..., 5$ are unknown constants to be determine.

We further require that the values of the first-, second-, third- and fourth-order derivatives be the same for the pair of segments that join at each point (x_l, u_l).

To derive expression for the coefficients of (4) in terms of $u_{l-\frac{1}{2}}$, $u_{l+\frac{1}{2}}$, $M_{l-\frac{1}{2}}$, $M_{l+\frac{1}{2}}$, $F_{l-\frac{1}{2}}$ and $F_{l+\frac{1}{2}}$ we first denote:

$$\begin{array}{ll} (i) & S_l(x_{l-\frac{1}{2}}) = u_{l-\frac{1}{2}}, \\ (ii) & S_l(x_{l+\frac{1}{2}}) = u_{l+\frac{1}{2}}, \\ (iii) & S_l''(x_{l-\frac{1}{2}}) = M_{l-\frac{1}{2}}, \\ (iv) & S_l''(x_{l+\frac{1}{2}}) = M_{l+\frac{1}{2}}, \\ (v) & S_l^{(4)}(x_{l-\frac{1}{2}}) = F_{l-\frac{1}{2}}, \\ (vi) & S_l^{(4)}(x_{l+\frac{1}{2}}) = F_{l+\frac{1}{2}}. \end{array} \tag{5}$$

From algebraic manipulation, we get:

$$\begin{aligned} a_l^{(0)} &= \tfrac{1}{768}[5h^4(F_{l-\frac{1}{2}} + F_{l+\frac{1}{2}}) - 48h^2(M_{l-\frac{1}{2}} + M_{l+\frac{1}{2}}) + 384(u_{l-\frac{1}{2}} + u_{l+\frac{1}{2}})], \\ a_l^{(1)} &= \tfrac{1}{5760h}[7h^4(F_{l+\frac{1}{2}} - F_{l-\frac{1}{2}}) + 240h^2(M_{l-\frac{1}{2}} - M_{l+\frac{1}{2}}) + 5760(u_{l+\frac{1}{2}} - u_{l-\frac{1}{2}})], \\ a_l^{(2)} &= \tfrac{1}{32}[-h^2(F_{l-\frac{1}{2}} + F_{l+\frac{1}{2}}) + 8(M_{l-\frac{1}{2}} + M_{l+\frac{1}{2}})], \\ a_l^{(3)} &= \tfrac{1}{144h}[h^2(F_{l-\frac{1}{2}} - F_{l+\frac{1}{2}}) + 24(M_{l+\frac{1}{2}} - M_{l-\frac{1}{2}})], \\ a_l^{(4)} &= \tfrac{1}{48}(F_{l-\frac{1}{2}} + F_{l+\frac{1}{2}}), \\ a_l^{(5)} &= \tfrac{1}{120h}(F_{i+\frac{1}{2}} + F_{l-\frac{1}{2}}), \end{aligned}$$

where $l = 1, 2, ..., N$.

The continuity of first derivative implies

$$M_{l-\frac{3}{2}} + 22M_{l-\frac{1}{2}} + M_{l+\frac{1}{2}} = \tfrac{h^2}{240}(7F_{l-\frac{3}{2}} - 254F_{l-\frac{1}{2}} + 7F_{l+\frac{1}{2}}) + \tfrac{24}{h^2}(u_{l-\frac{3}{2}} - 2u_{l-\frac{1}{2}} + u_{l+\frac{1}{2}}), \quad l = 2(1)N-1, \tag{6}$$

and continuity of third derivative implies

$$M_{l-\frac{3}{2}} - 2M_{l-\frac{1}{2}} + M_{l+\frac{1}{2}} = \tfrac{h^2}{24}(F_{l-\frac{3}{2}} + 22F_{l-\frac{1}{2}} + F_{l+\frac{1}{2}}), \quad i = 2(1)N-1. \tag{7}$$

Subtracting (7) from (6) and simplifying we obtain

$$M_{l-\frac{1}{2}} = \tfrac{1}{h^2}(u_{l-\frac{3}{2}} - 2u_{l-\frac{1}{2}} + u_{l+\frac{1}{2}}) - \tfrac{h^2}{1920}(F_{l-\frac{3}{2}} + 158F_{l-\frac{1}{2}} + F_{l+\frac{1}{2}}). \tag{8}$$

