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A Qualitative Description of the Dynamics in the Interactions between Three Populations: Pollinators, Plants, and Herbivores

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Abstract—In population dynamics the study of both, the abundance and the spatial distribution of the populations in a given habitat, is a fundamental issue a From ecological point of view, the determination of the factors influencing such changes involves important problems. In this paper a mathematical model to describe the temporal dynamic and the spatiotemporal dynamic of the interaction of three populations (pollinators, plants and herbivores) is presented. The study we present is carried out by stages: 1. The temporal dynamics and 2. The spatio-temporal dynamics. In turn, each of these stages is developed by considering three cases which correspond to the dynamics of each type of interaction. For instance, for stage 1, we consider three ODE nonlinear systems describing the pollinator-plant, plant-herbivore and plant-pollinator-herbivore, interactions, respectively. In each of these systems different types of dynamical behaviors are reported. Namely, transcritical and pitchfork bifurcations, existence of a limit cycle, existence of a heteroclinic orbit, etc. For the spatiotemporal dynamics of the two mathematical models a novel factor are introduced. This consists in considering that both, the pollinators and the herbivores, move towards those places of the habitat where the plant population density is high. In mathematical terms, this means that the diffusive part of the pollinators and herbivores equations depend on the plant population density. The analysis of this part is presented by considering pairs of populations, i. e., the pollinator-plant and plant-herbivore interactions and at the end the two mathematical model is presented, these models consist of two coupled nonlinear partial differential equations of reaction-diffusion type. These are defined on a rectangular domain with the homogeneous Neumann boundary conditions. We focused in the role played by the density dependent diffusion term into the coexistence of the populations. For both, the temporal and spatio-temporal dynamics, a several of numerical simulations are included.

Keywords—Bifurcation, heteroclinic orbits, steady state, traveling wave.

I. Introduction

THE objective of this paper is to study the pollinator-plant-herbivore interaction and to understand the factors that influence the changes of these populations. These changes may be observed in the density, distribution, etc., of the above mentioned populations. Several authors address this issue by considering the interaction of two or three populations from different points of view see e.g. [2], [5], [6], [9]- [12], [14]- [16]. Throughout this paper we will study this issue based on some mathematical models that consist of coupled nonlinear differential equations. These mathematical models are derived by using some facts ecologically supported (see [4] and [10]). Such models are characterized by the interaction terms that are described by a Holling response of type II. These models have been studied by several authors, (see [2], [6], [8]- [10], [16]).

The structure of this study will be as follows: In the second section we are going to study the pollinator-plant interaction by the means of two models. In first model we will show the qualitative behavior of the temporal dynamics of such interaction. The analysis we present here complements and extends the analysis carried out by other authors (see [2] and [10]). To study the spatiotemporal dynamics of this populations, we will present a second model. This model considers the movement made by the pollinators towards those places of the habitat where the plant population density is high. Further in this paper, we will include some results which come from our numerical simulations.

In the following section we are going to study the plant-herbivore interaction and will be developed into two parts. During the first part we will show the qualitative behavior of temporal dynamic by using a mathematical model. The novelty of this model lies in the fact that the authors in [9] consider the plant-herbivore interaction as a predator-prey interaction types. The second part, for the study of spatiotemporal dynamics, we present the mathematical model that considers the basic principle of the movement executed by the pollinators those places of the habitat where the plant population density is high, is presented. The results of the numerical simulations based on the investigation of the coexistence between both populations show very interesting results.

In the fourth and last section we will show the results brought by the qualitative analysis of the temporal dynamics of interaction made by the three populations by using a mathematical model. The construction of model can be seen in [2] and [9]. It is important to realize that despite the fact that it is difficult to model two populations, to include a third population its a even more difficult the task. For instance, the incorporation a third species (predator or other) for avoid the unlimited growth reflected in the model of mutualistic populations, may be unrealistic (see [1]). On the other hand, the idea of incorporating a third population to a mutualistic model in order to study the effects that may be originated, could be considered a more realistic perspective, see [2], [15] and [16]. This is the innovation of the proposed model a fresh

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point of view to avoid staying in a non realistic situation.

Finally, we will present the conclusions of the analysis focusing on the coexistence of the three populations. In each section we will give the biological interpretation of the results.

II. The Mathematical Model of Pollinator-Plant Interaction

The temporal dynamics of the pollinator-plant mathematical models have been raised from two points of view, which are: Deterministic (see [11] and [12]) and probabilistic (see [14]) models. Later, the mathematical models were constructed to describe such interaction including the spatial part (see [9] and [5]). We will focus on the deterministic models.

This section is divided into two parts. In the first part we begin by exposing the results of different types of dynamical behaviors of a temporal model. Namely, transcritical and pitchfork bifurcations and the existence of heteroclinic orbits.

The second part is for the spatiotemporal dynamics a mathematical model with a novel factor is introduced. This takes into consideration that the pollinators, move towards those places in the habitat where the plant population density is high. In this part we will show some of the effects, obtained while considering the spatial part by the numerical simulation.

A. Temporal Dynamic

Most models of pollinator-plant interaction exhibit unrealistic behavior, such as an unlimited growth of both populations, or that some terms do not have biological meaning, to name a few (see [1] and [11]). The model we are going to study next incorporates several characteristics that are based on experimental observations (see [4] and [10]). This model does not represents an unrealistic behavior. The construction of this model can be seen in [2] and [10]. Let $a = a(t)$ and $p = p(t)$ represent the population density of pollinators and plants at time t, respectively. The mathematical model is:

$$\begin{aligned} \dot{a} &= a(k-a) + \frac{k_2\sigma\mu^2 ap}{1+\phi\sigma\mu^2 p} \\ \dot{p} &= -\gamma p + \frac{k_1\sigma\mu ap}{1+\phi\sigma\mu^2 p} \end{aligned} \tag{1}$$

where the dot on a and p denotes the derivative with respect to time. All the parameters appearing in system (1) have an important ecological interpretation, (see [9] and [10]). Let

$$\mu_1 = \frac{\gamma}{\sigma k_1 k} \quad \text{and} \quad \mu_0 = \frac{4\gamma k_2}{\phi k_1 \sigma \left(k + \frac{k_2}{\phi}\right)^2}$$

Then, Table I shows the existence of the equilibrium points, based on the parameter values of the system (1).

All the points of table (I) exists in the positive quadrant, except when:

- If $\mu_1 = \mu$ and $k < \dfrac{k_2}{\phi}$, then (a_1^*, p_1^*) is in the positive quadrant.
- If $\mu_1 = \mu$ and $k > \dfrac{k_2}{\phi}$, then (a_1^*, p_1^*) is in the fourth quadrant.

TABLE I
The Equilibrium Points' Classification in System (1)

$\mu < \mu_0$		$(0,0)$	$(k,0)$		
$\mu = \mu_0$	If $k \geq \frac{k_2}{\phi}$	$(0,0)$	$(k,0)$		
	If $k < \frac{k_2}{\phi}$	$(0,0)$	$(k,0)$	(a_0^*, p_0^*)	
$\mu_0 < \mu < \mu_1$	If $k \geq \frac{k_2}{\phi}$	$(0,0)$	$(k,0)$		
	If $k < \frac{k_2}{\phi}$	$(0,0)$	$(k,0)$	(a_1^*, p_1^*)	(a_2^*, p_2^*)
$\mu_1 = \mu$	If $k = \frac{k_2}{\phi}$	$(0,0)$	$(k,0)$		
	If $k \neq \frac{k_2}{\phi}$	$(0,0)$	$(k,0)$	(a_1^*, p_1^*)	
$\mu_1 < \mu$	If $k \neq \frac{k_2}{\phi}$	$(0,0)$	$(k,0)$	(a_1^*, p_1^*)	(a_2^*, p_2^*)
	If $k = \frac{k_2}{\phi}$	$(0,0)$	$(k,0)$		

- If $\mu_1 < \mu$ and $k \neq \dfrac{k_2}{\phi}$, then (a_2^*, p_2^*) is in the fourth quadrant.

The classification of the local stability of each of the equilibrium points is:

- The equilibrium point $(0,0)$ is unstable (just like a saddle), for all values of the positive parameters.
- The stability of the equilibrium point $(k,0)$ is classified according to the associated eigenvalues of the Jacobian matrix of system (1), evaluated at the equilibrium point $(k,0)$. The Jacobian matrix is:

$$J_\mu = \begin{bmatrix} -k & kk_2\mu^2\sigma \\ 0 & kk_1\mu\sigma - \gamma \end{bmatrix} \tag{2}$$

The Jacobian matrix J_μ is an upper triangular matrix, which has as eigenvalue:

$$\lambda_1 = kk_1\mu\sigma - \gamma \quad \lambda_2 = -k \tag{3}$$

Thus, the stability of the equilibrium point $(k,0)$ is classified as follows:

  - If $\mu < \mu_1$, then the equilibrium point $(k,0)$ is locally asymptotically stable.
  - If $\mu = \mu_1$, then the equilibrium point $(k,0)$ is nonhyperbolic.
  - If $\mu > \mu_1$, then the equilibrium point $(k,0)$ is unstable, (saddle).

Using Table (I) and results from above, we conclude the following results.

Theorem 1: Let $\mu > 0$.

- If $k_2 \neq k\phi$, then in system (1) a transcritical bifurcation takes place. The point of bifurcation is: $((k,0);\mu_1)$.
- If $k_2 = k\phi$, then in system (1) a pitchfork bifurcation takes place. The point of bifurcation is: $((k,0);\mu_1)$.

Proof: To prove theorem (1), we will consider the system:

$$\begin{aligned} \dot{a} &= a(k-a)(1+\phi\sigma\mu^2 p) + k_2\sigma\mu^2 ap \\ \dot{p} &= -\gamma p(1+\phi\sigma\mu^2 p) + k_1\sigma\mu ap \end{aligned} \tag{4}$$

Systems (1) and (4) are topologically equivalent in $\mathbb{R}^2_+$. Later we apply the translation: $a_0 = a - k$, $p_0 = p$ and $\mu_{11} = \mu - \mu_1$. Finally, we apply a suitable linear transformation (see [7] and

[13]) to get a system of the form $(\dot{x} = Jx + F(x))$ as shown next:

$$\begin{aligned} \dot{a}_1 &= f_1(a_1, p_1, \mu_{11}) \\ \dot{p}_1 &= f_2(a_1, p_1, \mu_{11}) \end{aligned} \tag{5}$$

where[1] and μ_{11} is the new parameter. Now we include μ_{11} as a new variable in system (5), like:

$$\begin{aligned} \dot{a}_1 &= f_1(a_1, p_1, \mu_{11}) \\ \dot{p}_1 &= f_2(a_1, p_1, \mu_{11}) \\ \dot{\mu}_{11} &= 0 \end{aligned} \tag{6}$$

System (6) has an equilibrium point at $(a_1, p_1, \mu_{11}) = (0, 0, 0)$, The Jacobian matrix's eigenvalues of the linear approximation of the system (6) are 0, $-k$ and 0 and hence by the center manifold theorem, system (6) has a two dimensional center manifold and can be represented by the graph of a function depending on a_1 and μ_{11} variables. This is:

$$\begin{aligned} W^c = \{&(a_1, p_1, \mu_{11}) \in \mathbb{R} | p_1 = r_1(a_1, \mu_{11}), r_1(0,0) = 0, \\ &Dr_1(0,0) = 0\} \end{aligned} \tag{7}$$

where a_1 and μ_{11} the variables are small enough. Now we are going to find an approximation to the center manifold depending on the parameter μ_{11}. We suppose that the form of the center manifold is:

$$r_1(a_1, \mu_{11}) = c_1 a_1^2 + c_2 a_1 \mu_{11} + c_3 \mu_{11}^2 + \cdots \tag{8}$$

where the coefficients c_i are determined next. From (8), we have:

$$\dot{p}_1 = \frac{dr_1}{da_1}\dot{a}_1 + \frac{dr_1}{d\mu_{11}}\dot{\mu}_{11} \tag{9}$$

