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Analytical Slope Stability Analysis Based on the Statistical Characterization of Soil Shear Strength

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***Abstract*—**Increasing our ability to solve complex engineering problems is directly related to the processing capacity of computers. By means of such equipments, one is able to fast and accurately run numerical algorithms. Besides the increasing interest in numerical simulations, probabilistic approaches are also of great importance. This way, statistical tools have shown their relevance to the modelling of practical engineering problems. In general, statistical approaches to such problems consider that the random variables involved follow a normal distribution. This assumption tends to provide incorrect results when skew data is present since normal distributions are symmetric about their means. Thus, in order to visualize and quantify this aspect, 9 statistical distributions (symmetric and skew) have been considered to model a hypothetical slope stability problem. The data modeled is the friction angle of a superficial soil in Brasilia, Brazil. Despite the apparent universality, the normal distribution did not qualify as the best fit. In the present effort, data obtained in consolidated-drained triaxial tests and saturated direct shear tests have been modeled and used to analytically derive the probability density function (PDF) of the safety factor of a hypothetical slope based on Mohr-Coulomb rupture criterion. Therefore, based on this analysis, it is possible to explicitly derive the failure probability considering the friction angle as a random variable. Furthermore, it is possible to compare the stability analysis when the friction angle is modelled as a Dagum distribution (distribution that presented the best fit to the histogram) and as a Normal distribution. This comparison leads to relevant differences when analyzed in light of the risk management.

***Keywords*—**Statistical slope stability analysis, Skew distributions, Probability of failure, Functions of random variables.

I. Introduction

GEOTECHNICAL engineers deal with a considerable complexity which comes from the natural variability of the materials involved in their interventions. Such variability tends to be more intense when deeply weathered soils are taken into account. This is the case of the Brazilian soils present in the superficial layers of Brasilia's stratigraphic profile. The latter consist of a porous clay whose layers tend be as deep as 6 to 12 meters. This material is characterized by its high void ratio (1 to 3), low specific weight (13 to 17 kN/m³) and the high content of iron and aluminum oxi-hidroxides, which comes from the weathering process.

A good alternative to study this variability is the use of statistical tools, which have had a growing relevance in several engineering analyses. In [8], the authors presented the statistical evaluation of slope stability in rock, making the analysis of the spatial variability of structural features, of the properties of the rock mass and of the likelihood values of seismic forces and intensities of local rainfall. A statistical analysis of the intensity of rainfall in a slope stability was also conducted in [2].

Reference [5], instead of using the finite element method, used probabilistic Monte Carlo method to develop a reliable design for the mechanical stability of earth walls. In relation to soil-nail walls, [15] studied the reliability of load factors and strength. Taking advantage of the scope of reliability, the study in drilled wells was explored by [16]. An alternative that has been widely used to study the variability of soil is the use of analytical methods, such as the ones shown in [7], [11], [18], and more recently, [13].

Regarding the statistical distributions applied to study random variables in geotechnics, the most widely used distribution is the normal one, which is probably the most applied distribution in science due to its universality. However, despite being a widespread distribution in geotechnics, this type of distribution is symmetric, failing to represent the true behavior of the data, which is usually asymmetric. Moreover, its probability density function (PDF) tend to not represent adequately the real physical behavior of the parameters as its support is the whole real axis while some geotechnical properties only have physical meaning for positive values. On the other hand, many geotechnical parameters may be considered as normal random variables when little is known about the properties studied. This approach is valid because in such cases this may be the only alternative to provide risk assessment. It is important to mention that if the distribution of the data studied is symmetric about its average, the normal distribution can be truncated to limit the response only to positive values.

For this paper, a compilation of information regarding the strength of the local soil has been done using as a basis the works [3], [4] and [9]. The authors considered consolidated-drained triaxial tests and saturated direct shear tests, performed in the topsoil of the Federal District. Based on local experience, it was considered that in the first six meters the soil has similar characteristics. Thus, only the corresponding fraction of the data has been analyzed.

