

# Recovering the Boundary Data in the Two Dimensional Inverse Heat Conduction Problem Using the Ritz-Galerkin Method

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**Abstract**—This article presents a numerical method to find the heat flux in an inhomogeneous inverse heat conduction problem with linear boundary conditions and an extra specification at the terminal. The method is based upon applying the satisfier function along with the Ritz-Galerkin technique to reduce the approximate solution of the inverse problem to the solution of a system of algebraic equations. The instability of the problem is resolved by taking advantage of the Landweber's iterations as an admissible regularization strategy. In computations, we find the stable and low-cost results which demonstrate the efficiency of the technique.

**Keywords**—Inverse problem, parabolic equations, heat equation, Ritz-Galerkin method, Landweber iterations.

## I. INTRODUCTION

**H**EAT conduction problems are of vital importance in many areas of applied sciences [1], [8] e.g. heat exchangers, mathematical finance, various chemical and biological systems [1], [5], [6]. The variety of applications has drawn attention of many researchers which in turn led to significant progress in this area [3], [4]. However, multi-dimensional problems with irregular domains have not examined sufficiently [7], [13]. The organization of heat conduction problems is twofold:

- If the heat flux or temperature histories are known as functions of time, the problem of finding temperature distribution is interpreted as a direct problem.
- If the description of a surface condition is sufficient, the problem of approximating temperature histories from transient temperature measurements at some interior locations is called the inverse problem.

Of particular interest to us is the inverse problem for the heat equation which describes the heat conduction procedure in a given inhomogeneous medium.

In the rectangle  $\Omega_2 = \{(x, y, t) \in \mathbf{R}^3 \mid 0 \leq x \leq 1, 0 \leq y \leq 1, 0 < t \leq T\}$ , we seek the approximation of the solution of the Cauchy problem described by the following heat conduction equation in two dimensions [15]:

$$\frac{\partial A(x, y, t)}{\partial t} = \frac{\partial^2 A(x, y, t)}{\partial x^2} + \frac{\partial^2 A(x, y, t)}{\partial y^2} + F(x, y, t), \quad \text{in } \Omega_2, \quad (1)$$

with the initial condition:

$$A(x, y, 0) = f(x, y), \quad (x, y) \in [0, 1] \times [0, 1], \quad (2)$$

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and the known boundary data:

$$A(0, y, t) = h_1(y, t), \quad A(1, y, t) = h_2(y, t), \quad (3)$$

$$\frac{\partial A(x, y, t)}{\partial y} \Big|_{y=1} = \varphi(x, t), \quad (x, y, t) \in [0, 1] \times [0, 1] \times (0, T]. \quad (4)$$

This problem is specified by:

$$A(x^*, y^*, t) = E(t) + \alpha(t), \quad t \in [0, T], \quad (5)$$

at the interior location  $(x^*, y^*) \in (0, 1) \times (0, 1)$  and so that the boundary condition  $\frac{\partial A(x, y, t)}{\partial y} \Big|_{y=0} = g(x, t)$ ,  $(x, t) \in [0, 1] \times (0, T]$  is recovered.

The problem is ill-posed in the sense that the solution (if it exists) does not depend continuously on the data [2], [9], [10]. Here, assuming the sufficient constrains for the existence and uniqueness of the solution for the problem, our main struggle is to present an appropriate numerical scheme for approximating the solution.

In this paper we present an appropriate numerical scheme to approximate the solution provided that a unique solution exists. This paper is arranged as follows: In Section II, we implement the proposed technique and in Section III, we report our numerical findings. Section IV consists of a brief summary.