We substitute (8) in (5) and using (9) we obtain:

$$f_2(a_1, r_1(a_1, \mu_{11}), \mu_{11}) = \frac{dr_1}{da_1} f_1(a_1, r_1(a_1, \mu_{11}), \mu_{11}) \tag{10}$$

this is,

$$\frac{dr_1}{da_1} f_1(a_1, r_1(a_1, \mu_{11}), \mu_{11}) - f_2(a_1, r_1(a_1, \mu_{11}), \mu_{11}) = 0 \tag{11}$$

equating terms of same power to zero, we find:

$$\begin{aligned} c1 &= -\frac{k2(\gamma + kk_1\mu_{11}\sigma)^4(\gamma(k_2 - k\phi) + b^*)}{k^4 k_1^4 \sigma^2 (2k_1\mu_{11}\sigma + 1)} \\ c2 &= 0 \\ c3 &= 0 \\ b^* &= k^2\phi(1 + k_1\mu_{11}\sigma) - \gamma k k_1 \mu_{11}\phi\sigma. \end{aligned}$$

Substituting these values into (8), we obtain:

$$r_1(a_1, \mu_{11}) = c_1 a_1^2 + \cdots \tag{12}$$

subsequently, by substituting (12) into (6), we obtain the reduced vector field on the center manifold:

$$\begin{aligned} \dot{a}_1 &= f_{11}(a_1, r_1(a_1, \mu_{11}), \mu_{11}) \\ \dot{\mu}_1 &= 0 \end{aligned} \tag{13}$$

[1]Here f_1 and f_2 are not written because they are very large expressions.

where

$$\begin{aligned} f_{11} &= a_1 \frac{2k^6 k_1^6 \mu_{11}^2 \sigma^4 + k^6 k_1^5 \mu_{11}\sigma^3}{k^5 k_1^4 \sigma^2 (2k_1\mu_{11}\sigma + 1)} \\ &+ a_1^2 \frac{k^3 k_1^2 \sigma (2k_1\mu_{11}\sigma + 1)(\gamma + kk_1\mu_{11}\sigma)^2 c^*}{k^5 k_1^4 \sigma^2 (2k_1\mu_{11}\sigma + 1)} \\ &+ \cdots \end{aligned}$$

where $c^* = \gamma(k_2 - k\phi) + kk_1\mu_{11}\sigma(k_2 - \gamma\phi)$, with

$$f_{11}(0,0) = 0 \tag{14}$$

$$\frac{\partial f_{11}}{\partial a_1}(0,0) = 0 \tag{15}$$

$$\frac{\partial f_{11}}{\partial \mu_{11}}(0,0) = 0 \tag{16}$$

$$\frac{\partial^2 f_{11}}{\partial \mu_{11} \partial a_1}(0,0) = kk_1\sigma \tag{17}$$

$$\frac{\partial^2 f_{11}}{\partial a_1^2}(0,0) = \frac{2\gamma^3(k_2 - k\phi)}{k^2 k_1^2 \sigma} \tag{18}$$

$$\frac{\partial^3 f_{11}}{\partial a_1^3}(0,0) = \frac{-6\gamma^5\phi^2}{k^2 k_1^4 \sigma^2} \tag{19}$$

Since the all parameters' values are positive, then $kk_1\sigma \neq 0$ and $\frac{-6\gamma^5\phi^2}{k^2 k_1^4 \sigma^2} \neq 0$.

If $k_2 - k\phi \neq 0$, then (14)-(18) show that the orbit structure near $((0,0); \mu_{11})$ of system (5) is qualitatively the same as the orbit structure near $((0,0); \mu_{11})$ of $\dot{a}_1 = \mu_{11}a_1 \mp a_1^2$. Therefore, transcritical bifurcation takes place in system (1).

If $k_2 - k\phi = 0$, and (14)-(19) show that the orbit structure near $((0,0); \mu_{11})$ of system (5) is qualitatively the same as the orbit structure near $((0,0); \mu_{11})$ of $\dot{a}_1 = \mu_{11}a_1 \mp a_1^3$. Therefore pitchfork bifurcation takes place in system (1). ■

In Fig. 1 we observed the transcritical bifurcation of system (1) for some values of the parameters[2]. These values were taken from [9]. Now we state and show of the following theorem.

Theorem 2: If $k < \frac{k_2}{\phi}$, then in system (1) a saddle-node bifurcation takes place. The bifurcation point is $((a_0^*, p_0^*); \mu_0)$, where

$$a_0^* = \frac{1}{2}\left(k + \frac{k_2}{\phi}\right), \quad p_0^* = \frac{k_1^2\sigma(k_2 + k\phi)^3(k_2 - k\phi)}{16\gamma^2 k_2^2 \phi^3}.$$

Proof: To prove bifurcation we using the Sotomayor's theorem (see [7], pp $338 - 339$). The equilibrium point is (a_0^*, p_0^*), The bifurcation parameter is μ and the bifurcation value is $\mu = \mu_0$. Let F be the vector. This vector is right hand side of system (1), namely,

$$F((a,p);\mu) = \begin{bmatrix} a(k-a) + \dfrac{k_2\sigma\mu^2 ap}{1 + \phi\sigma\mu^2 p} \\ -\gamma p + \dfrac{k_1\sigma\mu ap}{1 + \phi\sigma\mu^2 p} \end{bmatrix} \tag{20}$$

the derivative of vector F with respect to μ is:

$$F_\mu((a,p);\mu) = \begin{bmatrix} \dfrac{2ak_2\mu p\sigma}{(p\phi\sigma\mu^2 + 1)^2} \\ -\dfrac{ak_1 p\sigma(\mu^2 p\phi\sigma - 1)}{(p\phi\sigma\mu^2 + 1)^2} \end{bmatrix} \tag{21}$$

[2]We used the software MATLAB to obtain the numerical solutions for all the ODE systems presented in this paper, also we used symbolic computation.

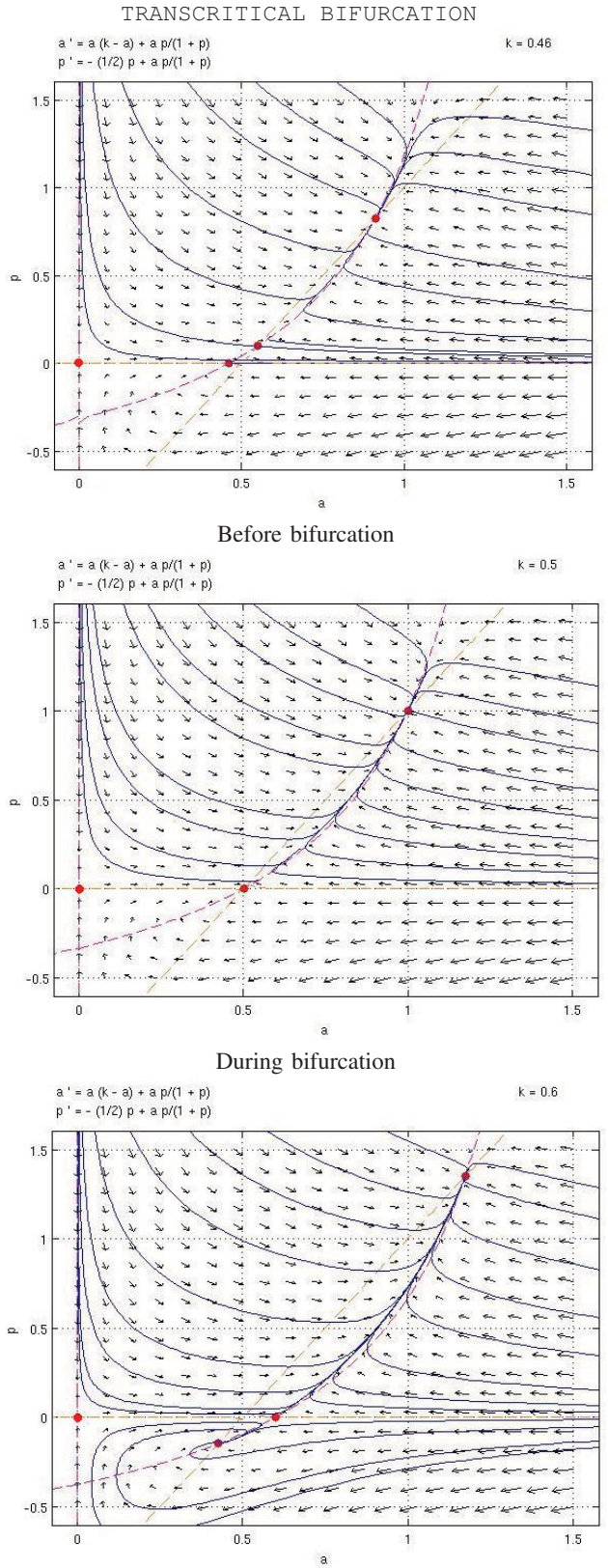


Fig. 1 Phase Portraits of the system (1)

The Jacobian matrix of the system (1) at equilibrium point $((a_0^*, p_0^*); \mu_0)$ is:

$$J_{\mu_0} = \begin{bmatrix} -\frac{k}{2} - \frac{k_2}{2\phi} & \frac{2\gamma^2 k_2 \phi}{k_1^2 \sigma (k_2 + k\phi)} \\ \frac{k_1^2 \sigma (k_2 + k\phi)^2 (k_2 - k\phi)}{8\gamma k_2^2 \phi^2} & -\frac{\gamma(k_2 - k\phi)}{2k_2} \end{bmatrix}$$

The eigenvalues and eigenvectors of J_{μ_0} are:

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= -\frac{k_2^2 + \gamma\phi(k_2 - k\phi) + k k_2 \phi}{2k_2\phi} \\ \upsilon_1 &= \begin{pmatrix} \frac{4\gamma^2 k_2 \phi^2}{\sigma k_1^2 (k_2 + k\phi)^2} \\ 1 \end{pmatrix} \\ \upsilon_2 &= \begin{pmatrix} -\frac{4\gamma k_2^2 \phi}{k_1^2 \sigma (k_2^2 - k^2\phi^2)} \\ 1 \end{pmatrix} \end{aligned}$$

The eigenvectors of $(J_{\mu_0})^t$ are:

$$\begin{aligned} \omega_1 &= \begin{pmatrix} \frac{k_1^2 \sigma (k_2^2 - k^2\phi^2)}{4\gamma k_2^2 \phi} \\ 1 \end{pmatrix} \\ \omega_2 &= \begin{pmatrix} -\frac{\sigma k_1^2 (k_2 + k\phi)^2}{4\gamma^2 k_2 \phi^2} \\ 1 \end{pmatrix} \end{aligned}$$

Executing the following calculations:

$$\omega_1^t F_\mu((a,p);\mu)\big|_{(a_0^*, p_0^*, \mu_0^*)} = \frac{k_1^3 \sigma^2 (k_2 + k\phi)^5 (k_2 - k\phi)}{64\gamma^2 k_2^2 \phi^4}$$

$$\omega_1^t [D^2 F((a,p);\mu)(v_1,v_1)]\big|_{(a_0^*, p_0^*, \mu_0^*)} = -\frac{8\gamma^3 \phi^3 (k_2 - k\phi)}{k_1^2 \sigma (k_2 + k\phi)^3} \quad (22)$$

where $\omega_1^t F_\mu((a,p,h);\mu)$ and $\omega_1^t [D^2 F((a,p,h);\mu)(v_1,v_1)]$ were defined in [7].

If $k < \frac{k_2}{\phi}$, then the both results of (22) are different from zero. Therefore by Sotomayor's theorem there exists a node-saddle bifurcation in system (1). ■

For some parameter values, the global behavior of system (1) is shown in the following theorem.

Theorem 3: If $\mu = \mu_0$ and $k < \frac{k_2}{\phi}$, then in system (1) exists a heteroclinic trajectory. This trajectory connects the unique nonhyperbolic equilibrium point (a_0^*, p_0^*) with the equilibrium point $(k, 0)$.

