

# Integer Programming Model for the Network Design Problem with Facility Dependent Shortest Path Routing

Taehan Lee

*Abstract*—We consider a network design problem which has shortest routing restriction based on the values determined by the installed facilities on each arc. In conventional multicommodity network design problem, a commodity can be routed through any possible path when the capacity is available. But, we consider a problem in which the commodity between two nodes must be routed on a path which has shortest metric value and the link metric value is determined by the installed facilities on the link. By this routing restriction, the problem has a distinct characteristic. We present an integer programming formulation containing the primal-dual optimality conditions to the shortest path routing. We give some computational results for the model.

*Keywords*—Integer programming, multicommodity network design, routing, shortest path.

## I. INTRODUCTION

NETWORK design is one of the most intensively studied topics in combinatorial optimization. The main concern is on the amount of capacity or facility to be installed on the links of underlying network such that the given traffic requirements among nodes are satisfied. Minimum cost multicommodity network design problem is to find a minimum cost facility installation to ensure that all given commodities can flow simultaneously [3], [5], [6]. In particular, the minimum cost multicommodity network design problem with discontinuous step increasing cost functions is a basic model in telecommunication network design [2], [4], [8], [10]. In the problem, the commodities can be routed any possible route under the capacity of installed facilities.

Lee et al. [9] suggested a network design problem which has a routing restriction depend on the installed link facilities for a internet network design. When a physical network, commodities and available link facilities are given, the problem selects the facility to be installed on each link. But, the commodity between two nodes must be routed on a path that has shortest metric value determined by the link facilities. The installed link facilities must accommodate all traffic requirements and the objective is to minimize the cost of the installed facilities. Under this routing restriction, the problem has a special structure and it is hard to handle. Lee et al. [9] showed that it is hard to define a neighborhood structure of a feasible solution and then they proposed a genetic algorithm to solve it.

T. Lee is with the Department of Industrial and Information Systems Engineering / RCIT, Chonbuk National University, Jeonju, 54896, Korea (e-mail: myth0789@jbnu.ac.kr).

Kara and Verter [7] performed a study on the hazmat network design problem which has the similar routing restriction [7]. When the possible road network and the required transportations (commodities) of hazmat material are given, the problem selects arcs to be opened for hazmat transportation. Then, the routing of hazmat commodities may be routed through a shortest length path on the opened road network. The objective is to minimize the total risk. Then, the problem selects the arcs to be opened under the shortest path routing restriction. They give a bilevel mathematical formulation for the problem. But the model cannot be solved directly because the model has too many variables and constraints. They give an algorithm which constructs a tree network and add some more arcs to the tree.

In this paper, we give an integer programming formulation for the problem and some computational results for randomly generated instances. We call the problem network design problem with facility dependent shortest path routing (NDPFDR).

## II. PROBLEM DESCRIPTION

In this section, we give a detailed explanation of the problem considered in this paper. When physical network  $G$ , traffic (commodity) matrix and the link facilities are given, the problem is to find minimum cost link facility installation plan to meet the traffic requirements. In typical network design problems, the routing for traffic requirements does not depend on the type of the link facility. So, we can select the link facility for each link to accommodate all traffic requirements with minimum cost. But, in some applications, the routing of traffic is restricted. We consider a problem has the routing restriction such that the route for a commodity between two nodes should be a shortest metric path when the link metric values are given by the installed link facilities. Then, the routes for commodities can be determined only after the link facility is decided for each link. This routing restriction makes the problem differ from typical network design problems.

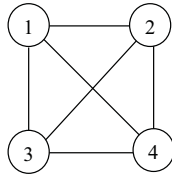
When a feasible facility installation is given, let's consider the case that an installed facility on a link is changed to another one. Under no routing restriction, if the changed facility has a larger capacity then the changed solution is also a feasible solution. On the other hand, the changed solution can be an infeasible solution to NDFDRP. Moreover, a solution whose all links have the maximum capacity facility can be an infeasible solution.

$$D = \begin{pmatrix} 0 & 4 & 6 & 5 \\ 4 & 0 & 5 & 7 \\ 6 & 5 & 0 & 3 \\ 5 & 7 & 3 & 0 \end{pmatrix}$$

(a) Traffic matrix

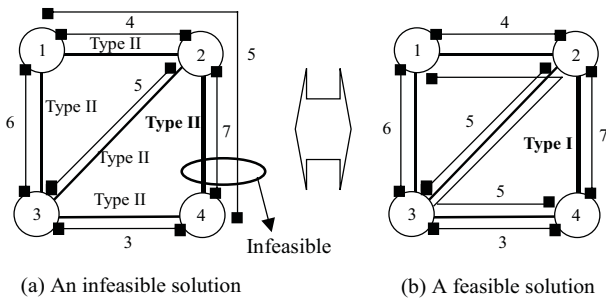
Type	Capacity	Metric	Cost
I	5	10	10
II	10	5	20