## II. SOLUTION OF INVERSE PROBLEM VIA RITZ-GALERKIN METHOD

Assume that the given functions satisfy some consistent conditions such as:

$$h_1(y, 0) = f(0, y), \quad h_2(y, 0) = f(1, y), \quad \varphi(x, 0) = f_y(x, 1), \quad (6)$$

$$E(0) + \alpha(0) = f(x^*, y^*).$$

Since (1)-(5) are linear, we employ the classical Ritz-Galerkin method [11]-[13]. Hence, we define:

$$B_1(x, y, t) = (y - 1)\varphi(x, t), \quad (7)$$

$$B_2(x, y, t) = h_1(y, t) + x(h_2(y, t) - h_1(y, t)), \quad (8)$$

$$B_3(x, y, t) = B_1(x, y, t) + B_2(x, y, t) - (y - 1) \frac{\partial B_2(x, y, t)}{\partial y} \Big|_{y=1}, \quad (9)$$

the satisfier function which fulfills (2)-(4) is:

$$SF2(x, y, t) = B_3(x, y, t) + f(x, y) - B_3(x, y, 0), \quad (10)$$

and the Ritz-type approximation for  $A(x, y, t)$  is sought in the form of the truncated series

$$\overline{A(x, y, t)} = \sum_{i_1, i_2, i_3=1}^{N_1, N_2, N_3} c_{i_1, i_2, i_3} t x(x-1)(y-1)^2 \psi_{i_1, i_2, i_3}(x, y, t) + SF2(x, y, t). \quad (11)$$

The expansion coefficients  $c_{i_1, i_2, i_3}$  are determined by the simultaneous Galerkin equations [12], [16]:

$$GES : \begin{cases} \langle Residual(x, y, t), \Psi_{i_1, i_2, i_3}(x, y, t) \rangle = 0, \\ (Residual(x^*, y^*, t_{i_3})) = E(t_{i_3}) + \alpha(t_{i_3}), \end{cases} \quad (12)$$

where  $\langle, \rangle$  denotes the inner product defined by

$$\langle f, g \rangle = \int_{[0, T]} \int_{[0, 1]} \int_{[0, 1]} f(x, y, t) g(x, y, t) dx dy dt,$$

$i_{s \in \{1, 2, 3\}} = \overline{1, N_s - \frac{s(s-1)(s-2)r}{6}}$ ,  $i_3 = \overline{1, N_1 N_2 r}$  and  $\delta(t - t_{i_3})$  is the Dirac delta corresponding to the grids come from the domain of the problem,  $\Psi_{i_1, i_2, i_3}(x, y, t) = \psi_{i_1}(x) \psi_{i_2}(y) \psi_{i_3}(t)$ ,  $i_{j \in \{1, 2, 3\}} = \overline{1, N_j}$  are the basis functions and

$$Residual(x, y, t) := \frac{\partial \overline{A(x, y, t)}}{\partial t} - \frac{\partial^2 \overline{A(x, y, t)}}{\partial x^2} - \frac{\partial^2 \overline{A(x, y, t)}}{\partial y^2} - F(x, y, t), \quad (13)$$

or equivalently

$$Residual(x, y, t) := \overline{A(x, y, t)} - f(x, y) - \int_{[0, t]} \left\{ \frac{\partial^2 \overline{A(x, y, s)}}{\partial x^2} + \frac{\partial^2 \overline{A(x, y, s)}}{\partial y^2} + F(x, y, s) \right\} ds. \quad (14)$$

By applying the Galerkin equations (12), we obtain a linear system of equations  $Ac = b$  which will be solved by the Landweber's iterations to find the stable solution [8]. Following, we introduce the basic Gronwall's lemma [14] formulated in its simplest form which will be used in the sequel.

**Lemma 1:** For a function  $g(t)$  satisfying the inequality:

$$\frac{dg}{dt} \leq ag(t) + b(t), \quad t > 0, \quad a = const, \quad b(t) \geq 0, \quad (15)$$

the following estimate holds:

$$g(t) \leq \exp(at)(g(0) + \int_{[0, t]} \exp(-a\tau)b(\tau)d\tau).$$

**Theorem 1:** Consider  $\overline{A(x, y, t)}$  as the regularized solution for  $A(x, y, t)$  obtained by (7)-(13) corresponding to inverse problem. Then the existence of a priori information

$$\sup_{x \in (0, 1)} |A(x^*, y^*, t) - \frac{\partial A(x, 0, t)}{\partial y}|_{y=0} \leq \delta^*(t), \quad |\delta^*(t)| < \infty,$$

about the solution guarantees solution stability with respect to the right-hand side and boundary input data. Here  $A(x^*, y^*, t) = E(t) + \alpha(t)$  is exactly given as the overdetermination (5).