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TABLE I
STATISTICAL DISTRIBUTIONS USED TO FIT THE DATA STUDIED

Distributions	Parameters	Probability Density Function	Cumulative Density Function
Normal Distribution	μ, σ	$\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$	$\frac{1}{2} Erfc\left[\frac{-x+\mu}{\sqrt{2}\,\sigma}\right]$
Student's T Distribution	μ, σ, ν	$\frac{\left(\frac{\nu}{\nu+\frac{(x-\mu)^2}{\sigma^2}}\right)^{\frac{1+\nu}{2}}}{\sigma\sqrt{\nu}\ \mathrm{B}\left[\frac{\nu}{2},\frac{1}{2}\right]}$	$\begin{cases} \frac{\mathrm{B}\left[\frac{\nu\,\sigma^2}{(x-\mu)^2+\nu\,\sigma^2},\frac{\nu}{2},\frac{1}{2}\right]}{2\,\mathrm{B}\left[\frac{\nu}{2},\frac{1}{2}\right]} & x>\mu \\ \frac{1}{2}\left(1+\frac{\mathrm{B}\left[\frac{(x-\mu)^2}{(x-\mu)^2+\nu\,\sigma^2},\frac{1}{2},\frac{\nu}{2}\right]}{\mathrm{B}\left[\frac{1}{2},\frac{\nu}{2}\right]}\right) & x\le\mu \end{cases}$
Cauchy Distribution	a, b	$\frac{1}{b\,\pi\left(1+\frac{(-a+x)^2}{b^2}\right)}$	$\frac{1}{2}+\frac{ArcTan\left[\frac{-a+x}{b}\right]}{\pi}$
LogNormal Distribution	μ, σ	$\begin{cases} \frac{e^{-\frac{(-\mu+log[x])^2}{2\sigma^2}}}{\sqrt{2\pi}\,x\,\sigma} & x>0 \\ 0 & x\le 0 \end{cases}$	$\begin{cases} \frac{1}{2} Erfc\left[\frac{\mu-Log[x]}{\sqrt{2}\,\sigma}\right] & x>0 \\ 0 & x\le 0 \end{cases}$
LogLogistic Distribution	γ, σ	$\begin{cases} \frac{x^{-1+\gamma}\gamma\,\sigma^{-\gamma}}{\left(1+\left(\frac{x}{\gamma}\right)^{\gamma}\right)^2} & x>0 \\ 0 & x\le 0 \end{cases}$	$\begin{cases} \frac{1}{1+\left(\frac{x}{\sigma}\right)^{-\gamma}} & x>0 \\ 0 & x\le 0 \end{cases}$
Gamma Distribution	α, β	$\begin{cases} \frac{e^{-\frac{x}{\beta}}x^{-1+\alpha}\beta^{\alpha}}{\Gamma(\alpha)} & x>0 \\ 0 & x\le 0 \end{cases}$	$\begin{cases} 1-\frac{\Gamma\left[\alpha,\frac{x}{\beta}\right]}{\Gamma[\alpha]} & x>0 \\ 0 & x\le 0 \end{cases}$
Weibull Distribution	α, β	$\begin{cases} \frac{e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\alpha\left(\frac{x}{\beta}\right)^{-1+\alpha}}{\beta} & x>0 \\ 0 & x\le 0 \end{cases}$	$\begin{cases} 1-e^{-\left(\frac{x}{b}\right)^{a}} & x>0 \\ 0 & x\le 0 \end{cases}$
Dagum Distribution	p, a, b	$\begin{cases} a\,b^{-a\,p}p\,x^{-1+a\,p}\left(1+\left(\frac{x}{b}\right)^{a}\right)^{-1-p} & x>0 \\ 0 & x\le 0 \end{cases}$	$\begin{cases} \left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p} & x>0 \\ 0 & x\le 0 \end{cases}$
Singh Maddala Distribution	q, a, b	$\begin{cases} a\,b^{-a}q\,x^{-1+a}\left(1+\left(\frac{x}{b}\right)^{a}\right)^{-1-q} & x>0 \\ 0 & x\le 0 \end{cases}$	$\begin{cases} 1-\left(1+\left(\frac{x}{b}\right)^{a}\right)^{-q} & x>0 \\ 0 & x\le 0 \end{cases}$

The hypothetical slope proposed refers to a soil mass 2 meters high and with specific saturated weight of 16 kN/m³. The method of analysis chosen is the infinite slope formulation, considering the cohesion as a constant and the friction angle as a statistical distribution (firstly a normal and then another one that best fits the histogram of the data).