(b) Link facilities



(c) Physical network

Fig. 1 Example: Input data



(a) An infeasible solution

(b) A feasible solution

Fig. 2 Two solutions to the example

Lee et al. [9] gave an example in Fig. 1 to show this special structure of the problem. Suppose that the traffic matrix and physical network are given like in Figs. 1 (a) and (c) and we consider two link facilities whose cost, metric and capacity are given in Fig. 1 (b). Now, we consider two solutions in Fig. 2. Fig. 2 (a) is a solution such that the link facility which has the larger capacity (type II) is installed on every links. But the solution is infeasible because the capacity of link (2, 4) is not sufficient under the shortest metric routing. On the other hand, Fig. 2 (b) is a solution such that the smaller capacity (type I) is installed on the link (2, 4). But the solution is a feasible solution. This is caused by the routing rule. The solution Fig. 2 (a) also has the capacity sufficient to accommodate all traffic if the route for the traffic between node 1 and 4 is the same as that in Fig. 2 (b). This means that a solution obtained by increasing a link capacity from a feasible solution could be an infeasible solution.

It is assumed that at most one link facility can be installed on a link. It could be thought as an unreasonable assumption. But, link facility  $t$  does not mean one real facility but a combination of several real facilities. In other word, there are some candidates of facility combinations and we select one among them for each link.

### III. MATHEMATICAL FORMULATION

In this section, we propose an integer programming formulation for NDPFDR and give some comments. First, we

define notations and decision variables.

$V$  : set of nodes in  $G$

$E$  : set of undirected links  $(i,j)$  in  $G$

$D$  : traffic matrix

$T$  : set of link facilities

$c^t$  : cost of the type  $t$  facility,  $t = 1, 2, \dots, |T|$

$l_e^t$  : metric of the type  $t$  facility when installed at link  $e$ ,  $e \in E, t = 1, 2, \dots, |T|$

$u^t$  : capacity of the type  $t$  facility,  $t = 1, 2, \dots, |T|$ . (Assume that  $u^t < u^{t+1}$ )

$o_k$  : source node of commodity  $k \in K$

$d_k$  : destination node of commodity  $k \in K$

$t_k$  : traffic requirements for  $k$

$w^t$  : metric value of facility  $t$

$y_{ij}^t$  : Binary variable denoting whether the facility  $t$  is installed on link  $(i,j) \in E$

$z_{ij}^k$  : Binary variable denoting whether the shortest path for commodity  $k$  passes arc  $(i,j)$

$x_i^k$  : (distance) Label of node  $i$  for demand pair  $k$

Note that the links to be installed a facility are undirected and a commodity flows through a directed path on the network. Thus, the possible directed arc set is denoted by  $A$  and then  $A$  contains two directed arc  $(i,j), (j,i)$  for each link  $(i,j) \in E$ . With the above notations and decision variables, we can formulate (NDPFDR) as follows.

(NDPFDR)

$$\begin{aligned} \min \quad & \sum_{t \in T} \sum_{e \in E} c^t y_e^t \\ \text{s.t.} \quad & \sum_{t \in T} y_e^t = 1 \end{aligned} \quad (1)$$

$$x_j^k - x_i^k \leq \sum_{t \in T} w^t y_{ij}^t + M(1 - \sum_{t \in T} y_{ij}^t) \quad \forall e \in E \quad (2)$$

$$x_i^k - x_j^k \leq \sum_{t \in T} w^t y_{ji}^t + M(1 - \sum_{t \in T} y_{ji}^t) \quad \forall k \in K, (i,j) \in E \quad (3)$$

$$x_j^k - x_i^k \geq \sum_{t \in T} w^t y_{ij}^t - M(1 - z_{ij}^k) \quad \forall k \in K, (i,j) \in E \quad (4)$$

$$x_i^k - x_j^k \geq \sum_{t \in T} w^t y_{ji}^t - M(1 - z_{ji}^k) \quad \forall k \in K, (i,j) \in E \quad (5)$$

$$\sum_{(o_k, j) \in A} z_{o_k j}^k - \sum_{(j, o_k) \in A} z_{j o_k}^k = 1 \quad \forall k \in K \quad (6)$$

$$\sum_{(d_k, j) \in A} z_{d_k j}^k - \sum_{(j, d_k) \in A} z_{j d_k}^k = -1 \quad (7)$$

$$\forall k \in K$$

$$\sum_{(i, j) \in A} z_{ij}^k - \sum_{(j, i) \in A} z_{ji}^k = 0 \quad (8)$$

$$\forall k \in K, i \in N \setminus \{o_k, d_k\}$$

$$\sum_{k \in K} t^k (z_{ij}^k + z_{ji}^k) \leq \sum_{t \in T} u^t y_{ij}^t \quad (9)$$