Proof: To prove this theorem, we define:

$$\Omega_0 = \{(a,p) | k \le a \le a_0^*, 0 \le p \le p_0^*\}$$

In a straightforward calculation is easy to verify that the vector field of the system (1) in the boundary of Ω_0 is pointing inwards the region Ω_0. On the other hand:

- $(k, 0)$, is an equilibrium point asymptotically stable. The subspace of the equilibrium point is generated by vectors

$$E^s = \left\{ (1,0), \left(\frac{-16\gamma^2 k k_2^3 \phi^2}{k_1^2 \sigma (k_2 + k\phi)^4}, -k + \frac{\gamma (k_2 - k\phi)^2}{(k_2 + k\phi)^2} \right) \right\}. \quad (23)$$

- (a_0^*, p_0^*), is a saddle-node equilibrium point. The equilibrium point has two subspaces: a stable one and a central one, generated by:

$$E^s = \left\{ \left(\frac{-4\gamma k_2^2 \phi}{k_1^2 \sigma (k_2^2 - k^2 \phi^2)}, 1 \right) \right\} \quad (24)$$

and

$$E^c = \left\{ \left(\frac{4\gamma^2 k_2 \phi^2}{k_1^2 \sigma (k_2 + k\phi)^2}, 1 \right) \right\} \quad (25)$$

respectively. The center manifold w^c is tangent to E^c. It is easy to see that E^c is within Ω_0.

System (1) does not have a periodic trajectory into Ω_0, (see [2]).

Let $\varphi(t)$ the trajectory, which is tangent to E^c, furthermore, we have that the trajectory, $\varphi(t)$, is pointing inwards Ω_0 and since the vector field is continuous and does not have a equilibrium point into $\Omega_0 - \{(k,0); (a_0^*, p_0^*)\}$, neither periodic trajectory into Ω_0 and equilibrium point $(k, 0)$ is stable, then by Poincaré-Bendixson theorem, the trajectory $\varphi(t)$ when $t \to \infty$ is going to $(k, 0)$. Therefore, system (1) has a heteroclinic trajectory that connects the nonhyperbolic equilibriums (a_0^*, p_0^*) and $(k, 0)$. This trajectory exist during the saddle-node bifurcation, see the second phase portrait of Fig. 1. ■

For other parameter values, the global dynamic of system (1) is shown in the following theorem.

Theorem 4: For each parameters values satisfying $\mu = \mu_1$ and $k < \frac{k_2}{\phi}$, then the system (1) has a heteroclinic trajectory that connecting the nonhyperbolic equilibrium point $(k, 0)$ with the unique hyperbolic equilibrium point (a_1^*, p_1^*), where $a_1^* = \frac{k_2}{\phi}$ and $p_1^* = \frac{k k_1^2 \sigma (k_2 - k\phi)}{\gamma^2 \phi^2}$.

Proof: To prove theorem (4), let us define the region

$$\Omega_1 = \{(a,p) | k \le a \le a_1^*, 0 \le p \le p_1^*\}$$

By a straightforward calculation is easy to see that the vector field of system (1) in the boundary of Ω_1 is points inwards the region Ω_1. Furthermore

- $(k, 0)$ is a nonhyperbolic equilibrium point. The eigenvalues and eigenvectors of the Jacobian matrix evaluated at $(k, 0)$ are:

$$\begin{aligned} \lambda_1 &= 0 \qquad \lambda_2 = -k \\ \upsilon_1 &= \begin{pmatrix} \frac{\gamma^2 k_2}{k^2 k_1^2 \sigma} \\ 1 \end{pmatrix} \quad \upsilon_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned} \quad (26)$$

  Note that the vector υ_1 is inwards Ω_1.

- (a_1^*, p_1^*) is an hyperbolic equilibrium point. The eigenvalues and eigenvectors of the Jacobian matrix evaluated at (a_1^*, p_1^*) are:

$$\begin{aligned} \lambda_1 &= \frac{-k_2^2 - \gamma\phi(k_2 - k\phi) + r}{2k_2\phi} \\ \lambda_2 &= \frac{-k_2^2 - \gamma\phi(k_2 - k\phi) - r}{2k_2\phi} \end{aligned} \quad (27)$$

  where

$$r = \sqrt{k_2^4 - \gamma\phi(k_2 - k\phi)(2k_2^2 + \gamma k\phi^2 - \gamma k_2 \phi - 4k k_2 \phi)}$$

Note that the real parts of eigenvalues λ_1 and λ_2 are negative, therefore the equilibrium point is stable[3].

Let $\varphi(t)$ the trajectory which is tangent to E^c, furthermore, we have that the trajectory, $\varphi(t)$, is pointing inwards Ω_1 and since the vector field is continuous and does not have an equilibrium point into $\Omega_1 - \{(k,0);(a_1^*,p_1^*)\}$, neither closed trajectory into Ω_1 and equilibrium point (a_1^*,p_1^*) is stable, then by the Poincaré-Bendixson theorem, the trajectory $\varphi(t)$, tends to (a_1^*,p_1^*) when $t \to \infty$. Therefore, system (1) has a heteroclinic trajectory to connect the nonhyperbolic equilibrium point $(k,0)$ with the equilibrium point (a_0^*,p_0^*). ■

The following result exhibits the global dynamics of system (1).

Theorem 5: For μ and k satisfying $\mu_1 < \mu$ and $k \neq \frac{k_2}{\phi}$, the system (1) has a heteroclinic trajectory connecting the equilibrium point $(k,0)$ with the nontrivial equilibrium (a_1^*,p_1^*).

Proof: Let $\mu_1 < \mu$ and $k \neq \frac{k_2}{\phi}$, then the system has three equilibrium point. They are:

- $(0,0)$ is a saddle point.
- $(k,0)$ is a saddle point. The eigenvalues and eigenvectors of the Jacobian matrix evaluated at $(k,0)$ are:

$$\begin{array}{rclcrcl} \lambda_1 &=& kk_1\mu\sigma - \gamma & & \lambda_2 &=& -k \\ \upsilon_1 &=& \begin{pmatrix} kk_2\mu^2\sigma \\ k + kk_1\mu\sigma - \gamma \end{pmatrix} & & \upsilon_2 &=& \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{array}$$

- (a_1^*,p_1^*), is global attractor in $\mathbb{R}^2_+$.

Note that υ_1 is in the positive quadrant. Since

$$\begin{array}{rcl} \mu_1 < \mu & \Longrightarrow & \dfrac{\gamma}{\sigma k k_1} < \mu \\ & \Longrightarrow & 0 < k + kk_1\mu\sigma - \gamma \end{array}$$

hence υ_1 lies the positive quadrant.

Let υ_1 is the tangent of the trajectory $\varphi(t)$ at equilibrium point $(k,0)$. The trajectory $\varphi(t)$ is tangent to υ_1 and furthermore this trajectory is inside a positive quadrant. On the other hand since (a_1^*,p_1^*) is a global attractor, then by the Poincaré-Bendixson theorem, $\varphi(t)$ when $t \to \infty$ is going to (a_1^*,p_1^*). Therefore, if $\mu_1 < \mu$, then the system (1) has a heteroclinic trajectory. This trajectory connects the equilibrium $(k,0)$ and (a_1^*,p_1^*), after a transcritical bifurcation takes place. ■

B. Spatiotemporal Dynamic

In order to study the spatial effects when pollinators and plants interact in the habitat, we consider the following hypotheses: (1) the movement of pollinators is towards those places of the habitat where the plant population density is high. (2) The plant population does not move, but its spatial distribution changes because of the interaction with the pollinator, and (3) The temporal dynamic is given by system (1).

A mathematical model which incorporates the above hypotheses is

$$\begin{array}{rcl} \dfrac{\partial a}{\partial t} &=& D_1 div\left[f_1(p) \bigtriangledown a\right] + a(k-a) + \dfrac{k_2\sigma\mu^2 ap}{1+\phi\sigma\mu^2 p} \\ \dfrac{\partial p}{\partial t} &=& -\gamma p + \dfrac{k_1\sigma\mu ap}{1+\phi\sigma\mu^2 p} \end{array} \quad (28)$$

where $a \equiv a(\vec{x},t)$ and $p \equiv p(\vec{x},t)$ are the pollinators and plants densities at the position $\vec{x} \in \Omega$, at time t. Homogeneous Neumann boundary conditions on the boundary $\partial\Omega$ and $f_1(p) > 0$ with $f_1(0) \geq 0$ are consider.

Note the function f_1 measures the intensity with the pollinator moves toward those places of high plant density. Actually this is a novel ingredient int he study we present. Since for instance in [9] the authors consider constant diffusion coefficient for the pollinators.

With the purpose of finding the numerical solutions of the system (28), we will consider the region $\Omega = \{(x,y)|0 < x < 10, 0 < y < 10\}$, with homogeneous Neumann boundary conditions. The initial conditions we consider are:

$$\begin{array}{rcl} a(x,y,0) &=& p_1^*(1 + 0.01sin(x+y)cos(x-y)) \\ p(x,y,0) &=& a_1^*(1 + 0.01sin(2x)) \end{array}$$

where (a_1^*,p_1^*) is the positive stable equilibrium point of system (1).

In order to have an idea of the spatial effects when pollinators move towards those places of the habitat where the plant population density is high, we consider functions f_1 of the form $f_1(p) = a_1 p$, where $a_1 > 0$. We carried out several numerical simulations[4].

In Figs. 2 and 3, the results of our numerical simulations are show. There, the color scale represents a heat-likes spectrum, in which the purple color represent low density populations, while red color corresponds to the high populations density. This color scale will be used throughout the text.

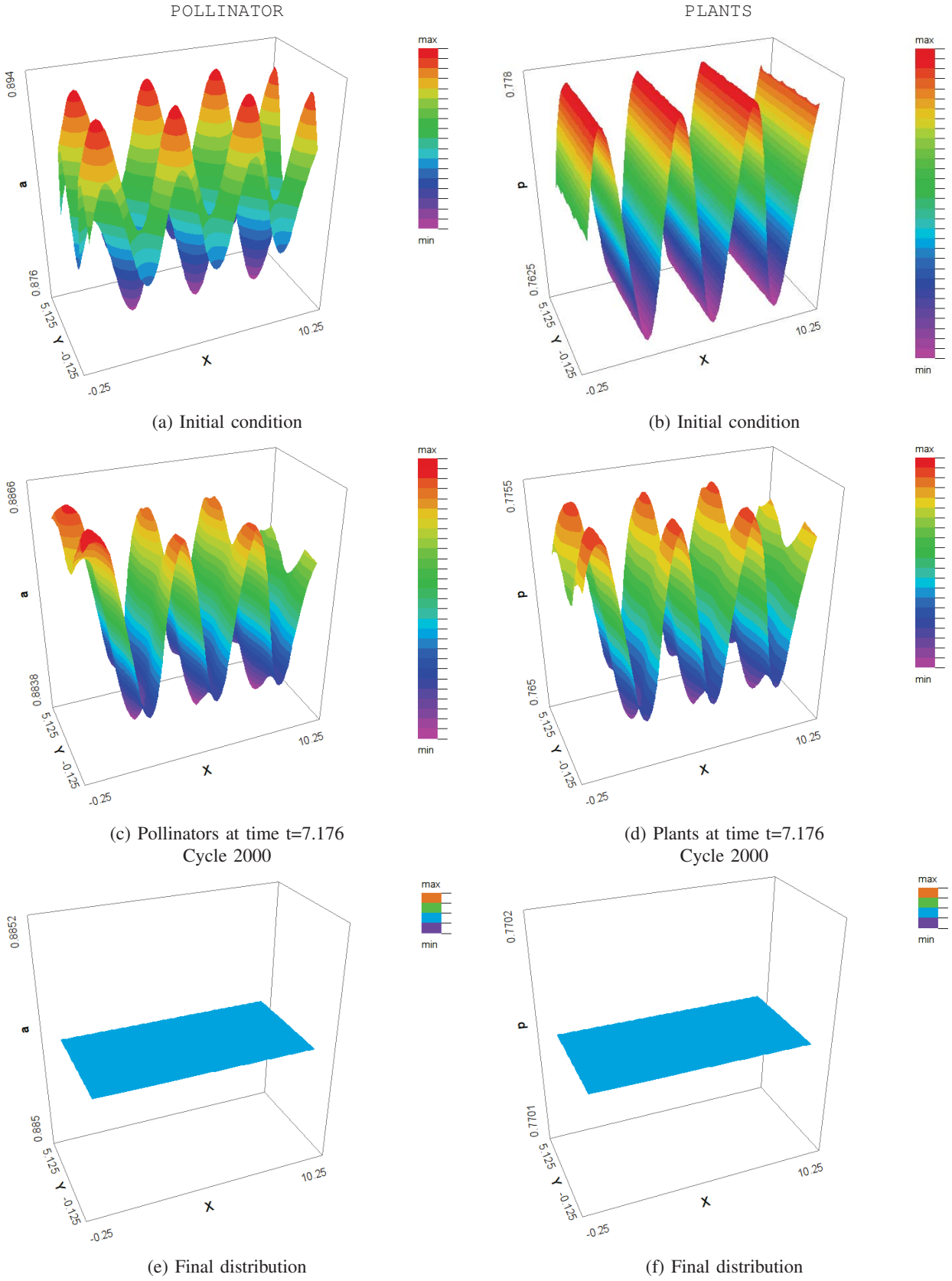
By selecting various values of a_1 such that $0 < a_1 < 0.5$, all the numerical simulations show: (1) The existence of a transient of the pollinator and plant populations where share the same habitat, see 2(c) and 2(d). (2) The steady and homogeneous state acts as an attractor in the space of solutions of the system. The final distribution of both populations become homogeneous. This can be seen in the last two figures 2(e) and 2(f).