II. STATISTICAL DISTRIBUTION FOR FRICTION ANGLE DATA

In order to define the statistical distribution that best fits the histogram of the data obtained in laboratory, nine symmetrical and asymmetrical distributions have been used, including symmetric and skew distributions, such as Dagum's [10]. In this context, Table I shows probability density functions (PDF) and the cumulative density function (CDF) of the tested statistical distributions.

It can be seen in Table I that normal, Student's T and Cauchy distributions are not suitable for most geotechnical parameters including the friction angle because their range includes all values of x in real axis, in contrast to other distributions that are only defined for positive values.

III. ADHERENCE TESTS FOR STATISTICAL DISTRIBUTIONS

To check the suitability of the models above to the friction angle data, best fitting analyzes were performed. The statistical tests of Anderson-Darling [1] and Cramer-von Mises [6] have been applied to the problems by means of built-in functions in the computer software *Mathematica* [12].

Both cited tests belong to the class of quadratic statistics EDF (tests based on empirical distribution function). Thus, it is considered that a set of data (x1, x2, x3, ..., xn) is represented by an empirical function $F_n(x)$, related to its cumulative density function (CDF), which may be expressed as in (1) and compared to the CDF's of the models, F(x).

$$F_n(x) = \frac{1}{N}\sum_{i=1}^{N}\left\lceil\frac{1+\operatorname{sgn}(x-x_i)}{2}\right\rceil \tag{1}$$

where ⌈x⌉ is a function that provides the smallest integer that is not smaller than x and sgn is the sign function which is +1 when x is positive and -1 otherwise. In other words, the empirical cumulative distribution ranges from 0 to 1 and increases by 1 / N for each value in the data set.

Being both $F_n(x)$ and F(x) known, the EDF test statistics, which gives the distance between both CDFs, is given as:

$$TS = N\int_{-\infty}^{\infty}\left[F_n(x)-F(x)\right]^2 w(x)\,d\,F(x) \tag{2}$$

When w(x) in (2) is equal to 1, the Cramer von Mises test statistics (TS) is presented. On the other hand, in case of w (x) = F (x) [1-F (x)], it refers to the Anderson-Darling TS. The main difference between them is that Anderson-Darling method assigns more weight to differences in the tails of the distribution. From the manner in which (2) is presented, it is clear that the higher the value of the test statistics, the worse the fit to the data model.

IV. STATISTICAL ANALYSIS OF THE FRICTION ANGLE DATA

The compiled shear strength data of the material studied have been obtained from the results of theses and dissertations approved by the Graduate Program in Geotechnical Engineering from the University of Brasilia.

Results of triaxial tests and direct shear tests were obtained by [3], [4] and [9] in samples collected in various parts of the

Federal District. To equalize the data, in the present paper, the results used are the ones corresponding to the consolidated-drained and saturated tests, in addition to excluding two extreme erroneous values. The data considered corresponds to samples of the first six meters from the soil mass, being such samples classified as porous clay. Table II summarizes the values obtained as peak resistances in failure envelopes.

TABLE II
FRICTION ANGLE DATA

30	26,7	30,8
31,3	30,3	27,5
32,3	31,7	33,3
29	30,4	38,8*
23,2	30	46,3*
35	25	26
25	32	29
28	28	32
32	33	25
33	31	-

*Extreme data excluded from the analysis for not being adequate to the local reality.