Elimination of M_l's between (7) and (8) leads to the following useful relation:

$$u_{l-\frac{5}{2}} - 4u_{l-\frac{3}{2}} + 6u_{l-\frac{1}{2}} - 4u_{l+\frac{1}{2}} + u_{l+\frac{3}{2}} = \tfrac{h^4}{1920}(F_{l-\frac{5}{2}} + 236F_{l-\frac{3}{2}} + 1446F_{l-\frac{1}{2}} + 236F_{l+\frac{1}{2}} + F_{l+\frac{3}{2}}), \quad l = 3(1)N-2. \tag{9}$$

III. Numerical Method

Let the region $R = [L_0, L_1] \times [0, \infty]$, be discretized by a set of points $R_{h,k}$ which are the vertices of a grid points $(x_{l-\frac{1}{2}}, t_j)$, where $x_{l-\frac{1}{2}} = (l - \frac{1}{2})h$, $l = 1, 2, ..., N$, $Nh = L_1 - L_0$, and $t_j = jk$, $j = 0, 1, 2, 3$. The quantities h and k are mesh size in the space and time directions respectively.

We develop an approximation for (1) in which the time derivative is replaced by a finite difference and the space derivative by the quintic spline function approximation. We need the following finite difference approximation for the time partial derivatives of u. Let:

$$\overline{u}_{tt}{}_l^j = \frac{\delta_t^2}{k^2(1+\sigma\delta_t^2)} u_l^j, \tag{10}$$

where σ is a parameter such that the finite difference approximation to the time derivative is $O(k^2)$ for arbitrary σ and of $O(k^4)$ for $\sigma = \frac{1}{12}$. If we choose $\sigma = \frac{1}{4}$ and $\sigma = \frac{1}{6}$ the finite difference approximations reduce to parametric cubic and cubic spline relations respectively.

At the grid point (l, j), the differential equation (1) may be discretized by:

$$\overline{u}_{tt}{}_l^j + \mu_l \overline{u}_{xxxx}{}_l^j = f_l^j. \tag{11}$$

Equation (11) is then discretized and written in the form

$$\frac{\delta_t^2}{k^2(1+\sigma\delta_t^2)} u_l^j + \mu_l F_l^j = f_l^j, \tag{12}$$

where $F_l^j = S_l^{(4)}(x_l, t_j)$ is the fourth derivative of spline at (x_l, t_j), δ_t is the central difference operator with respect to t so that $\delta_t^2 u_l^j = u_l^{j+1} - 2u_l^j + u_l^{j-1}$. From (12) we have

$$F_l^j = \frac{1}{\mu_l}(f_l^j - \frac{\delta_t^2}{k^2(1+\sigma\delta_t^2)} u_l^j), \tag{13}$$

then we have

$$F_{l\pm\frac{1}{2}}^j = \frac{1}{\mu_{l\pm\frac{1}{2}}}(f_{l\pm\frac{1}{2}}^j - \frac{\delta_t^2}{k^2(1+\sigma\delta_t^2)} u_{l\pm\frac{1}{2}}^j), \tag{14}$$

$$F_{l\pm\frac{3}{2}}^j = \frac{1}{\mu_{l\pm\frac{3}{2}}}(f_{l\pm\frac{3}{2}}^j - \frac{\delta_t^2}{k^2(1+\sigma\delta_t^2)} u_{l\pm\frac{3}{2}}^j), \tag{15}$$

and

$$F_{l-\frac{5}{2}}^j = \frac{1}{\mu_{l-\frac{5}{2}}}(f_{l\frac{5}{2}}^j - \frac{\delta_t^2}{k^2(1+\sigma\delta_t^2)} u_{l-\frac{5}{2}}^j). \tag{16}$$