A second numerical simulation was carried out for the values of a_1 are such that $0.5 < a_1$. The result show: (1) The transient of the pollinator population tends to occupy the entire habitat, on the other hand, the plant population occupies the habitat gradually, see 3(c) and 3(d). (2) The steady and homogeneous state acts as an attractor in the space of solutions of system (28), that is the final distribution of both populations, becomes homogeneous, see 3(e) and 3(f). This is shown in Figs. 3.

[3]The characteristic polynomial is:

$$p(\lambda) = \lambda^2 + \frac{k_2^2 + \gamma\phi(k_2 - k\phi)}{k_2\phi}\lambda + \frac{\gamma(k_2 - k\phi)^2}{k_2\phi}$$

[4]We used the FlexPDE software to obtain the numerical solutions for all the PDE systems presented in this paper



(a) Initial condition

(b) Initial condition

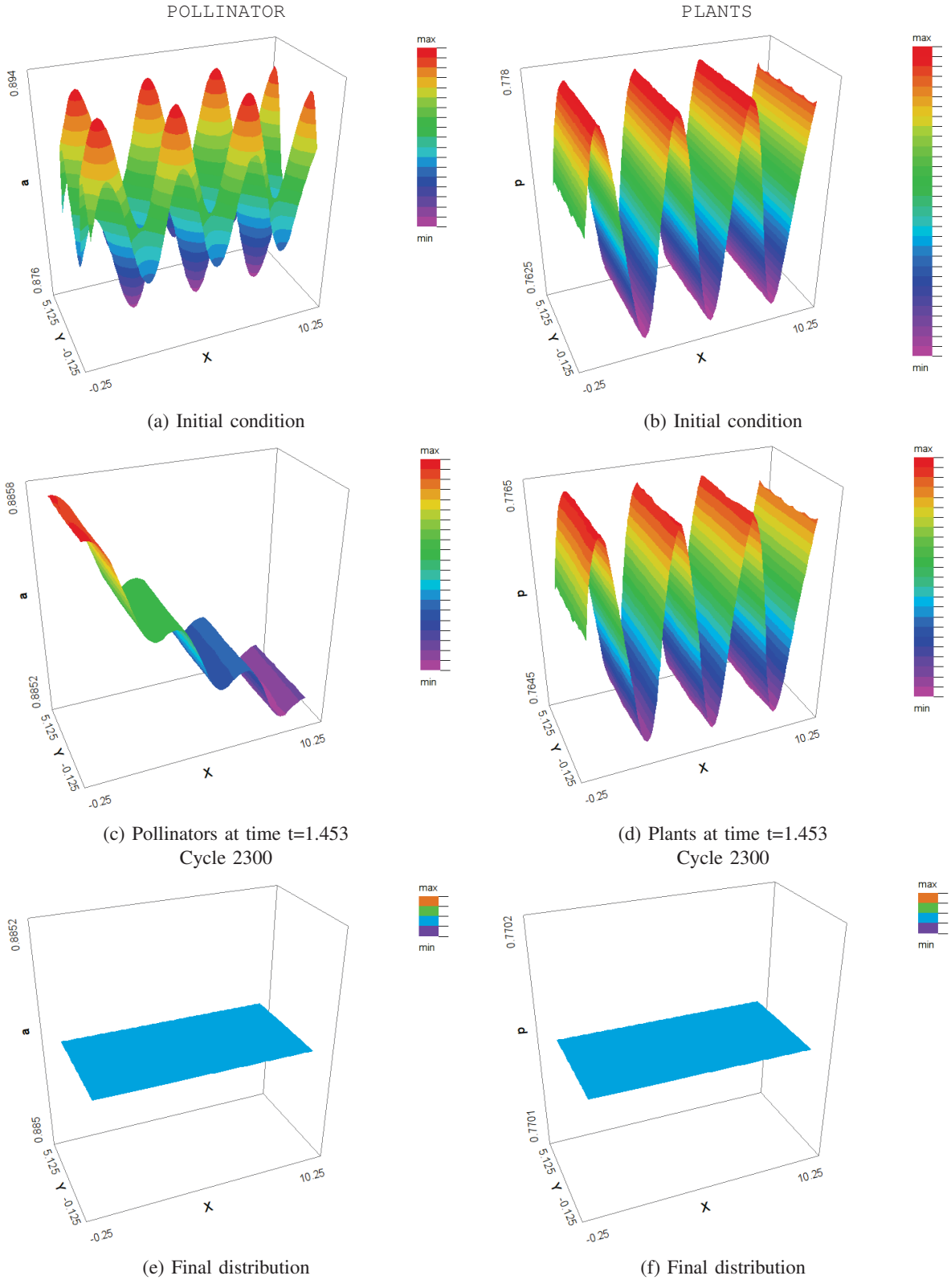
(c) Pollinators at time t=7.176 Cycle 2000

(d) Plants at time t=7.176 Cycle 2000

(e) Final distribution

(f) Final distribution

Fig. 2 Numerical simulation of system (28), with $D_1 = 1$ and $f_1(p) = 0.1p$



(a) Initial condition

(b) Initial condition

(c) Pollinators at time t=1.453 Cycle 2300

(d) Plants at time t=1.453 Cycle 2300

(e) Final distribution

(f) Final distribution

Fig. 3 Numerical simulation of system (28), with $D_1 = 1$ and $f_1(p) = 10p$

The interpretation of the above results is as following: In both cases, the populations coexist but the way how they are distributed in the habitat is different. This happens during transients.

The difference that we observe during transients may be due to the fact that the diffusion coefficient depends directly on the density of the plant population, said fact allows us to obtain different ways that this populations distribute themselves during transients.

III. The Mathematical Model of Plant-Herbivore Interaction

This section is devoted to describe the qualitative behavior of the plant-herbivore interaction by a mathematical model. This type of model has the attracted in attention recent publications, (see [2], [8] and [9]). Since this relation type is determinant for the existence of life. Without the presence of the plant or herbivore population the food chain as we know it will break. This event involves a radical change in the environment. Other factors may also alter the environment such as scarcity or abundance of water.

This section begins by presenting the results of different types of dynamical behaviors of a temporal model. These include transcritical bifurcations and existence of a heteroclinic orbit.

Later, for the spatiotemporal dynamics of the mathematical model a novel factor is introduced. This consists in considering that the herbivores, move towards those places of the habitat where the plant population density is high. In this part we show some effects which are obtained considering the spatial part of the numerical simulation.

A. Temporal Dynamic

A very particular characteristic of the model that we will see next is that this model is of predator-prey type. Since, from the viewpoint of the authors in [2] and [9], the herbivore is benefited from plants, but the plants are harmed by the presence of herbivores. The interaction term is described by a Holling response of type II. We denote by $h(t)$ the herbivore population density at the time t. The mathematical model is:

$$\begin{aligned} \dot{p} &= p\left(1-\frac{p}{k}\right)-\frac{ph}{1+p} \\ \dot{h} &= -\alpha\beta h+\beta\frac{ph}{1+p} \end{aligned} \tag{29}$$

where α, β, and k are positive parameters.

The equilibrium points of the system (29) are: $(0,0)$, $(k,0)$ and $(\tilde{p},\tilde{h})$, where

$$\tilde{p}=\frac{\alpha}{1-\alpha} \qquad \tilde{h}=(1+\tilde{p})\left(1-\frac{\tilde{p}}{k}\right)$$

To continue we will show the classification of each equilibrium point.

- The equilibrium point $(0,0)$ is unstable (saddle), for all $k>0$, $\alpha>0$ and $\beta>0$.
- The local stability analysis at the equilibrium point $(k,0)$ given us:
  - If $\frac{\alpha}{1-\alpha}<k$, then the equilibrium point $(k,0)$ is a stable node.
  - If $\frac{\alpha}{1-\alpha}>k$, then the equilibrium point $(k,0)$ is a saddle.
  - If $k=\frac{\alpha}{1-\alpha}$, then the equilibrium point $(k,0)$ is nonhyperbolic.

The local dynamics is given by the following theorem.

Theorem 6: In system (29) transcritical bifurcation takes place. The bifurcation point is $((k_0,0);k_0)$, where

$$k_0=\frac{\alpha}{1-\alpha}$$

Proof: To prove, we will consider the next system

$$\begin{aligned} \dot{p} &= p\left(1-\frac{p}{k}\right)(1+p)-ph \\ \dot{h} &= -\alpha\beta h(1+p)+\beta ph \end{aligned} \tag{30}$$

The systems (30) and (29) are topologically equivalent in $\mathbb{R}_+{}^2$, later we apply the translation: $p_0=p-k$, $h_0=h$ and $k_{11}=k-k_0$ and obtain:

$$\begin{aligned} \dot{p_0} &= \frac{(\alpha+k_{11}+p_0-\alpha k_{11}-\alpha p_0)b}{(\alpha-1)(\alpha+k_{11}-\alpha k_{11})} \\ \dot{h}_0 &= -\beta(\alpha-1)(k_{11}+p_0)h_0 \end{aligned} \tag{31}$$

where

$$b=((1+k_{11})p_0+(\alpha+k_{11})h_0+(1-\alpha)p_0^2-\alpha k_{11}(h_0+p_0))$$

Later, we apply a linear transformation suitable (see [7] and [13]) to system (31), which is transformed $(\dot{x}=Jx+F(x))$ as follows:

$$\begin{aligned} \dot{p}_1 &= f_1(p_1,h_1,k_{11}) \\ \dot{h}_1 &= f_2(p_1,h_1,k_{11}) \end{aligned} \tag{32}$$

where[5] k_{11} is the new parameter. Now we include a k_{11} as a new variable in system (32), this is:

$$\begin{aligned} \dot{p}_1 &= f_1(p_1,h_1,k_{11}) \\ \dot{h}_1 &= f_2(p_1,h_1,k_{11}) \\ \dot{k}_{11} &= 0 \end{aligned} \tag{33}$$

System (33) has an equilibrium point at $(p_1,h_1,k_{11})=(0,0,0)$. The eigenvalues of the Jacobian matrix are 0, $\frac{1}{\alpha-1}$, 0, and by the center manifold theorem, system (33) has a center manifold two dimensional and can be represented by a graph with p_1 and k_{11} variables, this is:

$$\begin{aligned} W^c = \{&(p_1,h_1,k_{11})\in\mathbb{R}|h_1=r_1(p_1,k_{11}),r_1(0,0)=0, \\ &Dr_1(0,0)=0\} \end{aligned} \tag{34}$$

for p_1 and k_{11} small enough. Now we are going to find an approximation to center manifold depending on parameter k_{11}. We suppose that the form of center manifold is:

$$r_1(p_1,k_{11})=c_1p_1^2+c_2p_1k_{11}+c_3k_{11}^2+\cdots \tag{35}$$

where the coefficients c_i will be determined later on. From (34) we have

$$\dot{h}_1=\frac{dr_1}{dp_1}\dot{p}_1+\frac{dr_1}{dk_{11}}\dot{k}_{11} \tag{36}$$

[5]Here f_1 and f_2 are not written because they are very large expressions.