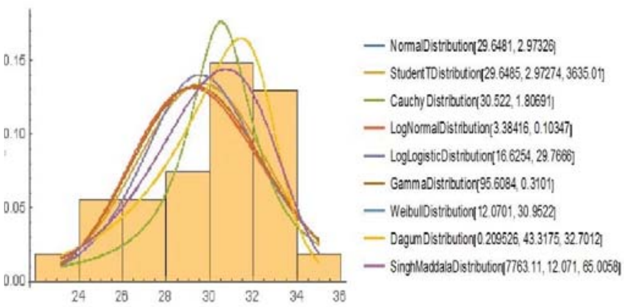

Fig. 1 PDF of the fitted statistical distributions and data histogram

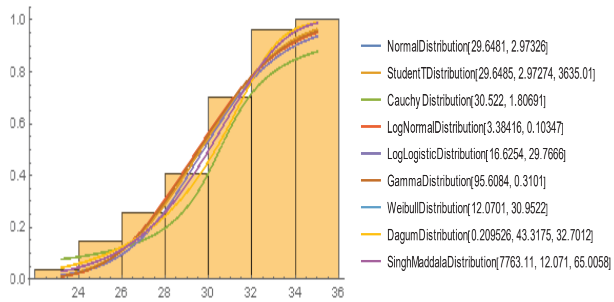

Fig. 2 CDF of the fitted statistical distributions and data histogram

Through the software Mathematica, best fit anlyses have been carried out for each of the nine distributions analyzed. To verify the accuracy of each distribution with respect to the experimental data, adeherence tests proposed by Anderson-Darling and Cramér-von Mises have been used. The results of these tests are shown in Table III and in Figs. 1-4.

The p values of the statistical analyses are related to the likelihood of these tests provide different results due to accident or due to a significant fact, i.e., in technical terms it can be said that the null hypothesis is rejected when p value is less than a given threshold (typically in the order of 5% or 1%). In this context, it may be seen that all the distributions tested showed satisfactory p values.

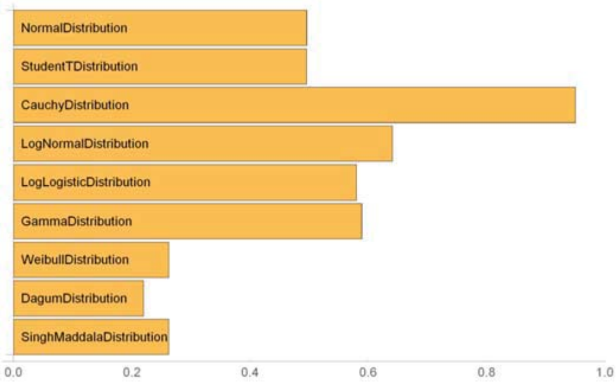

Fig. 3 TS of Anderson-Darling tests

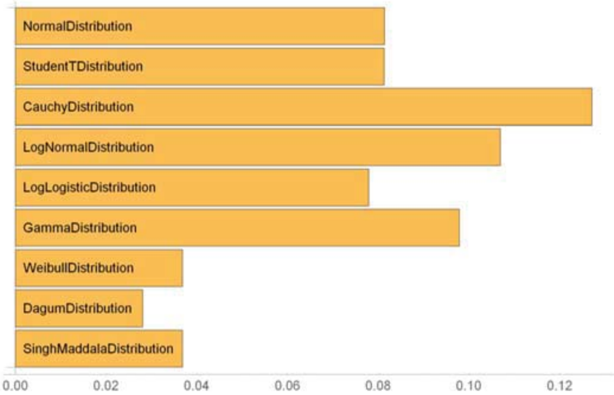

Fig. 4 TS of Cramér-von Mises tests

Since the hypothesis tests are not conclusive to choose the best fit distribution, one may look for other fitting evidences, such as directly evaluating the TS which are related to the the square of the difference between the observed and expected cummulative distributions. In this regard, according to Anderson-Darling methodology Dagum distribution is best fit because its TS is 0.219 while normal distribution TS's is of 0.495. Besides, in the Cramer-von Mises test the Dagum distribution also showed the lowest TS value (0.028) in comparison to the normal distribution TS's of 0.081.

Based on the fitting analysis performed, the Dagum distribution showed itself as the best fit for friction angle data. In this sense, the rupture of a hypothetical slope with 2 meters is analyzed based on the Mohr-Coulomb failure criterion, with an average cohesion value of 9kPa, and friction angle following a Dagum Distribution.

V. ANALYTICAL SLOPE STABILITY FOR THE FACTOR OF SAFETY

In the present effort, the factor of safety (FS) for an infinite slope situation is studied. Mathematically:

$$FS = \frac{c + (\gamma_{sat} - \gamma_w)\, h \cos^2(i) \tan(\phi)}{\gamma_{sat}\, h \sin(i) \cos(i)} = \frac{c + \eta \tan(\phi)}{\xi} \qquad (3)$$

in which γ_w is water's specific weight, γ_{sat}, c, and $\emptyset$ are soil's cohesion specific weight and friction angle, respectively; i and h are slope's inclination and height, respectively; $\eta = (\gamma_{sat} - \gamma_w)\, h \cos^2(i)$; $\xi = \gamma_{sat}\, h \sin(i) \cos(i)$.

