

On Bianchi Type Cosmological Models in Lyra's Geometry

R. K. Dubey

Abstract—Bianchi type cosmological models have been studied on the basis of Lyra's geometry. Exact solution has been obtained by considering a time dependent displacement field for constant deceleration parameter and varying cosmological term of the universe. The physical behavior of the different models has been examined for different cases.

Keywords—Bianchi type-I cosmological model, variable gravitational coupling (G) and Cosmological Constant term (β).

I. INTRODUCTION

WEYL [1] first try to unify gravitation and electromagnetism in single space-time geometry. He showed how one can introduce vector field in the Riemannian space time with an intrinsic geometric significance, but this theory was not accepted. Later Lyra's [2] proposed a modification of Riemannian geometry by introducing a gauge function into the structureless manifold that bears close resemblance to Weyl's geometry and removes non integrability condition arises in Weyl's geometry. In this way Riemannian geometry was given a new modification by Lyra's geometry Sen [3] found that the static with finite density in Lyra's modified Riemannian geometry is similar to the static Einstein model. Sen and Dunn [4] proposed a new scalar tensor theory of gravitation and constructed the field equations analogous to the Einstein's field equations based on the Lyra's Geometry.

Halford [5] has pointed out that the constant vector displacement field ϕ_i in Lyra's Geometry plays the role of cosmological constant in the normal general relativistic treatment. Further, it was shown by Halford that the scalar tensor treatment based on Lyra's Geometry predicts the same effects, within the observational limit, as in Einstein theory.

Several researchers have studied cosmological models based on Lyra's geometry with a constant displacement field. Beesham [6] considered FRW models with a time dependent field. Singh and Singh [7] have presented Bianchi type I, II and Kantowski-Sachs cosmological models with a time dependent displacement field and have made a comparative study of the Robertson-Walker Models with a constant deceleration parameter in Einstein's theory with cosmological term and in the cosmological theory based on Lyra's geometry. The essential difference between the cosmological theories based on Lyra's geometry and Riemannian geometry

lies in the fact that constant vector displacement field's β arises naturally from the concept of gauge function in Lyra's geometry where as constant Λ was introduced in adhoc fashion in the usual treatment. Some very recent works done on Lyra's geometry are given in [8] i.e. cosmology based on Lyra's geometry and in one model they have shown that the gauge function is large in the beginning but decreases with the evolution of the model. Pradhan and Pandey [9] and Rahaman et al. [8] studied some topological defects within the framework of Lyra's geometry. Bhowmik and Rajput [10] obtained anisotropic Bianchi type cosmological models on the basis of Lyra's geometry. Recently Reddy [11] examined the plane symmetric cosmic strings in Lyra's manifold. Faber and Guth [12] shown that at a very early stage, the universe might have been anisotropic. Dubey et al. [13] have studied anisotropic cosmological models with varying cosmological term. However in the course of evolution the universe has developed isotropy as we observe today. We have been motivated by this conjecture to study the anisotropic models of the universe in this paper.

In this paper our purpose is to derive Bianchi type I cosmological models with a time dependent displacement field. We have derived cosmological models by assuming constant deceleration parameter in Lyra's geometry. Also assuming $\lambda = 0, 1$, we have obtained different models and discussed the physical behavior of models.

II. METRIC AND FIELD EQUATIONS

We consider Bianchi type I metric:

$$ds^2 = dt^2 - A^2 dx^2 - B^2 dy^2 - C^2 dz^2 \quad (1)$$

where A, B and C are the functions of cosmic time t only.