Substituting (14)-(16) into useful relation (9) we obtain:

$$\begin{aligned} &(1920\sigma\lambda^2 + \tfrac{1}{\mu_{l-\frac{5}{2}}})u_{l-\frac{5}{2}}^{j+1} + (-7680\sigma\lambda^2 + \tfrac{236}{\mu_{l-\frac{3}{2}}})u_{l-\frac{3}{2}}^{j+1} + \\ &(11520\sigma\lambda^2 + \tfrac{1446}{\mu_{l-\frac{1}{2}}})u_{l-\frac{1}{2}}^{j+1} + (-7680\sigma\lambda^2 + \tfrac{236}{\mu_{l+\frac{1}{2}}})u_{l+\frac{1}{2}}^{j+1} + \\ &(1920\sigma\lambda^2 + \tfrac{1}{\mu_{l+\frac{3}{2}}})u_{l+\frac{3}{2}}^{j+1} + [1920(1-2\sigma)\lambda^2 - \tfrac{2}{\mu_{l-\frac{5}{2}}}]u_{l-\frac{5}{2}}^j \\ &+(-7680(1-2\sigma)\lambda^2 - \tfrac{472}{\mu_{l-\frac{3}{2}}})u_{l-\frac{3}{2}}^j + (11520(1-2\sigma)\lambda^2 \\ &-\tfrac{2892}{\mu_{l-\frac{1}{2}}})u_{l-\frac{1}{2}}^j + (-7680(1-2\sigma)\lambda^2 - \tfrac{472}{\mu_{l+\frac{1}{2}}})u_{l+\frac{1}{2}}^j + \\ &(1920(1-2\sigma)\lambda^2 - \tfrac{2}{\mu_{l+\frac{3}{2}}})u_{l+\frac{3}{2}}^j + (1920\sigma\lambda^2 + \tfrac{1}{\mu_{l-\frac{5}{2}}})u_{l-\frac{5}{2}}^{j-1} \\ &+(-7680\sigma\lambda^2 + \tfrac{236}{\mu_{l-\frac{3}{2}}})u_{l-\frac{3}{2}}^{j-1} + (11520\sigma\lambda^2 + \tfrac{1446}{\mu_{l-\frac{1}{2}}})u_{l-\frac{1}{2}}^{j-1} \\ &+(-7680\sigma\lambda^2 + \tfrac{236}{\mu_{l+\frac{1}{2}}})u_{l+\frac{1}{2}}^{j-1} + (1920\sigma\lambda^2 + \tfrac{1}{\mu_{l+\frac{3}{2}}})u_{l+\frac{3}{2}}^{j-1} \\ &= \tfrac{\sigma k^2}{\mu_{l-\frac{5}{2}}}(f_{l-\frac{5}{2}}^{j+1} + f_{l-\frac{5}{2}}^{j-1}) + \tfrac{236\sigma k^2}{\mu_{l-\frac{3}{2}}}(f_{l-\frac{3}{2}}^{j+1} + f_{l-\frac{3}{2}}^{j-1}) + \\ &\tfrac{1446\sigma k^2}{\mu_{l-\frac{1}{2}}}(f_{l-\frac{1}{2}}^{j+1} + f_{l-\frac{1}{2}}^{j-1}) + \tfrac{236\sigma k^2}{\mu_{l+\frac{1}{2}}}(f_{l+\frac{1}{2}}^{j+1} + f_{l+\frac{1}{2}}^{j-1}) + \\ &\tfrac{\sigma k^2}{\mu_{l+\frac{3}{2}}}(f_{l+\frac{3}{2}}^{j+1} + f_{l+\frac{3}{2}}^{j-1}) + \tfrac{k^2(1-2\sigma)}{\mu_{l-\frac{5}{2}}} f_{l-\frac{5}{2}}^j + \tfrac{236k^2(1-2\sigma)}{\mu_{l-\frac{3}{2}}} f_{l-\frac{3}{2}}^j + \\ &\tfrac{1446k^2(1-2\sigma)}{\mu_{l-\frac{1}{2}}} f_{l-\frac{1}{2}}^j + \tfrac{236k^2(1-2\sigma)}{\mu_{l+\frac{1}{2}}} f_{l+\frac{1}{2}}^j + \tfrac{k^2(1-2\sigma)}{\mu_{l+\frac{3}{2}}} f_{l+\frac{3}{2}}^j, \\ &l = 3(1)N-2, \end{aligned} \tag{17}$$

where $\lambda = \frac{k}{h^2}$ is the mesh ratio.