TABLE III
ANDERSON-DARLING AND CRAMÉR-VON MISES TESTS

Fitted Distributions	Anderson-Darling		Cramér-von Mises		Skewness
	TS	P-Value	TS	P-Value	
Normal (29,6; 2,9)	0,495	0,749	0,081	0,683	0
T-Student (29,6; 2,9; 3635)	0,495	0,749	0,081	0,684	0
Cauchy (30,5; 1,8)	0,950	0,382	0,127	0,467	Indetermined
LogNormal (3,3; 0,1)	0,640	0,608	0,106	0,551	0,312
LogLogistic (16,6; 29,7)	0,579	0,665	0,077	0,703	0,537
Gamma (95,6; 0,3)	0,588	0,656	0,097	0,595	0,204
Weibull (12,1; 30,9)	0,262	0,963	0,036	0,948	- 0,712
Dagum (0,2; 43,3; 32,7)	0,219	0,984	0,028	0,982	- 1,096
SinghMaddala (7763,1; 12; 65)	0,262	0,963	0,036	0,948	- 0,712

Equation (3) considers important parameters for the slope stability anlysis, thus it is chosen to analyze such phenomenon. This way, the probability of the random variable FS be less than or equal to fs is given as:

$$P(FS \le fs) = P\left(\frac{c + \eta \tan(\phi)}{\xi} \le fs\right) = P\left(\Phi \le \arctan\left(\frac{\xi\, fs - c}{\eta}\right)\right) \qquad (4)$$

Thus, based on (4), it is easy to get that:

$$P(FS \le fs) = \left[1 + \left(\frac{\arctan\left(\frac{\xi\, fs - c}{\eta}\right)}{b}\right)^{-a}\right]^{-p} \qquad (5)$$

where a, b and p are parameters of the Dagum distribution.

Based on (5), the probability density function (PDF) of the factor of safety may be obtained as:

$$f(fs) = \frac{d\, P(FS \le fs)}{d\, fs} = \frac{\xi\, a\, p \left[\frac{\arctan\left(\frac{\xi\, fs - c}{\eta}\right)}{b}\right]^{-a-1} \left[1 + \left(\frac{\arctan\left(\frac{\xi\, fs - c}{\eta}\right)}{b}\right)^{-a}\right]^{-p-1}}{b\, \eta \left(1 + \frac{(\xi\, fs - c)^2}{\eta^2}\right)} \qquad (6)$$

In a similar way, if the friction angle is chosen to be a normal distribution, the factor of safety's PDF may be given as:

$$f(fs) = \frac{d\, P(FS \le fs)}{d\, fs} = \frac{\xi\, e^{-\frac{\left(\mu + \arctan\left(\frac{\xi\, fs - c}{\eta}\right)\right)^2}{2\sigma^2}}}{\eta\, \sigma \sqrt{2\pi} \left(1 + \frac{(\xi\, fs - c)^2}{\eta^2}\right)} \qquad (7)$$

VI. DIFFERENCES BETWEEN CONSIDERING THE FRICTION ANGLE AS A DAGUM OR AS A NORMAL RANDOM VARIABLE

The universalization of the normal distribution and consequent widespread practical use is not always appropriate for the properties of the materials used in engineering. In the case of geotechnical engineering, it is clear that there is an asymmetrical behavior of the soil properties. In this research, it was shown that the friction angle showed better fit for Dagum distribution while compared to the Normal distribution. This fact does not qualify the Dagum distribution as the most suited to model geotechnical parameters. However, it brings out the importance of dealing with non-normal random variables. Fig. 5 schematically shows the concept of risk (probability of failure vs cost) for various types of interventions [17] and, based on these principles, Table IV shows the difference between considering that the friction angle follows a Normal or a Dagum distribution.