The field equations in normal gauge for Lyra's geometry as obtained by Sen [3] are

$$R_{ij} - \frac{1}{2} g_{ij} R + \frac{3}{2} \phi_i \phi_j - \frac{3}{4} g_{ij} \phi_k \phi^k = -8\pi G(t) T_{ij} \quad (2)$$

where ϕ_i the displacement vector and other symbols have their usual meanings. We consider the energy momentum tensors corresponding to perfect fluid as:

$$T_{ij} = (\rho + p) u_i u_j - p g_{ij} \quad (3)$$

The energy momentum conservation law is given by:

$$T^j_{i;j} = 0 \quad (4) \quad \text{From (6), (7), (8), (9), we obtain}$$

The time like displacement vector ϕ_i in (1) is given by

$$\phi_i = [\beta(t), 0, 0, 0] \quad (5) \quad \text{From (10) and (17):}$$

With (1), (3), (5), the field equation (2) and energy momentum conservation equation (4) becomes:

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} - \frac{3}{4}\beta^2 = 8\pi G(t)\rho \quad (6) \quad \text{Let us assume an equation of state:}$$

$$\frac{\dot{B}\dot{C}}{BC} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{3}{4}\beta^2 = -8\pi Gp \quad (7) \quad p = \lambda\rho, 0 \leq \lambda \leq 1 \quad (19)$$

$$\frac{\dot{C}\dot{A}}{CA} + \frac{\dot{C}}{C} + \frac{\dot{A}}{A} + \frac{3}{4}\beta^2 = -8\pi Gp \quad (8) \quad \text{With help of (19), (10) becomes,}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{3}{4}\beta^2 = -8\pi Gp \quad (9) \quad \dot{\rho} + 3H\rho(1+\lambda) = 0 \Rightarrow \frac{\dot{\rho}}{\rho} = -3(1+\lambda)\frac{\dot{a}}{a}$$

$$\dot{\rho} + 3H(\rho + p) = 0 \quad (10) \quad \text{Integrating:} \quad \log \rho = -3(1+\lambda) \log a + \log \rho_0 \Rightarrow \rho = \rho_0 a^{-3(1+\lambda)} \quad (20)$$

where $H = \frac{\dot{a}}{a}$ = Hubble's Parameter.

Spatial volume as an average scale factor of model (1) may be defined as:

$$a = (ABC)^{\frac{1}{3}} \quad (11)$$

Hubble's parameter in anisotropic model may be defined as:

$$H = \frac{\dot{a}}{a} = \frac{1}{3} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \quad (12)$$

We take anisotropy σ^2 as:

$$\sigma^2 = \frac{1}{2} \left(\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2} \right) - \frac{\theta^2}{6} \quad (13)$$

Taking $\theta = 3H$ & $H = \frac{1}{3} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right)$, (13) reduces to:

$$\sigma^2 = \frac{1}{2} \left\{ \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right)^2 - 2 \left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} \right) \right\} - \frac{3}{2} H^2 \quad (14)$$

$$\sigma^2 = 3H^2 - \left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} \right)$$

Equations (6) and (14) give:

$$\sigma^2 = 3H^2 - \frac{3}{4}\beta^2 - 8\pi G\rho \quad (15)$$

From (7), (8), (9) & (14), we have:

$$2\dot{H} + 3H^2 + \frac{3}{4}\beta^2 = -8\pi Gp - \sigma^2 \quad (16)$$

$$8\pi G\dot{\rho} + 8\pi G\rho + \frac{3}{2}\beta^2 \left(\frac{\dot{\beta}}{\beta} + 3H \right) + 8\pi G(\rho + p)3H = 0 \quad (17)$$

$$\frac{3}{2}\beta^2 \left(\frac{\dot{\beta}}{\beta} + 3H \right) = -8\pi G\rho \quad (18)$$

where ρ_0 is integrating constant. Using (19); (16) and (17) becomes:

$$2\dot{H} + 3H^2 + \frac{3}{4}\beta^2 = -8\pi G\lambda\rho - \sigma^2 \quad (21)$$

$$8\pi G\dot{\rho} + 8\pi G\rho + \frac{3}{2}\beta\dot{\beta} + \frac{9}{2}\beta^2 H + 8\pi G(\rho + p)3H = 0 \quad (22)$$

Adding (7), (8), (9):

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} + 2 \left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} \right) + \frac{9}{4}\beta^2 = -24\pi Gp$$

Using (6), we have:

$$\Rightarrow 8\pi G\rho + 3\beta^2 + 2 \left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} \right) = -24\pi Gp$$

Using (6) and (12), this equation reduces to:

$$8\pi G\rho + 3\beta^2 + 2 \left(3\dot{H} + 9H^2 - 16\pi G\rho - \frac{3}{2}\beta^2 \right) = -24\pi Gp, \quad (23)$$

$$\Rightarrow 2\dot{H} + 6H^2 - \eta\rho(1-\lambda) = 0, \text{ where } \eta = 8\pi G \text{ (Using 19)}$$

here, β^2 in Lyra's geometry plays the role of cosmological constant as Λ in the normal general relativistic treatment.