Expanding (17) in Taylor series in terms of $u(x_l, t_j)$ and it's derivatives, and replacing the derivatives involving t by the relation

$$\frac{\partial^{l+j}u(x,t)}{\partial x^l \partial t^j} = -\frac{\partial^{l+2j}u(x,t)}{\partial x^{l+2j}}. \quad (18)$$

We obtain the following truncation error:

$$T_l^j = \frac{1}{24}h^2 D_x^6 - \frac{1}{48}h^3 D_x^7 + \frac{1}{148}h^4 D_x^8 - \frac{1}{607}h^5 D_x^9 + \frac{1}{2315}h^3 D_x^{10} + ..., \quad (19)$$

where $D_x^q = (\frac{\partial^q u}{\partial x^p})_l^j$.

By choosing suitable values of σ we obtain the following methods:

(1) If we choose $\sigma \neq \frac{1}{12}$ in (17) we obtain various schemes of $O(k^2 + h^2)$.

(2) If we choose $\sigma = \frac{1}{12}$ in (17) we obtain a scheme of $O(k^4 + h^2)$.

IV. Stability Analysis and Convergence

The aim of this section is to obtain a valid stability condition of the scheme (17). First we will prove the following Theorem.

Theorem 1. *The scheme (17) for solving (1) is unconditionally stable if $\sigma \geq \frac{1}{4}$, and conditionally stable if $\sigma < \frac{1}{4}$.*

Proof: By using Von Newmann's method. We may assume that the solution of (17) at the grid point (l, j) is of the form:

$$u_l^j = \xi^j e^{li\theta}, \quad (20)$$

where $i = \sqrt{-1}$, θ is real and ξ in general is complex. Substituting (20) in homogenous part of (17), we obtain the following characteristic equation

$$\xi^2 + 2\gamma\xi + 1 = 0, \quad (21)$$

where

$$\gamma = \frac{15360\lambda^2 \sin^4(\frac{\theta}{2})}{\rho} - 1,$$

with

$$\rho = 8(3840\sigma\lambda^2 + \frac{1}{\mu_{l-\frac{5}{2}}} + \frac{1}{\mu_{l+\frac{3}{2}}})\sin^4(\frac{\theta}{2}) - 8(\frac{1}{\mu_{l-\frac{5}{2}}} + \frac{59}{\mu_{l-\frac{3}{2}}} + \frac{59}{\mu_{l+\frac{1}{2}}} + \frac{1}{\mu_{l+\frac{3}{2}}})\sin^2(\frac{\theta}{2}) + (\frac{1}{\mu_{l-\frac{5}{2}}} + \frac{236}{\mu_{l-\frac{3}{2}}} + \frac{1446}{\mu_{l-\frac{1}{2}}} + \frac{236}{\mu_{l+\frac{1}{2}}} + \frac{1}{\mu_{l+\frac{3}{2}}}).$$

We have to show the roots of (21) must be in unite circle.

We apply Routh-Hurwitz Criterion to (21) and get the necessary and sufficient conditions for (17) to be stable as:

$$-1 \leq 1 - \frac{15360\lambda^2 \sin^4(\frac{\theta}{2})}{\rho} \leq 1.$$

After simplification from the left inequality we obtain

$$16[960(4\sigma - 1)\lambda^2 + \frac{1}{\mu_{l-\frac{5}{2}}} + \frac{1}{\mu_{l+\frac{3}{2}}}]\sin^4(\frac{\theta}{2}) - 8(\frac{1}{\mu_{l-\frac{5}{2}}} + \frac{59}{\mu_{l-\frac{3}{2}}} + \frac{59}{\mu_{l+\frac{1}{2}}} + \frac{1}{\mu_{l+\frac{3}{2}}})\sin^2(\frac{\theta}{2}) + (\frac{1}{\mu_{l-\frac{5}{2}}} + \frac{236}{\mu_{l-\frac{3}{2}}} + \frac{1446}{\mu_{l-\frac{1}{2}}} + \frac{236}{\mu_{l+\frac{1}{2}}} + \frac{1}{\mu_{l+\frac{3}{2}}}) \geq 0. \quad (22)$$

In the case $\sigma \geq \frac{1}{4}$, the scheme (17) is unconditionally stable and if $\sigma < \frac{1}{4}$ the scheme is conditionally stable. ■

By using Lax theorem we conclude that: The scheme (17) is convergent as long as stability criterion is satisfied.