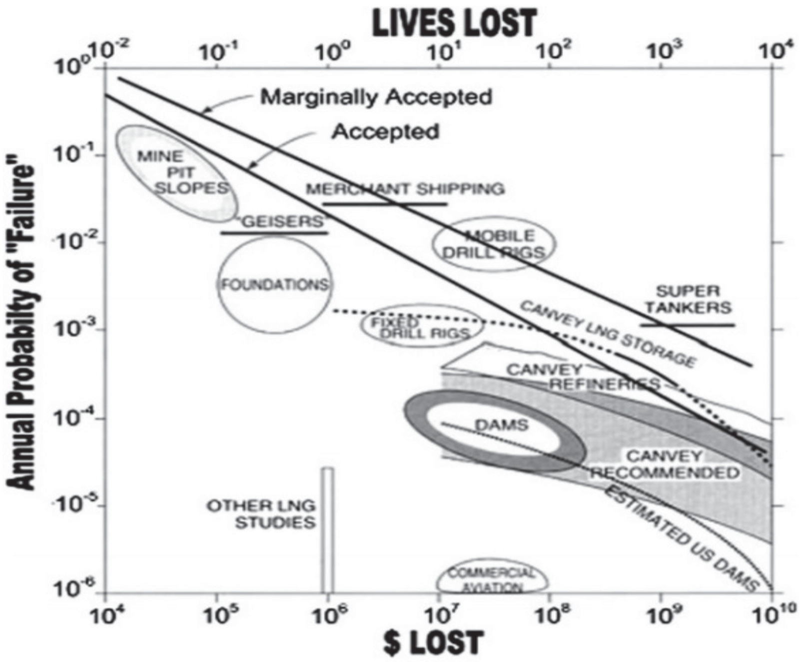

Fig. 5 Tolerable risk level (adapted [14])

Based on the data presented in Table IV, it can be seen that choosing the normal distribution is quite conservative. This represents, in the case of hypothetical slope, a significant difference in physical design (the slope inclination) of the proposed structures for each type of analyzed intervention. Fig. 6 illustrates the difference between Normal and Dagum distributions. In such figure, one is able to visualize the slope inclination and the correspondent probability of failure. Therefore, it becomes apparent that, within acceptable limits of probability of failure, the normal distribution has considerable excess conservatism

TABLE IV
DIFFERENCES BETWEEN MODELLING THE FRICTION ANGLE AS A NORMAL AND A DAGUM RANDOM VARIABLE: A HYPOTHETICAL SLOPE ANALYSIS

Intervention type	Tolerable risk	Maximum slope inclination - Dagum	Maximum slope inclination - Normal
Minning Slopes	5×10^{-2}	29º	5º
Foundations	5×10^{-3}	26º	1º
Dams	5×10^{-5}	23º	~ 0º

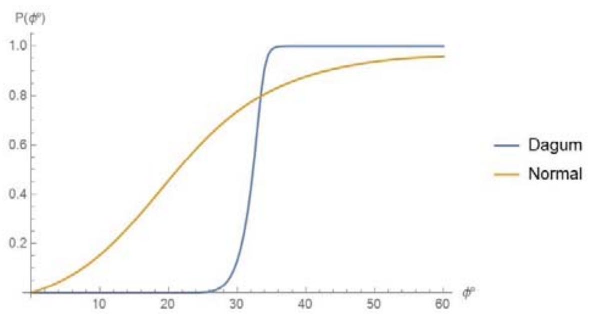

Fig. 6 Slope Inclination and the corresponding probability of failure

VII. Conclusion

Risk management and probabilistic analyzes are believed to be fundamental tools of the so-called new engineering, being highlighted in this research their relevance in geotechnical engineering. Currently, with the extinction of the zero risk concept in engineering interventions and with the need to quantify the risks to which the structures are subjected, the statistical treatment of data has been shown to be an efficient tool to address such issues. In this context, by considering the significant growth of the application of statistics to engineering problems, this research emphasizes such trend by presenting a statistical slope statistical analysis.

The results of this research contribute towards a reflection regarding indiscriminate use of the normal distribution in practical engineering problems. It is worth noticing that, in some cases, using the normal distribution does not necessarily have significant distortions with respect to best fit distribution and therefore may be considered in some cases. On the other hand, it is important to emphasize that the properties of materials often demand models that have asymmetric behavior, which leads the present research to conclude that each random variable must be treated with a distribution that appropriately represents its physical and statistical properties.

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