### III. NUMERICAL EXAMPLES

This example [15] illustrates the applicability of the proposed method for solving the inverse problem with input data given by:

$$h_i(y, t) = iyt^2, \quad i \in \{1, 2\}, \quad \varphi(x, t) = (x+1)t^2, \quad (16)$$

$$E(t) = \frac{3t^2}{4}, \quad (x^*, y^*) = \left(\frac{1}{2}, \frac{1}{2}\right), \quad T = 1.$$

The unknown data which is sought, is given by:

$$g(x, t) = (x+1)t^2, \quad (x, t) \in [0, 1] \times (0, 1]. \quad (17)$$

We use the scheme presented by (7)-(14) to solve this problem numerically and for the exact data i.e.  $\alpha(t) = 0$  we derive the exact solution. It can be expected because the analytical solution of this problem is separable with respect to the variables  $x, y, t$  and our procedure employs the satisfier function which produces accurately the exact solution. As the second example, we consider the inverse heat conduction problem with the exact solution:

$$A(x, y, t) = \sin(x+y+t) - (y-1)^2 \cos(t), \quad 0 \leq t \leq 1, \quad (18)$$

and we are going to find the approximation to the boundary data:

$$g(x, t) = \cos(x+t) + 2 \cos(t), \quad (x, t) \in [0, 1] \times (0, 1], \quad (19)$$

using the extra condition given at the  $(x^*, y^*) = (\frac{1}{4}, \frac{1}{4})$ .

First we contaminate the boundary data using the function:

$$\alpha(t) = \lambda\% \sin\left(\frac{t}{(\lambda\%)^2}\right), \quad \lambda\% = \lambda \times 10^{-2} \in \{3, 5, 9\} \times 10^{-2},$$

and then use the Bernstein multi-scaling functions and the Landweber's iterations with the parameters  $N_1 = N_2 = 1$ ,  $N_3 = 20$ ,  $a = 1$ ,  $m = 10^4$ . we find the the approximations depicted by Figs. 1(a)-1(c). It is seen as  $\lambda$  decreases, the accuracy of the approximations gradually becomes better.

### IV. CONCLUSION

The paper addresses a technique for boundary identification of the two dimensional inverse heat conduction problems. Focusing on the numerical aspects, the discontinuity between the input and output data has been discussed. Appropriate application of the Ritz-Galerkin method along with an iterative procedure, namely, the Landweber's iterations, yield satisfactory productions that contain the qualifications of accurate, stable and low cost numerical solution.

#### APPENDIX A

##### PROOF OF THE THEOREM 1

Generally speaking, the approximate solution (11) satisfies the following partial differential equation

$$\frac{\partial \overline{A(x, y, t)}}{\partial t} - \left( \frac{\partial^2 \overline{A(x, y, t)}}{\partial x^2} + \frac{\partial^2 \overline{A(x, y, t)}}{\partial y^2} \right) = \overline{F(x, y, t)}, \quad (20)$$