V. Numerical Illustrations

We applied the presented schemes to the following fourth-order initial boundary value problems. The presented scheme (17) is an implicit three level scheme, to start any computation, it is necessary to know the solution of u, at $t = -k$. The solution at $t = -k$ may be approximated by using Taylor series expansion of U_l^{-1} and U_l^0 and using the differential equation (1).

Since the initial values of u and u_t are known explicitly at $t = 0$. This implies all their successive tangential derivatives are known at $t = 0$. So that the values of $u, u_x, u_{xx}, ..., u_t, u_{tx}, ...$, are known at $t = 0$.

Following [20] by the help of Taylor expansion, a second-order approximation to u at $t = -k$ can be written as

$$U_l^{-1} = U_l^0 - k(U_t)_l^0 + \frac{k^2}{2}(U_{tt})_l^0 + \mathcal{O}(k^3). \quad (23)$$

Using the initial values, in (1), we can obtain

$$(U_{tt})_l^0 = [f(x,t) - \mu(x)U_{xxxx}(x,t)]_l^0. \quad (24)$$

Thus by using (23) and (24), we may obtain the numerical solution of u at $t = -k$ as follows

$$U_l^{-1} = \phi(lh) - k\psi(lh) + \frac{k^2}{2}[f(lh,0) - \mu(lh)\phi_{xxxx}(lh)] + \mathcal{O}(k^3). \quad (25)$$

The relation (17) gives $N - 4$ linear algebraic equations in the N unknowns $u_{l-\frac{1}{2}}$, for $l = 1, ..., N$. We need four more equations, two at each boundary, for the unique solution of $u_{l-\frac{1}{2}}$, $l = 1, ..., N$.

To use the schemes of order $\mathcal{O}(k^2 + h^2)$ and $\mathcal{O}(k^4 + h^2)$, we use the following equations at boundaries:

$$-6u_0^j + 10u_{\frac{1}{2}}^j - 5u_{\frac{3}{2}}^j + u_{\frac{5}{2}}^j = -\frac{5}{4}h^2(u_{xx})_0^j, \quad l = 1, \quad (26)$$

$$\frac{22}{5}u_0^j - 9u_{\frac{1}{2}}^j + 8u_{\frac{3}{2}}^j - \frac{22}{5}u_{\frac{5}{2}}^j + u_{\frac{7}{2}}^j = -\frac{1}{4}h^2(u_{xx})_0^j, \quad l = 2, \quad (27)$$

$$u_{N-\frac{7}{2}}^j - \frac{22}{3}u_{N-\frac{5}{2}}^j + 8u_{N-\frac{3}{2}}^j + 9u_{N-\frac{1}{2}}^j + \frac{22}{5}u_N^j = -\frac{1}{4}h^2(u_{xx})_N^j, \quad l = N - 2, \quad (28)$$

$$u_{N-\frac{5}{2}}^j - 5u_{N-\frac{3}{2}}^j + 10u_{N-\frac{1}{2}}^j - 6u_N^j = -\frac{5}{4}h^2(u_{xx})_N^j, \quad l = N - 1. \quad (29)$$

The resulting linear system of equations can be solved.