We substitute (35) into (33) and using (36), we obtain:

$$f_2(p_1, r_1(p_1, k_{11}), k_{11}) = \frac{dr_1}{dp_1} f_1(p_1, r_1(p_1, k_{11}), k_{11}) \quad (37)$$

equivalently

$$\frac{dr_1}{dp_1} f_1(p_1, r_1(p_1, k_{11}), k_{11}) - f_2(p_1, r_1(p_1, k_{11}), k_{11}) = 0 \quad (38)$$

simplifying and equating the coefficients of the same order in p_1, we find:

$$\begin{aligned} c1 &= \frac{(\alpha-1)(\alpha+k_{11}-\alpha k_{11})^2 + b^*}{(1-k_{11}(\alpha-1)(\beta-\alpha\beta+1))^2 c^*} \\ c2 &= 0 \\ c3 &= 0 \\ b^* &= \beta(\alpha-1)(\alpha-\alpha^2)(\alpha+k_{11}-\alpha k_{11}) \\ c^* &= (1-k_{11}(\alpha-1)(2\beta-2\alpha\beta+1)). \end{aligned}$$

Now let us substitute in (35) the coefficient values founded above, we obtain:

$$r_1(p_1, k_{11}) = c_1 p_1^2 + \cdots \quad (39)$$

By substituting (39) into (33), we obtain the reduced vector field

$$\begin{aligned} \dot{p}_1 &= f_{11}(p_1, r_1(p_1, k_{11}), k_{11}) \quad (40) \\ \dot{k}_{11} &= 0 \end{aligned}$$

where

$$\begin{aligned} f_{11} &= p_1 [\beta k_{11}(1-\alpha)] \\ &+ p_1^2 \left[\frac{\beta(\alpha-1)(\alpha+k_{11}-\alpha k_{11})}{k_{11}-\alpha k_{11}+\beta k_{11}-2\alpha\beta k_{11}+\alpha^2\beta k_{11}+1} \right] \\ &+ \cdots \end{aligned}$$

with

$$\begin{aligned} f_{11}(0,0) &= 0 \quad (41) \\ \frac{\partial f_{11}}{\partial p_1}(0,0) &= 0 \quad (42) \\ \frac{\partial f_{11}}{\partial k_{11}}(0,0) &= 0 \quad (43) \\ \frac{\partial^2 f_{11}}{\partial k_{11} \partial p_1}(0,0) &= -\beta(\alpha-1) \quad (44) \\ \frac{\partial^2 f_{11}}{\partial p_1^2}(0,0) &= 2\alpha\beta(\alpha-1) \quad (45) \end{aligned}$$

Since $0 < \alpha < 1$, then (41)-(45) show that the orbit structure near $((k,0);k_0)$ is qualitatively the same as the orbit structure near $((0,0);k_{11})$ of $\dot{p}_1 = k_{11}p_1 \mp p_1^2$. Therefore there exists a transcritical bifurcation in system (29). ■

Fig. 4 shows the phase portrait of the system (29) for some values of the parameters. Here, we complete the stability analysis of the equilibrium point $(\tilde{p}, \tilde{h})$, when there exists in the positive quadrant. These are the results:

1) If $k > 1$, $0 < \alpha < 1$ and $k > \frac{\alpha}{1-\alpha}$, then
   - If $k \leq \frac{1+\alpha}{1-\alpha}$, then $(\tilde{p}, \tilde{h})$ is asymptotically stable (see [9]).

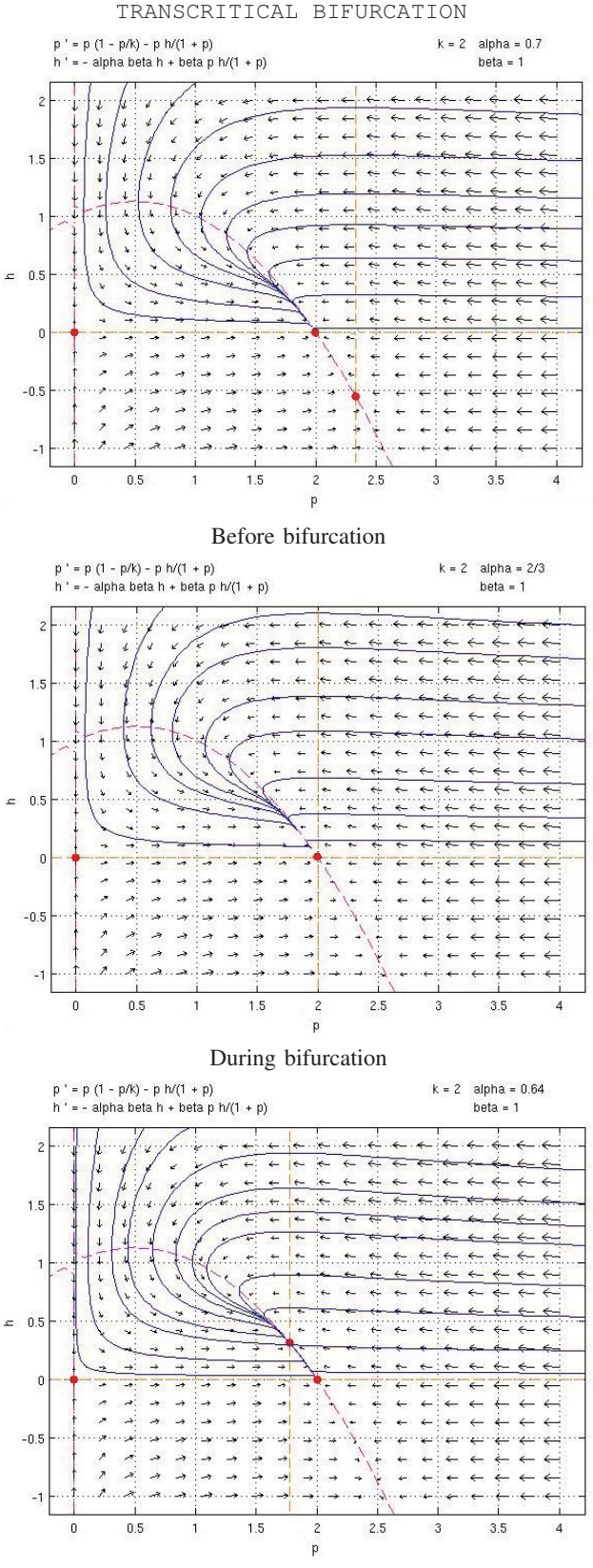


Fig. 4 Phase Portraits of system (29)

- If $k > \dfrac{1+\alpha}{1-\alpha}$, then $(\tilde{p},\tilde{h})$ is unstable and a stable limit cycle emerges from a Hopf bifurcation (see [3] and [9]).

2) If $k = 1$ and $0 < \alpha \leq \frac{1}{2}$, then $(\tilde{p},\tilde{h})$ exists in the positive quadrant, where

$$\tilde{p} = \frac{\alpha}{1-\alpha} \qquad \tilde{h} = \frac{1-2\alpha}{(1-\alpha)^2}$$

This point can be a stable node or transcritical.

To exhibit the global dynamics of system (29), we state and show the following theorem.

Theorem 7: For each k such that $k \in \left(\dfrac{\alpha}{1-\alpha}, \dfrac{1+\alpha}{1-\alpha}\right)$, the system (29) has a heteroclinic trajectory. This trajectory connecting the equilibrium points $(k,0)$ and $(\tilde{p},\tilde{h})$.

Proof: Let k such that $k \in \left(\dfrac{\alpha}{1-\alpha}, \dfrac{1+\alpha}{1-\alpha}\right)$. then the equilibrium point $(k,0)$ is a saddle and the equilibrium point $(\tilde{p},\tilde{h})$ is a global attractor (see [9]).

We consider the eigenvector v_2 associated with the positive eigenvalue and let $\varphi(t)$ the trajectory, which is tangent to v_2 at equilibrium point. Since v_2 is inwards the positive quadrant then the trajectory $\varphi(t)$ when $t \to \infty$ will converge to $(\tilde{p},\tilde{h})$. Therefore, if k satisfying the hypothesis then the system (29) has a heteroclinic trajectory. ■

Fig. 5 shows an heteroclinic orbit of the system (29) for some specific parameters values satisfying the condition of the Theorem (7). Note that this trajectory exists after of the transcritica bifurcacion, see the second phase portrait of the Fig. 4.

B. Spatiotemporal Dynamic

To study the spatial effects when plants and herbivores interact in the same habitat, we consider the next hypotheses: (1) the herbivores move towards those places of the habitat where the plant population density is high. (2) The plant population does not move, but its spatial distribution changes because of the interaction with the herbivores, and (3) the temporal dynamic is given by system (29).

A mathematical model that reflects the above hypotheses is

$$\begin{aligned} \frac{\partial p}{\partial t} &= p\left(1-\frac{p}{k}\right) - \frac{m_1 p h}{s+p} \\ \frac{\partial h}{\partial t} &= D_2 div\left[f_2(p) \bigtriangledown h\right] + \frac{m_2 p h}{s+p} - \eta h, \end{aligned} \tag{46}$$

where $p \equiv p(\vec{x},t)$ and $h \equiv h(\vec{x},t)$ are the plants and herbivores densities at the position $\vec{x} \in \Omega$, at time t. Homogeneous Neumann boundary conditions on $\partial\Omega$ are consider. The function f_2 is such that $f_2(p) > 0$ and $f_2(0) \geq 0$ for $p \geqslant 0$. This function measures the intensity with the herbivore moves toward those places of high plant density. As a matter of fact this is a novel ingredient in the present paper.

We obtain the plant-herbivore model proposed in [9], when $f_2(p) = 1$. In order to find the numerical solutions of the system (46), we are going to consider the rectangular habitat Ω, with initial conditions given by

$$\begin{aligned} p(x,y,0) &= p_1^*(1+0.2sin(x+y)cos(x-y)) \\ h(x,y,0) &= h_1^*(1+0.4sin(2x)) \end{aligned}$$

where (p_1^*, h_1^*) is the unstable equilibrium point of the system (28). This point is obtained with the parameters values for which the system (28) has a stable limit cycle.

To have an idea of the spatial effects when the movement of herbivores is towards those places of the habitat where the plant population density is high, we consider the function f_2 of the form $f_2(p) = b_1 p$ with $b_1 > 0$. After we carried out several numerical simulation we obtained the following observations:

By selecting different values of b_1 such that $0 < b_1 < 0.1$. we observe in the numerical simulation that the herbivore population moves to those places where the plant population density is high. This originates a decreasing in the plants population density where there is a high density of herbivores and increasing plant where there are few herbivores, resulting in a traveling wave like behavior. See Figs. 6 (a)-(d) and 7 (a)-(d). On the other hand, if b_1 is such that $b_1 \in (5,10)$, then the result of the numerical simulation is carried out and we can observe that the population coexist through periodic oscillations, see Figs. 8 (a)-(d) and 9 (a)-(d).