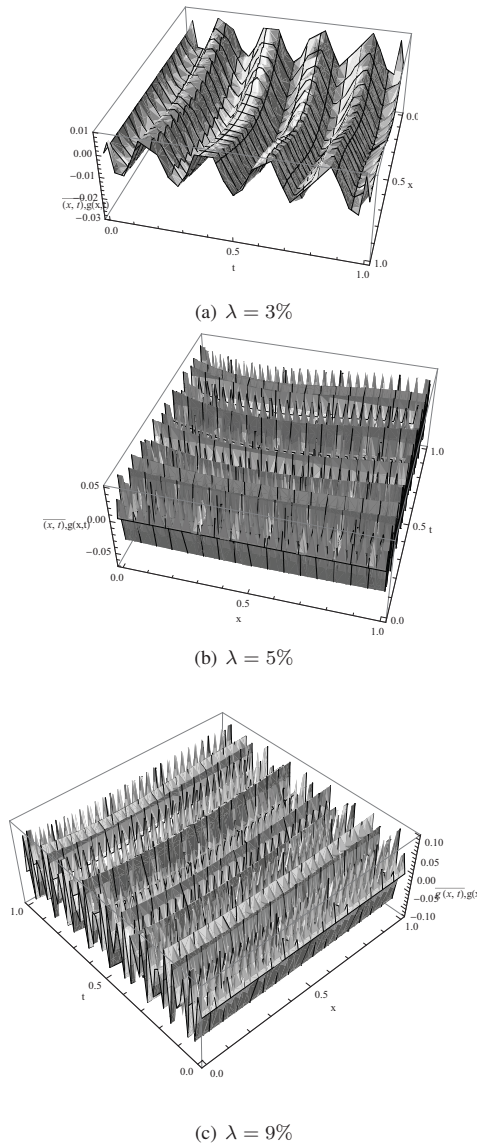


Fig. 1: Graphs of the absolute error between the exact and approximate solutions, i.e.  $|g(x, t) - \bar{g}(x, t)|$  obtained by the Ritz-Galerkin method discussed in Example 2 for noisy data

which by subtracting (20) from (1) and setting  $e(x, y, t) = A(x, y, t) - \bar{A}(x, y, t)$ ,  $\delta F(x, y, t) = F(x, y, t) - \bar{F}(x, y, t)$  we get

$$\frac{\partial e(x, y, t)}{\partial t} - \left( \frac{\partial^2 e(x, y, t)}{\partial x^2} + \frac{\partial^2 e(x, y, t)}{\partial y^2} \right) = \delta F(x, y, t), \quad (21)$$

with the following homogeneous initial and boundary conditions

$$e(x, y, 0) = e(0, y, t) = e(1, y, t) = \frac{\partial e(x, y, t)}{\partial y} \Big|_{y=1} = 0. \quad (22)$$

Now for the functions given in the domain  $\Omega_2^* = (0, 1) \times (0, 1)$ , and vanishing at the boundary points of this domain i.e.  $\partial\Omega_2^*$ ,

we define a Hilbert space  $\mathcal{H} = \mathcal{L}_2(\Omega_2^*)$ , equipped with the norm

$$\|v\|_{\mathcal{H}} = \langle v, v \rangle^{\frac{1}{2}} = \left( \int_{\Omega_2^*} v^2(x) dx \right)^{\frac{1}{2}}, \quad x \in \Omega_2^*. \quad (23)$$

Thus taking  $u \in \mathcal{H}$  and introducing the operator

$$\mathcal{F}u = - \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad 0 < x < 1, \quad 0 < y < 1, \quad (24)$$

taking into account (21)-(22) we have

$$\frac{de}{dt} + \mathcal{F}e = \delta F(t), \quad e(0) = 0, \quad 0 < t < T.$$

By multiplying (21) by  $e(t)$ , we have

$$\underbrace{\left\langle \frac{de}{dt}, e \right\rangle}_{=(III)} + \underbrace{\langle \mathcal{F}e, e \rangle}_{=(II)} = \underbrace{\langle \delta F, e \rangle}_{=(I)}. \quad (25)$$

Firstly, applying the Cauchy-Bunyakowsky inequality [14], we have:

$$(I) = \int_{[0,1]} \int_{[0,1]} \delta F(x, y, t) e(x, y, t) dx dy \leq \|\delta F\|_{\mathcal{H}} \|e\|_{\mathcal{H}}. \quad (26)$$

Secondly, applying integration by parts and taking the homogeneous conditions (22) we have

$$\begin{aligned} (II) &= - \int_{[0,1]} \int_{[0,1]} \frac{\partial}{\partial x} \left( \frac{\partial e}{\partial x} \right) e dx dy - \int_{[0,1]} \int_{[0,1]} \frac{\partial}{\partial y} \left( \frac{\partial e}{\partial y} \right) e dy dx \\ &= \int_{[0,1]} \int_{[0,1]} \left( \frac{\partial e}{\partial x} \right)^2 + \left( \frac{\partial e}{\partial y} \right)^2 dy dx + \int_{[0,1]} e(x, 0, t) \frac{\partial e}{\partial y} \Big|_{y=0} dx. \end{aligned} \quad (27)$$