Problem 1. Consider the non-homogenous linear fourth-order parabolic equation with constant coefficient

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = (\pi^4 - 1)\sin\pi x \cos t, \quad 0 \leq x \leq 1, \quad t \geq 0,$$

subject to the initial conditions

$$u(x,0) = \sin\pi x, \quad , u_t(x,0) = 0 \quad 0 \leq x \leq 1,$$

and with appropriate boundary conditions

$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad t \geq 0.$$

The exact solution for this problem is

$$u(x,t) = \sin\pi x \cos t.$$

We solve this problem with step size $h = 0.05$ and $k = 0.005$ giving $\lambda = 2$. By choosing $\sigma = \frac{1}{4}$ and $\sigma = \frac{1}{12}$ in scheme (17). The computed solutions are compared with the exact solution at grid points. The absolute errors are tabulate absolute errors in Table I. We have done all the computations over 10 time steps and then repeat the procedure for $h = 0.05$, $k = 0.00125$ and $\lambda = \frac{1}{2}$ over 16 time steps. We also compare our results with results in [10], [17], [18].

Problem 2. We consider the following homogenous fourth-order parabolic equation

$$\frac{\partial^2 u}{\partial t^2} + (\frac{1}{x} + \frac{x^4}{120})\frac{\partial^4 u}{\partial x^4} = 0, \quad \frac{1}{2} \le x \le 1, \quad t \ge 0,$$

subject to the initial conditions

$$u(x, 0) = 0, \quad , u_t(x, 0) = 1 + \frac{x^5}{120} \quad \frac{1}{2} \le x \le 1,$$

and with appropriate boundary conditions

$$u(\tfrac{1}{2}, t) = \tfrac{3841}{3840}\sin t, \quad u(1, t) = \tfrac{121}{120}\sin t,$$
$$u_{xx}(\tfrac{1}{2}, t) = \tfrac{1}{48}\sin t, \quad u_{xx}(\tfrac{1}{2}, t) = \tfrac{1}{6}\sin t, \quad t \ge 0.$$

The exact solution for this problem is

$$u(x, t) = (1 + \frac{x^5}{120})\sin t.$$

We solve this problem by using scheme (17) given in [9] and [1] with some step size $h\&k$ as:

(i) $h = 0.05$ and $k = 0.000125$ (80 time steps) giving $\lambda = 0.05$,

(ii) $h = 0.05$ and $k = 0.00025$ (40 time steps) giving $\lambda = 0.1$,

(iii) $h = 0.05$ and $k = 0.000625$ (16 time steps) giving $\lambda = 0.25$.

Absolute maximum relative errors at time $t = 0.01$ are given in Table II. Also, we compare our results with results in [1] and [9].

Problem 3. We consider the following homogeneous fourth-order parabolic equation

$$\frac{\partial^2 u}{\partial t^2} + (\frac{x}{\sin x} - 1)\frac{\partial^4 u}{\partial x^4} = 0, \quad 0 \le x \le 1, \quad t \ge 0,$$

subject to the initial conditions

$$u(x, 0) = x - \sin x, \quad , u_t(x, 0) = -(x - \sin x) \quad 0 \le x \le 1,$$

and with appropriate boundary conditions

$$u(0, t) = 0, \quad u(1, t) = e^{-t}(1 - \sin 1),$$
$$u_{xx}(1, t) = 0, \quad u_{xx}(1, t) = e^{-t}\sin 1, \quad t \ge 0.$$

The exact solution for this problem is

$$u(x, t) = (x - \sin x)e^{-t}.$$

We solve this problem with step size $h = 0.05$ and $k = 0.005$ giving $\lambda = 2$. By choosing $\sigma = \frac{1}{4}$ and $\sigma = \frac{1}{12}$ in scheme (17). The computed solutions are compared with the exact solution at grid points. The absolute errors are tabulate absolute errors in Table III. We have done all the computations over 10 time steps and then repeat the procedure for $h = 0.05$, $k = 0.00125$ and $\lambda = \frac{1}{2}$ over 16 time steps.