III. SOLUTION OF FIELD EQUATION

Assuming deceleration parameter constant, we consider:

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = n \text{ (constant)} \quad (24)$$

where n is constant and $n > -1$, $n \neq -1$

$$\text{From (24), } n \left(\frac{da}{dt} \right)^2 + a \left(\frac{d^2a}{dt^2} \right) = 0 \quad (25)$$

For integrating (25), let $\frac{da}{dt} = Z$, (25) reduces to, $nZ^2 + aZ \frac{dZ}{da} = 0$
integrating, $a = a_0 t^{\frac{1}{n+1}}$ (26) where a_0 is integrating constant at origin. Using (26), (20) becomes:

$$\rho = \rho_0 \left(a_0 t^{\frac{1}{n+1}} \right)^{-3(1+\lambda)} = \rho_1 t^{\frac{-3(1+\lambda)}{n+1}}, \quad (26)$$

where

$$\rho_1 = \rho_0 a_0^{-3(1+\lambda)} = \text{constant} \quad (27)$$

$$\text{From (23), } 2\dot{H} + 6H^2 = \eta\rho(1-\lambda)$$

$$\Rightarrow 2\frac{\ddot{a}}{a} + 4\frac{\dot{a}^2}{a^2} = \eta\rho(1-\lambda) \quad (\because H = \frac{\dot{a}}{a})$$

Integrating this equation using (26), and (27):

$$2(a^2\dot{a}) = \eta(1-\lambda)\rho_1 a_0 t^{\frac{-3\lambda+n+1}{n+1}} \Rightarrow \\ 8\pi G = \frac{2a_0(-3\lambda+n+1)}{(1-\lambda)\rho_1(n+1)} t^{\frac{1+3\lambda-2n}{n+1}} \Rightarrow G = G_0 t^{\frac{1+3\lambda-2n}{n+1}} \quad (28)$$

$$\text{where } G_0 = \frac{a_0(-3\lambda+n+1)}{4\pi(1-\lambda)\rho_1(n+1)} = \text{constant}$$

From (15): $\frac{3}{2}\beta\dot{\beta} + \frac{9}{2}\beta^2 \frac{\dot{a}}{a} = -8\pi G_0 \rho$. Using (26), (27), (28) in this equation, we have:

$$\frac{3}{2}\beta\dot{\beta} + \frac{9}{2}\beta^2 \frac{t^{-1}}{n+1} = -8\pi G_0 \rho_1 \frac{(1+3\lambda-2n)}{1+n} t^{-3} \Rightarrow \frac{3}{2}\beta\dot{\beta} + \frac{9}{2}\beta^2 \frac{1}{t(n+1)} = -\beta_0 t^{-3} \quad (29)$$

where $\beta_0 = 8\pi G_0 \rho_1 \frac{(1+3\lambda-2n)}{1+n} t = \text{constant}$. Let us consider, $\beta^2 = y$. Using above (29) reduces to:

$$\dot{y} + \frac{6}{(n+1)} \frac{y}{t} = -\frac{4}{3} \beta_0 t^{-3},$$

which is linear differential equation. So, solution is:

$$\beta^2 = -\frac{4}{3} \beta_0 t^{-2} + m t^{\frac{-6}{n+1}} \quad (30)$$

where m is integration constant. From (12) we have:

$$\sigma^2 = 3H^2 - \frac{3}{4}\beta^2 - 8\pi G\rho$$

Using (26), (27), (28) and (30):

$$\sigma^2 = \frac{m_1}{t^{\frac{6}{n+1}}}, \quad \text{where } m_1 \text{ is constant} \quad (31)$$

Equations (26)-(31) show that for $\frac{1}{2} < n+1 < 3(1+\lambda)$, and $\lambda < 1, G, \sigma^2, \beta^2$, to be positive and σ^2 to decay faster.