Finally, considering

$$\begin{aligned} (III) &= \int_{[0,1]} \int_{[0,1]} e \frac{de}{dt} dx dy = \frac{1}{2} \frac{d}{dt} \left( \int_{[0,1]} \int_{[0,1]} e^2 dx dy \right) \\ &= \|e\|_{\mathcal{H}} \frac{d}{dt} \|e\|_{\mathcal{H}}, \end{aligned}$$

we get

$$\begin{aligned} \|e\|_{\mathcal{H}} \frac{d}{dt} \|e\|_{\mathcal{H}} + \underbrace{\int_{[0,1]} \int_{[0,1]} \left( \frac{\partial e}{\partial x} \right)^2 + \left( \frac{\partial e}{\partial y} \right)^2 dy dx}_{\geq 0} \\ \leq \|\delta F\|_{\mathcal{H}} \|e\|_{\mathcal{H}} + \left| \int_{[0,1]} e(x, 0, t) \frac{\partial e}{\partial y} \Big|_{y=0} dx \right|. \end{aligned}$$

Applying the mean-value theorem for  $e(x, 0, t) \frac{\partial e}{\partial y} \Big|_{y=0}$  and taking  $x^\dagger \in (0, 1)$  we have

$$\begin{aligned} \left| \int_{[0,1]} e(x, 0, t) \frac{\partial e}{\partial y} \Big|_{y=0} dx \right| &= |e(x^\dagger, 0, t) \frac{\partial e(x^\dagger, y, t)}{\partial y} \Big|_{y=0}| \\ &\leq \|e\|_{\mathcal{H}} \left| \frac{\partial e(x^\dagger, y, t)}{\partial y} \Big|_{y=0} \right|. \end{aligned}$$

Considering the extra specification (5) and assuming the priori condition

$$\sup_{x \in (0,1)} \left| A(x^*, y^*, t) - \frac{\partial A(x, 0, t)}{\partial y} \Big|_{y=0} \right| \leq \delta^*(t),$$

we get

$$\left| \frac{\partial e(x^\dagger, y, t)}{\partial y} \Big|_{y=0} \right| \leq (|\delta^*(t) - \overline{\delta^*(t)}| + |E(t) - \overline{E(t)}|),$$

where

$$\sup_{x \in (0,1)} \left| \overline{A(x^*, y^*, t)} - \frac{\partial \overline{A(x, 0, t)}}{\partial y} \Big|_{y=0} \right| \leq \overline{\delta^*(t)},$$

$$\overline{A(x^*, y^*, t)} = \overline{E(t)} + \alpha(t).$$

Consequently

$$\left\| \frac{de}{dt} \right\|_{\mathcal{H}} \leq \|\delta F(t)\|_{\mathcal{H}} + \|\delta^*(t) - \overline{\delta^*(t)}\|_{\mathcal{H}} + \|E(t) - \overline{E(t)}\|_{\mathcal{H}},$$

by using the Gronwal's lemma with

$$a = 0, \quad b(t) = \|\delta F(t)\|_{\mathcal{H}} + \|\delta^*(t) - \overline{\delta^*(t)}\|_{\mathcal{H}} + \|E(t) - \overline{E(t)}\|_{\mathcal{H}},$$

one obtains

$$\|e(t)\|_{\mathcal{H}} \leq \int_{[0,t]} (\|\delta F(s)\|_{\mathcal{H}} + \|\delta^*(s) - \overline{\delta^*(s)}\|_{\mathcal{H}} + \|E(s) - \overline{E(s)}\|_{\mathcal{H}}) ds, \quad 0 \leq s \leq T.$$

For a given  $\epsilon > 0$ , such that  $\sup_{0 \leq t \leq T} \{\|\delta F(t)\|_{\mathcal{H}} + \|\delta^*(t) - \overline{\delta^*(t)}\|_{\mathcal{H}} + \|E(t) - \overline{E(t)}\|_{\mathcal{H}}\} \leq \epsilon$  we deduce

$$\|e(t)\|_{\mathcal{H}} \leq T\epsilon.$$

The latter inequality guarantees continuous dependence of the solution of problem (1)-(5) on the right-hand side and boundary conditions.

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