Problem 4. We consider the following non-homogeneous fourth-order parabolic equation

$$\frac{\partial^2 u}{\partial t^2} + (1+x)\frac{\partial^4 u}{\partial x^4} = (x^3 + x^4 - \frac{6}{7!}x^7)\cos t, \quad 0 \le x \le 1, \quad t \ge 0,$$

subject to the initial conditions

$$u(x, 0) = \frac{6}{7!}x^7, \quad , u_t(x, 0) = 0 \quad 0 \le x \le 1,$$

and with appropriate boundary conditions

$$u(0, t) = 0, \quad u(1, t) = \tfrac{6}{7!})\cos t,$$
$$u_{xx}(1, t) = 0, \quad u_{xx}(1, t) = \tfrac{1}{20}\cos t, \quad t \ge 0.$$

The exact solution for this problem is

$$u(x, t) = \frac{6}{7!}x^7\cos t.$$

We solve this problem with step size $h = 0.05$ and $k = 0.005$ giving $\lambda = 2$. By choosing $\sigma = \frac{1}{4}$ and $\sigma = \frac{1}{12}$ in scheme (17). The computed solutions are compared with the exact solution at grid points. The absolute errors are tabulate absolute errors in Table IV. We have done all the computations over 10 time steps and then repeat the procedure for $h = 0.05$, $k = 0.00125$ and $\lambda = \frac{1}{2}$ over 16 time steps.

TABLE I
ABSOLUTE ERRORS IN DISPLACEMENT FUNCTION $u(x,t)$, $h = 0.05$, IN PROBLEM 1

Methods	λ	Time steps	x=0.1	x=0.2	x=0.3	x=0.4
$O(k^2 + h^2)$	2	10	1.14(-5)	1.75(-5)	1.29(-5)	7.90(-6)
method	0.5	16	5.21(-6)	1.78(-6)	1.15(-6)	8.29(-7)
$O(k^4 + h^2)$	2	10	8.23(-6)	7.47(-6)	1.21(-6)	9.08(-7)
method	0.5	16	4.51(-6)	1.42(-6)	1.08(-6)	8.24(-7)
Evans &	2	10	2.20(-4)	4.10(-4)	5.40(-4)	6.20(-4)
Yousif [10]	0.5	16	2.50(-5)	4.70(-5)	6.60(-5)	7.80(-5)
Aziz	2	10	9.30(-6)	8.00(-6)	2.80(-6)	1.00(-6)
et. al [17]	0.5	16	9.20(-6)	7.90(-6)	2.80(-6)	9.80(-7)
Khan	2	10	1.87(-6)	2.13(-5)	1.49(-5)	8.60(-6)
et. al [18]	0.5	16	9.07(-6)	7.79(-6)	2.75(-6)	1.01(-6)

TABLE II
MAXIMUM RELATIVE ERROR MODULI AT TIME $t = 0.01$ IN SOLUTION OF PROBLEM 2

λ	$O(k^2 + h^2)$	$O(k^4 + h^2)$	In [9]	In [1]
0.05	3.86(-7)	2.87(-7)	1.90(-6)	3.30(-7)
0.1	3.82(-7)	2.73(-7)	7.20(-7)	3.30(-7)
0.25	3.41(-7)	2.42(-7)	4.10(-7)	3.30(-7)

TABLE III
ABSOLUTE ERRORS IN DISPLACEMENT FUNCTION $u(x,t)$, $h = 0.05$, IN PROBLEM 3

Methods	λ	Time steps	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$
$O(k^2 + h^2)$	2	10	1.06(-7)	3.47(-8)	8.65(-8)	2.18(-7)
method	0.5	16	1.02(-7)	1.69(-8)	6.35(-8)	1.36(-8)
$O(k^4 + h^2)$	2	10	1.04(-7)	3.28(-8)	8.40(-8)	2.11(-7)
method	0.5	16	1.01(-7)	1.30(-8)	6.12(-8)	1.27(-8)

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:10, No:8, 2016

TABLE IV
ABSOLUTE ERRORS IN DISPLACEMENT FUNCTION $u(x,t)$, $h = 0.05$, IN PROBLEM 4

Methods	λ	Time steps	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$
$O(k^2 + h^2)$	2	10	3.29(-9)	3.40(-9)	1.53(-8)	7.79(-8)
method	0.5	16	1.37(-9)	1.71(-10)	3.43(-10)	5.23(-10)
$O(k^4 + h^2)$	2	10	2.21(-9)	4.59(-10)	4.75(-9)	6.22(-9)
method	0.5	16	1.18(-9)	1.34(-10)	3.39(-10)	5.05(-10)

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