From (26) and (28), we find that this solution has only decelerating expansion with decaying $G(t)$.

From (30), for $n \neq -1$, β^2 is seen to be negative at the very early stage after the birth of the universe. Value of $\beta^2(t)$ decreases with the evolution of the universe.

From (28) & (30), we have that as $t \rightarrow \infty, G \rightarrow \infty$ and $\Lambda \rightarrow 0$, also scale factor becomes infinite. Therefore, we can say that model would essentially give an empty universe for large time t . The presence of a positive β^2 puts a restriction on the upper limit of anisotropy, whereas a negative Λ contributes to the anisotropy. The universe will be in a decelerating phase for negative Λ , and for positive Λ , the universe will slow its rate of decrease.

Further we will consider the different models for different values of λ & n .

Case Study: If $p = \rho = 0$, then from (23), we have:

$$\dot{H} + 3H^2 = 0 \\ \Rightarrow \frac{\dot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 = 0 \Rightarrow \frac{\dot{a}}{a} = -\frac{1}{3} \Rightarrow 3\frac{da}{a} = -dt$$

Integrating,

$$\log a^3 = -t + \log t_0 \Rightarrow a = a_1 t^{\frac{1}{3}}, \quad (32)$$

where a_1 is constant. Putting $p = \rho = 0$ in (15) and (18), we have:

$$\sigma^2 = 3H^2 - \frac{3}{4}\beta^2 \quad (33)$$

$$\frac{3}{2}\beta^2 \left(\frac{\dot{\beta}}{\beta} + 3H \right) = 0 \quad (34)$$

From (34), $\left(\frac{3}{4}\beta^2 \right)' + \frac{3}{4}\beta^2 \cdot 6H = 0$ (where dot denotes derivative)

$$\Rightarrow \left(\frac{3}{4}\beta^2 \right)' = -6\frac{\dot{a}}{a} \frac{3}{4}\beta^2$$

Integrating both sides, we have $\log \left(\frac{3}{4}\beta^2 \right) = \log a^{-6} + \log a_2$, a_2 is an integrating constant:

$$\beta^2 = \frac{4}{3} a_2 \left(a_1 t^{\frac{1}{3}} \right)^{-6}$$

Using (32):

$$\beta^2 = \frac{4}{3} a_2 a_1^{-6} t^{-2} \Rightarrow \beta^2 = \beta_1 t^{-2} \quad (35)$$

where β_1 is positive constant. From (33):

$$\begin{aligned} \sigma^2 &= 3H^2 - \frac{3}{4}\beta^2 \\ \sigma^2 &= \frac{1}{3}t^{-2} - \frac{3}{4}\beta_1 t^{-2} \quad (\text{Using (32) and (35)}) \\ \sigma^2 &= \left(\frac{1}{3} - \frac{3}{4}\beta_1 \right) t^{-2}, \quad \beta_1 < \frac{1}{3} \end{aligned} \quad (36)$$

Above models are empty universe model with decelerating expansion. In this case, $G(t)$ may have any constant expression. In above case anisotropy and gauge function β decreases from a large value with expansion of the universe.

IV. DISCUSSIONS

In this paper we have studied anisotropic Bianchi type-I cosmological models with varying cosmological term in Lyra's geometry. Singh and Desikan [14] have obtained models with both positive and negative $\beta_2(t)$. We have obtained the case of $\beta_2 > 0$ only because $\beta_2 < 0$ will give $\beta(t)$ into an imaginary quantity which is not physical. We find the scale factor by assuming deceleration parameter constant. For all models obtained σ^2 decays faster than ρ . But in stiff model rate of σ^2 & ρ are same. In all the models the universe evolves with decelerating expansion with decaying $G(t)$. For $p = \rho = 0$, the empty universe allows decelerating expansion with independent nature of $G(t)$.

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