$$\forall (i, j) \in E$$

$$y_{ij}^t, z_{ij}^k \in \{0, 1\}, x_i^k \geq 0$$

Constraints (1) mean that at most one facility can be installed on each link. Constraints (6)-(8) ensure that  $z_{ij}^k$ 's form a directed path for each commodity  $k \in K$ . Constraints (9) ensure that the total flow on a link does not exceed the capacity of installed link facility and a commodity can be flowed an arc only when a facility is installed on the link. By (6)-(8), a directed path from  $o_k$  to  $d_k$  is used for commodity  $k$  but it does not guarantee that the path is a shortest metric path. Constraints (2)-(5) ensure that the path has the shortest metric value for each commodity. Variables  $x_j^k (j \in N)$  are dual variables of the shortest path problem for commodity  $k$  and the constraints are the optimality condition for shortest path problem on directed network. (2) and (3) ensure that the solution is dual feasible to the shortest path problem for each commodity and (4) and (5) make the variables hold the complementary slackness conditions. Suppose that a facility is installed on a link, then a path for the commodity  $k$  satisfy the dual feasibility condition,  $x_j^k \leq x_i^k + \text{length of } (i, j)$  (link metric value determined by the facility installed on the link) by (2) and (3) for each direction. Moreover, the path pass through the link then the condition holds with equality by (4) and (5). For more details, refer to [1].

#### A. Strengthen the LP Relaxation

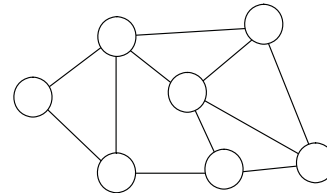
The LP relaxation by dropping the integrality condition from the above formulation gives a lower bound on the optimal objective value. Note that at most one of  $z_{ij}^k$  and  $z_{ji}^k$  can be 1 only when  $y_{ij} = 1$  because a path for a commodity passes a link only when a link facility is installed on the link. Thus, the following constraints are valid inequalities.

$$z_{ij}^k + z_{ji}^k \leq \sum_{t \in T} y_{ij}^t \quad \forall k \in K, (i, j) \in E \quad (10)$$

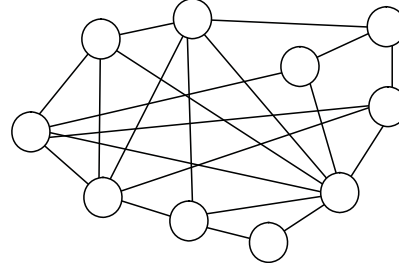
Moreover, we can easily show that the some solutions to the LP relaxation of (NDPFDR) cannot satisfy equation (10). Thus, we can strengthen the LP relaxation bound by adding (10).

#### IV. COMPUTATIONAL EXAMPLES

To test the proposed model, we apply the model on three small networks which has 7, 10, and 13 nodes, respectively. Two test network are in Fig. 3. We randomly generated the commodities between node in networks and solve the problem



(a) 7-node network



(b) 10-node network

Fig. 3 Test Networks

TABLE I  
EXPERIMENTAL RESULTS

$(n,  E ,  K )$	Opt	LP1	LP2	Time(sec)
(7,11,42)	100	54.9	86.0	11.14
(10,22,30)	105	51.0	83.83	131.84
(13,25,37)	135	71.5	115.5	264.404

with the proposed model. We used the callable library the callable library of CPLEX12.6 for c++ to solve the model. The results are summarized in Table I. In the table, the first column is the problem size. Three values mean the number of nodes, the number of arcs, and the number of commodities, respectively. The amount of each commodity is randomly generated between 1 to 5. We used three link facilities which has the following cost, metric value, and capacity.

$$\begin{aligned} l^1 &= 45, l^2 = 4, l^3 = 2 \\ c^1 &= 10, c^2 = 15, c^3 = 70 \\ u^1 &= 2, u^2 = 4, u^3 = 45 \end{aligned}$$

The second column in the table is the optimal objective value. The third and fourth column is the objective value of LP relaxations. The LP1 is the objective value from the LP relaxation without constraints (10) and LP2 is that from the LP relaxation contains constraints (10).

The results show that the model gives an optimal solution for each problem. The constraints (10) reduce the gap by about 30 %. But, we cannot solve the bigger problems because the computation time is much increased. Thus, some studies on primal heuristics and finding some valid inequalities would be a way to get some better results.

#### V. CONCLUSIONS

In this paper, we give an integer programming model for the network design problem which has the routing restriction. The

model contains primal variables, dual variables for the shortest path routing and the complementary slackness conditions to satisfy the routing restriction. We strengthened the model with a valid inequality and solve the model with CPLEX. When the number of commodities is increased, the model size is larger and it needs too much time to solve the model. Thus, the study on the polyhedron and some heuristic algorithms will be another research work.

#### ACKNOWLEDGEMENT

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