The observed effects on the numerical simulation are the following:

- Both populations coexist in the rectangular habitat.
- The way in which the populations distribute when they coexist, can be seen either a travelling wave like or as temporal periodic oscillations.
- The result of the previous behaviors is due to the dependence of the herbivores movement. Remember this it is towards place where the plant populations density is either high.

IV. The Mathematical Model of Pollinator-Plant-Herbivore Interaction

In this section, we focus in the study of the temporal dynamics of the pollinator-plant-herbivore interaction by a mathematical model. The following model has many important characteristics, some of which are, that the interaction terms of pollinator-plant and herbivore-plant are describe by a Holling response type II. In the pollinator-plant interaction both populations benefit from each other. In the plant-herbivore interaction, the first population is harmed, but the second population is benefited. The pollinator-herbivore interaction is an indirect relation. The construction of this model can be found in system [2] and [9]. The mathematical model is:

$$\begin{aligned} \dot{a} &= a\left(1-\frac{a}{k}\right) + \frac{g(h)k_2\sigma\mu^2 ap}{1+\phi\sigma\mu^2 p} \\ \dot{p} &= -\gamma p + \frac{g(h)k_1\sigma\mu ap}{1+\phi\sigma\mu^2 p} - \frac{m_1 ph}{s+p} \\ \dot{h} &= -\delta h + \frac{m_2 ph}{s+p}, \end{aligned} \tag{47}$$

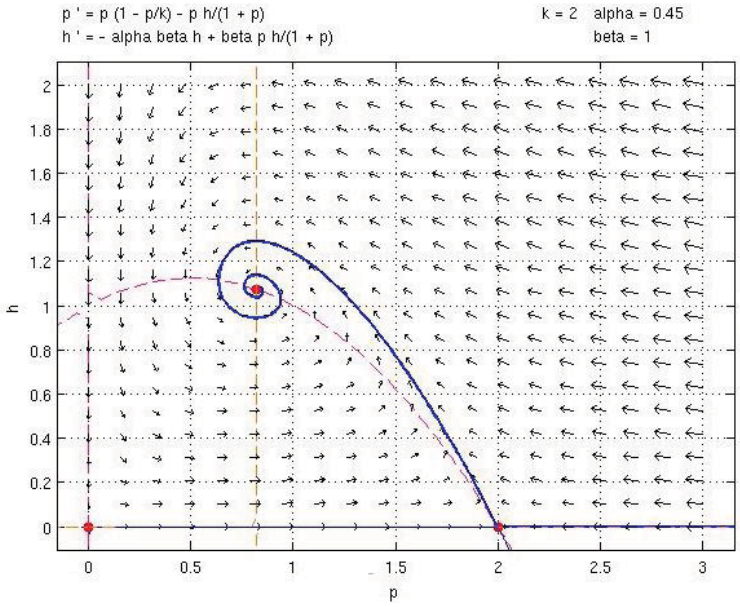


Fig. 5 Heteroclinic orbit of the system (29)

where the function $g \in C[0,\infty]$, $g(0) = 1$, $g'(h) \leq 0$ and $g(h) > 0 \ \forall \ h \geq 0$. The ecological interpretation of the function g is as following. The function g is the rate of visits of pollinators to plants which depends on the herbivore population density (see [9]).

The points $(0,0,0)$ and $(k,0,0)$ are equilibrium points for all values of the positive parameters of system (47). The stability of each equilibrium point is the next:

- The equilibrium point $(0,0,0)$ is unstable (saddle).
- The eigenvalues associated a linear approximation of the system (47) at equilibrium point $(k,0,0)$ are:

$$\lambda_1 = kk_1\mu\sigma - \gamma \quad \lambda_2 = -1 \quad \lambda_3 = -\delta$$

  Therefore the stability of the equilibrium point $(k,0,0)$ is classified as follows:
  - If $\mu < \dfrac{\gamma}{kk_1\sigma}$, then $(k,0,0)$ is a local attractor.
  - If $\mu > \dfrac{\gamma}{kk_1\sigma}$, then $(k,0,0)$ is a saddle point
  - If $\mu = \dfrac{\gamma}{kk_1\sigma}$, then $(k,0,0)$ is a nonhyperbolic equilibrium point.

With the following results we show the equilibrium point classification.

Theorem 8: Let $\mu_1 = \dfrac{\gamma}{kk_1\sigma}$. If $k_2 \neq \phi$, and $g(0)$, $g'(0)$, $g''(0)$ and $g'''(0)$ are defined, then in system (47) a transcritical bifurcation occur. The point $((k,0,0);\mu_1)$ is the bifurcation point.

Proof: This uses the Sotomayor's Theorem (see [7]). The equilibrium point is $(k,0,0)$, the bifurcation parameter is μ and the bifurcation value is: $\mu = \mu_1$. Let F be the vector field. Corresponding right hand side of system (47), i.e.,

$$F((a,p,h);\mu) = \begin{bmatrix} a\left(1-\dfrac{a}{k}\right) + \dfrac{g(h)k_2\sigma\mu^2 ap}{1+\phi\sigma\mu^2 p} \\ -\gamma p + \dfrac{g(h)k_1\sigma\mu ap}{1+\phi\sigma\mu^2 p} - \dfrac{m_1 hp}{s+p} \\ -\delta h + \dfrac{m_2 hp}{s+p} \end{bmatrix} \tag{48}$$

the derivative of vector F with respect to μ is:

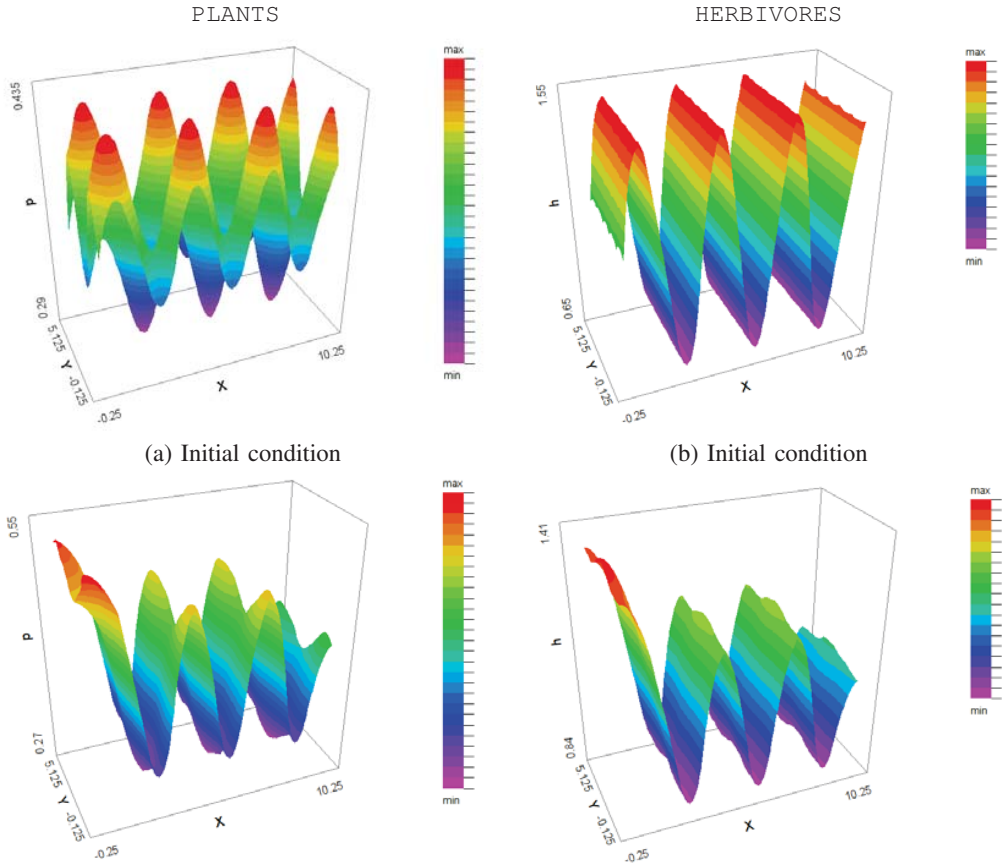
$$F_\mu((a,p,h);\mu) = \begin{bmatrix} g(h)\dfrac{2k_2\mu\sigma ap}{(1+\phi\sigma\mu^2 p)^2} \\ -g(h)\dfrac{k_1\sigma(\mu^2\phi\sigma p - 1)ap}{(1+\phi\sigma\mu^2 p)^2} \\ 0 \end{bmatrix} \tag{49}$$

The Jacobian matrix of the system (47) at the equilibrium point $((k,0,0);\mu_1)$ is:

$$J_{\mu_1} = \begin{bmatrix} -1 & \dfrac{\gamma^2 k_2}{kk_1^2\sigma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\delta \end{bmatrix} \tag{50}$$

The eigenvalue and eigenvector of J_{μ_1} are:

$$\begin{array}{ccccccc} \lambda_1 & = & 0 & \lambda_2 & = & -1 & \lambda_3 = -\delta \\ \upsilon_1 & = & \begin{pmatrix} \dfrac{\gamma^2 k_2}{kk_1^2\sigma} \\ 1 \\ 0 \end{pmatrix} & \upsilon_2 & = & \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \upsilon_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{array} \tag{51}$$



(a) Initial condition (b) Initial condition

(c) Plants at time t=15.19 (d) Herbivores at time t=15.19

Fig. 6 Numerical simulation of system (46) with $D_2 = 1$ and $f_2(p) = 0.1p$

The eigenvalues of $(J_{\mu_1})^t$ are:

$$\omega_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} \frac{-kk_1^2\sigma}{\gamma^2 k_2} \\ 1 \\ 0 \end{pmatrix} \quad \omega_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{52}$$

Now we carry out the following calculations:

- $$\omega_1^t F_\mu((a,p,h);\mu)\big|_{(k,0,0,\mu_1)} = 0 \tag{53}$$
- $$\omega_1^t [DF_{\mu_1}((a,p,h);\mu)v_1]\big|_{(k,0,0,\mu_1)} = kk_1\sigma \tag{54}$$
- $$\omega_1^t [D^2F((a,p,h);\mu)(v_1,v_1)]\big|_{(k,0,0,\mu_1)} = \frac{2\gamma^3(k_2-\phi)}{k^2k_1^2\sigma} \tag{55}$$

where $\omega_1^t[DF_{\mu_1}((a,p,h);\mu)v_1]$ is defined in [7]. Since (53) is zero, (54) is non zero and if $k_2 \neq \phi$, then (55) is not zero, then by Sotomayor's theorem there exists a transcritical bifurcation in system (47). ■

The nontrivial null-clines of system (47) are:

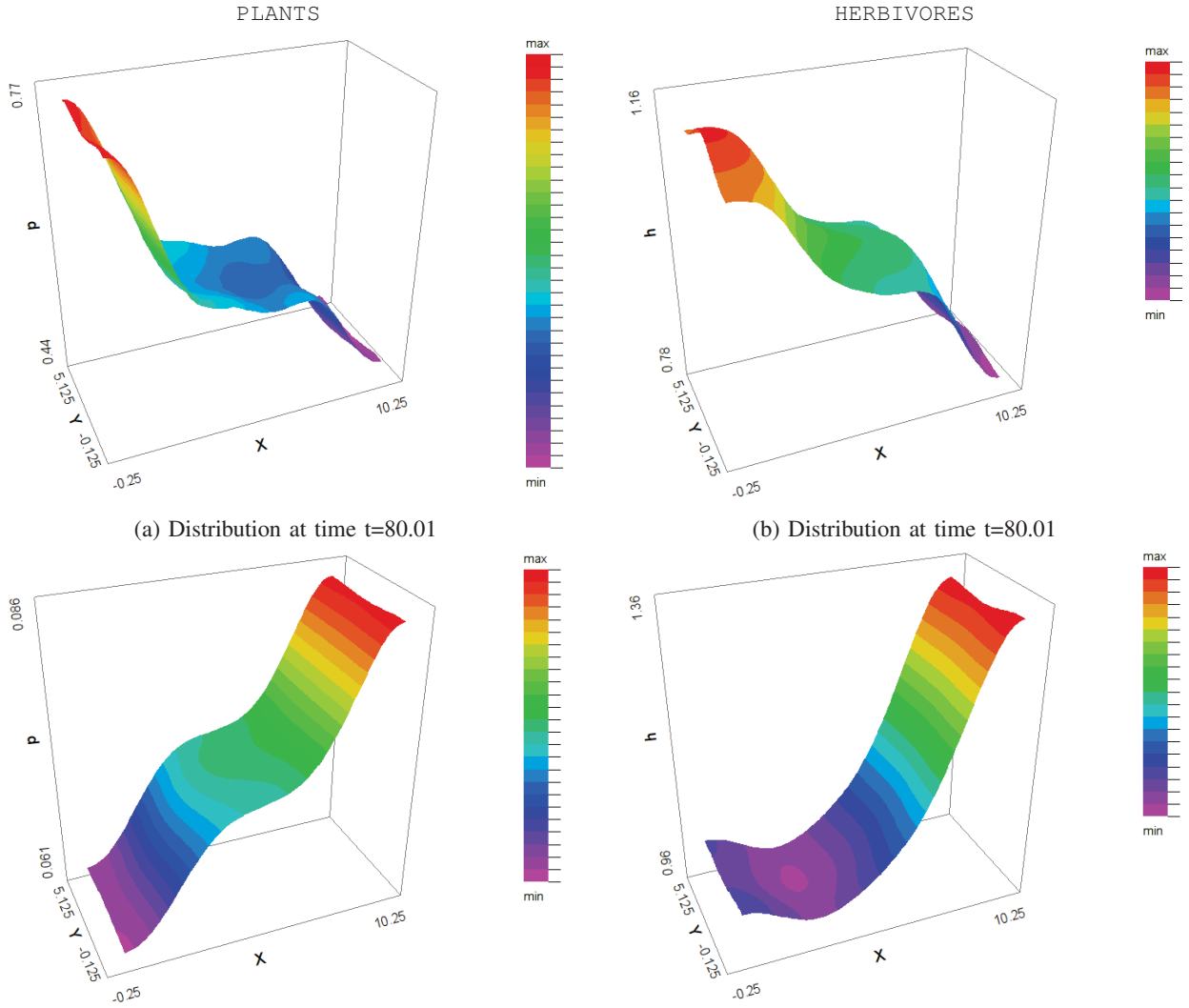
$$\begin{aligned} \left(1-\frac{a}{k}\right) + \frac{g(h)k_2\sigma\mu^2 p}{1+\sigma\phi\mu^2 p} &= 0 \\ -\gamma + \frac{g(h)k_1\sigma\mu a}{1+\sigma\phi\mu^2 p} - \frac{m_1 h}{s+p} &= 0 \\ -\delta + \frac{m_2 p}{s+p} &= 0 \end{aligned}$$

If $g(h) = 1$, then the intersection point of null-clines is

$$a^* = k + \frac{\delta k k_2 \mu^2 s\sigma}{\delta\phi s\sigma\mu^2 + m_2 - \delta} \qquad p^* = \frac{s\delta}{m_2-\delta}$$
$$h^* = \frac{s+p^*}{m_1}\left[-\gamma + \frac{k_1\mu\sigma a^*}{1+\sigma\phi\mu^2 p^*}\right]$$

The following theorem gives us conditions of a positive equilibrium.

Theorem 9: Let $g(h) = 1$. In system (47) the equilibrium point (a^*, p^*, h^*) given above is positive if and only if $m_2 > \delta$



(a) Distribution at time t=80.01

(b) Distribution at time t=80.01

(c) Distribution at time t=139.47

(d)Distribution at time t=139.47

Fig. 7 Numerical simulation of system (46) with $D_2 = 1$ and $f_2(p) = 0.1p$

and $k > k^*$, where

$$k^* = \frac{\gamma(m_2 + \delta(s\sigma\phi\mu^2 - 1))^2}{k_1\mu\sigma(m_2 - \delta)(\delta\phi s\sigma\mu^2 + m_2 - \delta + \delta k_2\mu^2 s\sigma)} \quad (56)$$

Proof: Let (a^*, p^*, h^*) be nontrivial equilibrium point, now we are going to analyze the cases next:

- a^* is positive for all value parameters.
- p^* is positive, if and only if, $m_2 > \delta$.
- h^* is positive, if and only if, $h^* > 0$, namely

$$\begin{aligned} &\Leftrightarrow \quad \frac{k_1\mu\sigma a^*}{1 + \sigma\phi\mu^2 p^*} > \gamma \\ &\Leftrightarrow \quad k_1\mu\sigma a^* > \gamma(1 + \sigma\phi\mu^2 p^*) \end{aligned}$$

substituting a^* and p^* into the last equation we arrive to the following inequality for k

$$\begin{aligned} &\Leftrightarrow \quad k > \frac{\gamma(m_2 + \delta(s\sigma\phi\mu^2 - 1))^2}{k_1\mu\sigma(m_2 - \delta)(\delta s\sigma\mu^2(\phi + k_2) + m_2 - \delta)} \\ &\Leftrightarrow \quad k > k^*. \end{aligned}$$

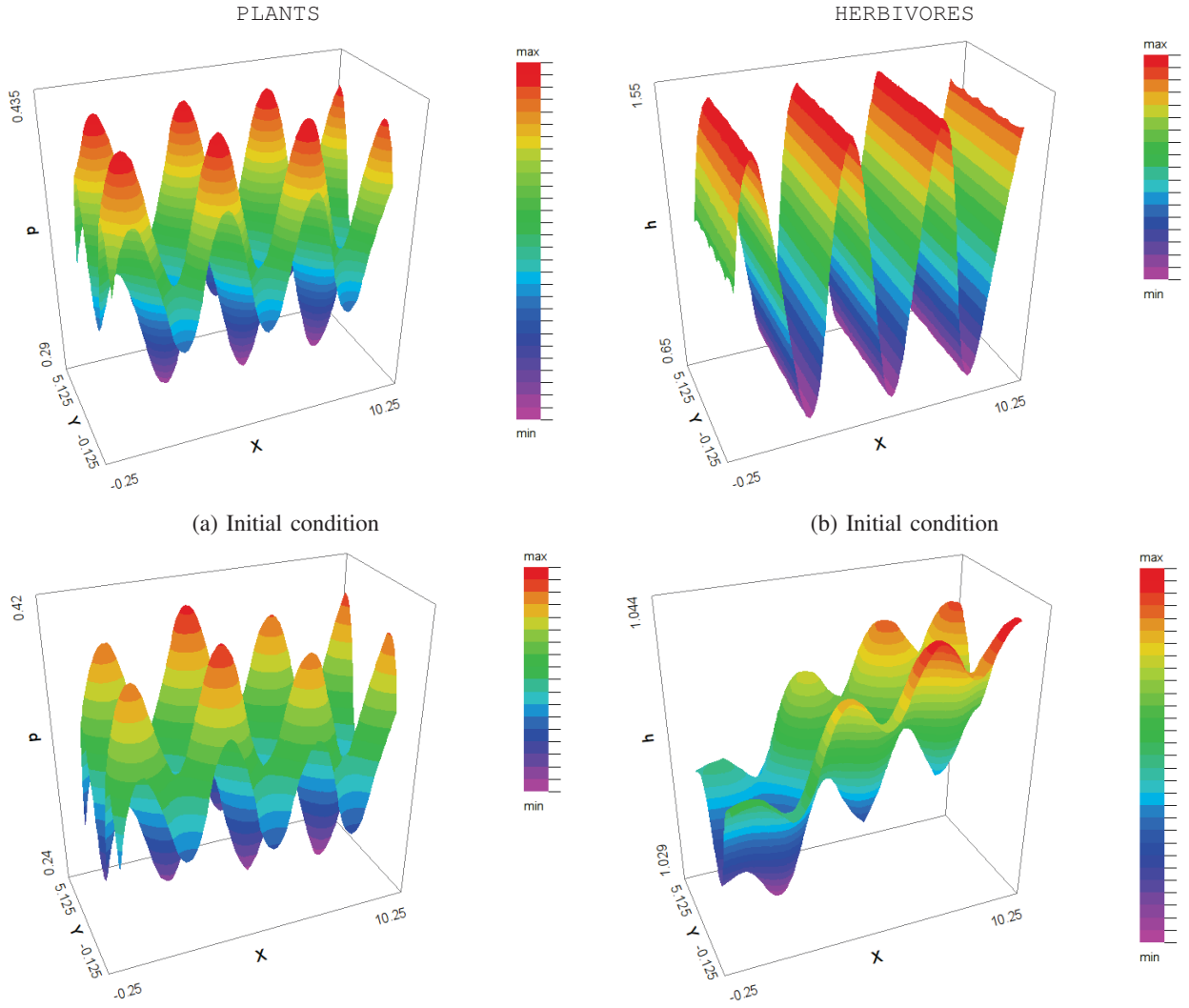
where:

$$k^* = \frac{\gamma(m_2 + \delta(s\sigma\phi\mu^2 - 1))^2}{k_1\mu\sigma(m_2 - \delta)(\delta s\sigma\mu^2(\phi + k_2) + m_2 - \delta)}.$$

Therefore, if $m_2 > \delta$ and $k > k^*$, then a equilibrium point exists (a^*, p^*, h^*) in the positive octant. ■

On the other hand, we have that the Jacobian matrix of system (47) at the equilibrium point (a^*, p^*, h^*) is:

$$J_1 = \begin{bmatrix} J_{11} & J_{12} & 0 \\ J_{21} & J_{22} & J_{23} \\ 0 & J_{32} & 0 \end{bmatrix} \quad (57)$$



(a) Initial condition

(b) Initial condition

(c) Plants at time t=11.61

(d) Herbivores at time t=11.61

Fig. 8 Numerical simulation of system (46) with $D_2 = 1$ and $f_2(p) = 10p$

where

$$
\begin{aligned}
J_{11} &= -\frac{\delta k_2 \mu^2 s\sigma}{\delta\phi s\sigma\mu^2 + m_2 - \delta} - 1 \\
J_{21} &= \frac{\delta k_1 \mu s\sigma}{\delta\phi s\sigma\mu^2 + m_2 - \delta} \\
J_{12} &= \frac{k k_2(\delta - m_2)\mu^2\sigma\eta}{(\delta\phi s\sigma\mu^2 + m_2 - \delta)^3} \\
J_{22} &= -\frac{\delta\gamma}{m_2} - \frac{\delta(k k_1(\delta - m_2)^2\mu\sigma\eta(\mu^2\phi s\sigma - 1))}{(\delta\phi s\sigma\mu^2 + m_2 - \delta)^3 m_2} \\
J_{32} &= \frac{(\delta - m_2)\gamma}{m_1} + \frac{k k_1(\delta - m_2)^2\mu\sigma\eta}{m_1(\delta\phi s\sigma\mu^2 + m_2 - \delta)^2} \\
J_{23} &= -\frac{\delta m_1}{m_2}
\end{aligned}
$$

and

$$\eta = (\delta k_2 s\sigma\mu^2 + \delta\phi s\sigma\mu^2 + m_2 - \delta).$$

The explicit form of characteristic polynomial is:

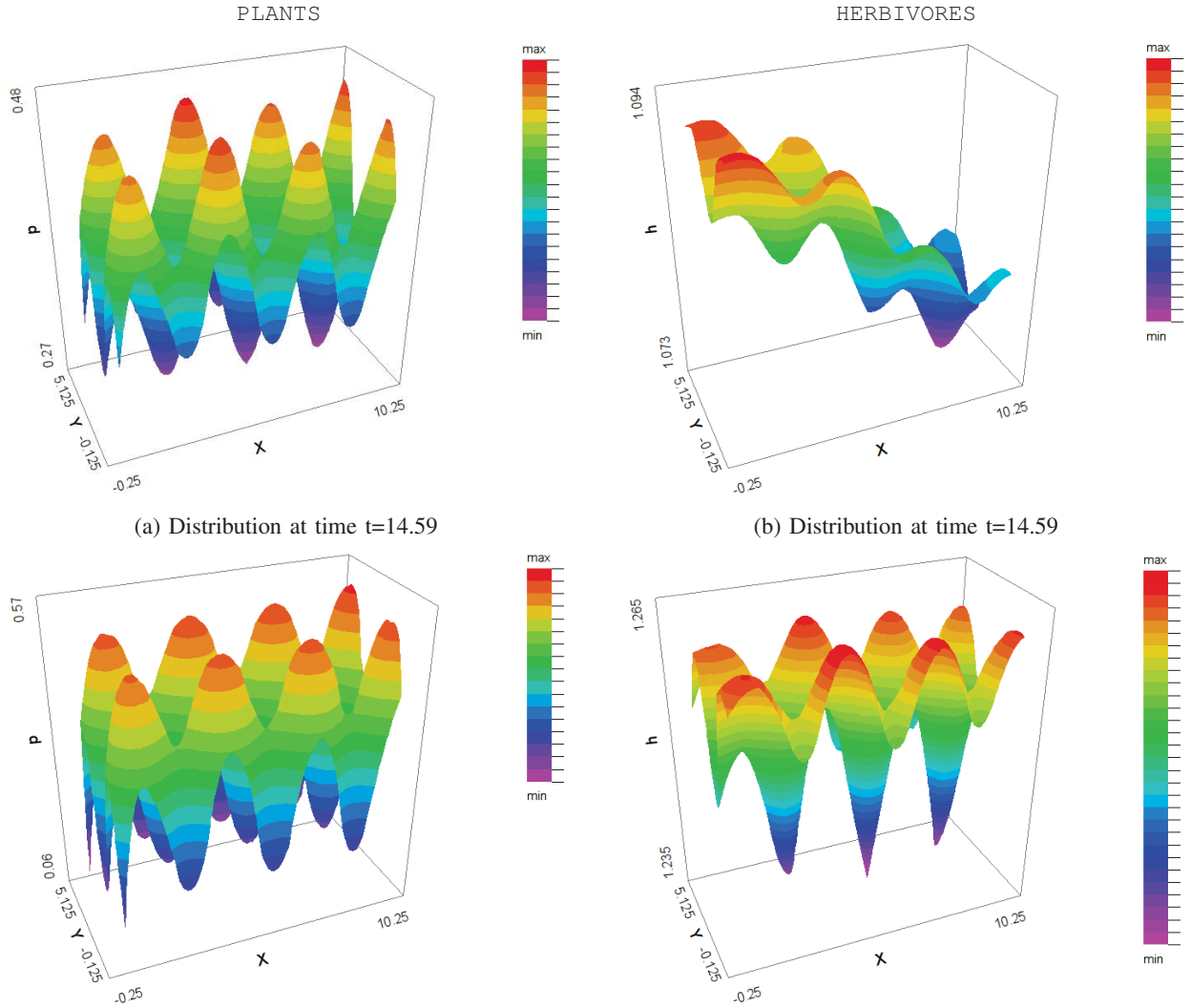
$$p(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 \tag{58}$$

where the coefficients a_i for $i = 1, 2, 3$, can be seen in Appendix (A).

Proposition 1: Let system (47) with $g(h) = 1$. If $k = k^*$, then (a^*, p^*, h^*) is a nonhyperbolic equilibrium point where

$$
\begin{aligned}
a^* &= \frac{\gamma(m_2 - \delta + \sigma\phi\mu^2\delta s)}{k_1\sigma\mu(m_2 - \delta)} \\
p^* &= \frac{\delta s}{m_2 - \delta} \\
h^* &= 0
\end{aligned}
$$

Proof: By a simple calculation, we have that the Jacobian



(a) Distribution at time t=14.59

(b) Distribution at time t=14.59

(c) Distribution at time t=50.74

(d) Distribution at time t=50.74

Fig. 9 Numerical simulation of system (46) with $D_2 = 1$ and $f_2(p) = 10p$

matrix of the system at the equilibrium point is

$$J_{k^*} = \begin{bmatrix} J_{11} & J_{12} & 0 \\ J_{21} & J_{22} & J_{23} \\ 0 & 0 & 0 \end{bmatrix} \tag{59}$$

By the structure having the matrix J_{k^*}, its eigenvalues and eigenvectors are:

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= \frac{1}{2}\left[-\frac{\delta\mu^2 s\sigma(k_2+\gamma\phi)}{\eta} - 1 + r_1\right] \\ \lambda_3 &= \frac{1}{2}\left[-\frac{\delta\mu^2 s\sigma(k_2+\gamma\phi)}{\eta} - 1 - r_1\right] \end{aligned} \tag{60}$$

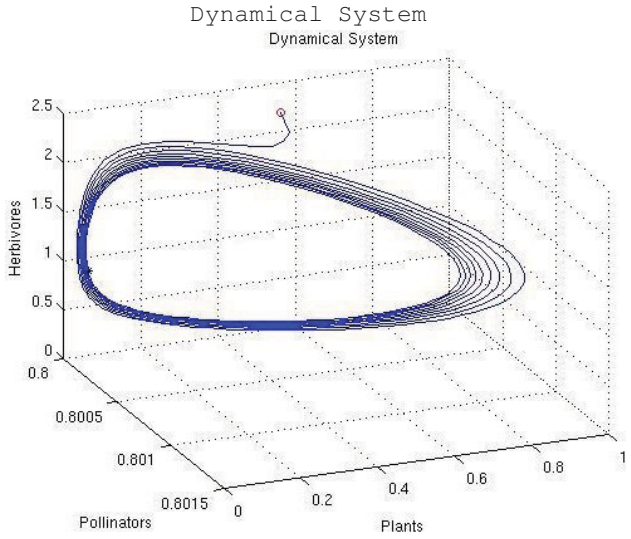
where

$$\begin{aligned} r_1 &= \sqrt{\left(\frac{\delta\mu^2 s\sigma(k_2+\gamma\phi)}{\eta} + 1\right)^2 - \psi} \\ \psi &= \frac{4\delta\gamma\mu^2 s\sigma[(\delta - m_2)(k_2 - \phi) + \delta\mu^2\phi s\sigma(k_2+\phi)]}{\eta^2} \end{aligned}$$

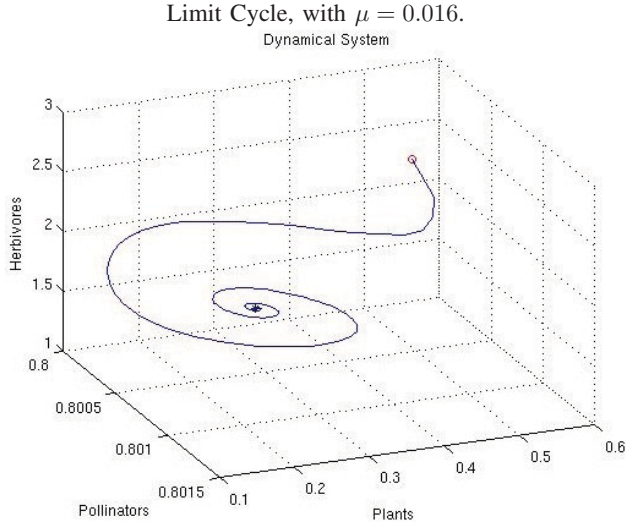
Because $\lambda_1 = 0$, the equilibrium (a^*, p^*, h^*) is a nonhyperbolic equilibrium point. ■

A set of numerical simulation of system (47) show the existence of the at least one equilibrium point in the positive octant. The equilibrium point can be a local attractor. For the other parameters values, the emergence of a limit cycle was also observed. See Fig. 10.

In both cases, the three populations coexist through periodic behaviors or damped oscillations.



Limit Cycle, with $\mu = 0.016$.



Global Attractor, with $\mu = 0.026$.

Fig. 10 Phase Portraits of the system (47), with $g(h) = 1$

V. Summary and Work in Progress

Along this paper, we focus our attention on studying the qualitative behavior of the three population (pollinators, plants and herbivores) mathematical model. This study involves both temporal and spatiotemporal dynamics were based on a situation more realistic situation. This is a list of our results and some research we think its worth to consider as continuation of study presented in this paper.

- We show the existence of some pretty interesting qualitative properties like bifurcation and heteroclinic orbits in the models raised by other authors to describe the temporal dynamics between pollinators, plants and herbivores.
  The results suggest the existence of a limit cycle a local attractor to the temporal dynamics of the three population; an issue which, the authors consider, will be really interesting to analyze.
- In this paper we just explored the spatiotemporal dynamics of two mathematical model. Namely for pollinator-plant and plant-herbivore with the novel ingredient: The plant density dependence of the diffusion term for the pollinator and herbivore. However it is worth to consider such a dynamics when the three populations interact. This aspects are under current investigations by the authors of this paper.
- The presented results of the spatiotemporal dynamics were obtained by considering the homogeneous Neumann boundary conditions. The authors consider that the possible changes that could be caused by using the homogeneous Dirichlet boundary conditions, can provide an interesting comparison between both sets of results. Actually this aspect is also under current investigation.

Appendix A
Determination of the Coefficients a_i in (59)

By using the MATLAB software we found the coefficients a_i in (59). These are:

$$\begin{aligned} a_2 &= \frac{\delta k k_1(\delta - m_2)^2\mu\sigma\eta(\mu^2\phi s\sigma - 1)}{(\delta\phi s\sigma\mu^2 + m_2 - \delta)^3 m_2} \\ &+ \frac{\delta\gamma}{m_2} + \frac{\delta k_2\mu^2 s\sigma}{\delta\phi s\sigma\mu^2 + m_2 - \delta} + 1 \end{aligned}$$

$$\begin{aligned} a_1 &= \frac{\delta\gamma(1+\delta-m_2)}{m_2} - \frac{\delta k k_1 k_2(\delta - m_2)^2\mu^3 s\sigma^2\eta}{(\delta\phi s\sigma\mu^2 + m_2 - \delta)^4} \\ &+ \frac{\delta^3 k k_1 k_2^2(\delta - m_2)^2\mu^5 s^2\sigma^3(\mu^2\phi s\sigma - 1)}{m_2(\delta\phi s\sigma\mu^2 + m_2 - \delta)^4} \\ &+ \frac{2\delta^2 k k_1 k_2(\delta - m_2)^2\mu^3 s\sigma^2(\mu^2\phi s\sigma - 1)}{m_2(\delta\phi s\sigma\mu^2 + m_2 - \delta)^3} \\ &+ \frac{\delta\mu\sigma(k k_1(\delta - m_2)^2 + \delta\gamma k_2\mu s)}{m_2(\delta\phi s\sigma\mu^2 + m_2 - \delta)} \\ &+ \frac{\delta k k_1(\delta - m_2)^2\mu\sigma(\delta k_2\mu^2 s\sigma + \mu^2\phi s\sigma - 1)}{m_2(\delta\phi s\sigma\mu^2 + m_2 - \delta)^2} \\ a_0 &= \frac{\delta(\delta - m_2)(m_2 - \delta + \delta k_2\mu^2 s\sigma + \delta\mu^2\phi s\sigma)b_0}{(\delta\phi s\sigma\mu^2 - \delta + m_2)^3 m_2} \\ b_0 &= \gamma(\delta\phi s\sigma\mu^2 - \delta + m_2)^2 + k k_1\mu\sigma(\delta - m_2)(b_1) \\ b_1 &= m_2 - \delta + \delta k_2\mu^2 s\sigma + \delta\mu^2\phi s\sigma \end{aligned}$$

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