

An Efficient Iterative Updating Method for Damped Structural Systems

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Abstract—Model updating is an inverse eigenvalue problem which concerns the modification of an existing but inaccurate model with measured modal data. In this paper, an efficient gradient based iterative method for updating the mass, damping and stiffness matrices simultaneously using a few of complex measured modal data is developed. Convergence analysis indicates that the iterative solutions always converge to the unique minimum Frobenius norm symmetric solution of the model updating problem by choosing a special kind of initial matrices.

Keywords—Model updating, iterative algorithm, damped structural system, optimal approximation.

I. INTRODUCTION

USING a finite element modeling, equation of motion of a linear elastic time-invariant structure with n degree of freedoms is given by

$$M_a \ddot{q}(t) + D_a \dot{q}(t) + K_a q(t) = 0. \quad (1)$$

The vector $q(t)$ represents the generalized coordinates of the system. M_a, D_a and K_a are, respectively, called the analytical mass, damping and stiffness matrices. Equation (1) is usually known as the finite element model (analytical model). If a fundamental solution to (1) is represented by $q(t) = xe^{\lambda t}$, then the scalar λ and the vector x must solve the quadratic eigenvalue problem

$$(\lambda^2 M_a + \lambda D_a + K_a)x = 0. \quad (2)$$

Complex numbers λ and nonzero complex vectors x for which this relation holds are, respectively, the eigenvalues and eigenvectors of the system. Equation (2) has $2n$ finite eigenvalues over the complex field, provided that the leading matrix coefficient M_a is nonsingular. It is known that the dynamical behavior of the differential system (1) usually can be interpreted via the eigenvalues and the corresponding eigenvectors of (2). The analytical model (1) is often validated by comparing their analytical modes of vibration with the results measured by modal testing. However, most modal data obtained by the finite element model don't agree with the experimental results. The lack of correlation between the analytical predictions and the experimental results can be traced to either experimental or modelling errors, or a combination of both. In order to obtain the most accurate and reliable model, the original analytical model (1) must be updated using the measured eigendata such that the agreement

between analytical predictions and test results is improved. The updated model may then be considered a better dynamical representation of the structure, and can be used with greater confidence for the analysis of the structure under different boundary conditions or with physical structural changes. The process is known as model updating.

In the past 40 years, various techniques for updating mass and stiffness matrices for undamped systems (i.e., $D_a = 0$) using measured response data have been discussed by [1]-[5]. For an account of the earlier methods, see [6], an integral introduction of the basic theory of finite element model updating is given. For damped structured systems, the theory and computation have been considered by [7]-[10]. All these existing methods are direct updating coefficient matrices, but the explicit solution is too difficult to be obtained by applying matrix computation techniques. We observe that the iterative methods for model updating have received little attention in these years. This paper we will offer a simple yet effective gradient based iterative algorithm to solve the damped structural model updating problem which can incorporate the measured eigendata into the finite element model to produce an adjusted model on the mass, damping and stiffness matrices that closely match the experimental modal data.

Let $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_p\} \in \mathbf{C}^{p \times p}$ and $X = [x_1, \dots, x_p] \in \mathbf{C}^{n \times p}$ be the measured eigenvalue and eigenvector matrices, where $p \ll n$ and both Λ and X are closed under complex conjugation in the sense that $\lambda_{2j} = \bar{\lambda}_{2j-1} \in \mathbf{C}$, $x_{2j} = \bar{x}_{2j-1} \in \mathbf{C}^n$ for $j = 1, \dots, l$, and $\lambda_k \in \mathbf{R}$, $x_k \in \mathbf{R}^n$ for $k = 2l + 1, \dots, p$. The problem of updating mass, damping and stiffness matrices simultaneously can be mathematically formulated as follows.

Problem P. Given real-valued symmetric matrices $M_a, D_a, K_a \in \mathbf{R}^{n \times n}$, find $(\hat{M}, \hat{D}, \hat{K}) \in \mathbf{S}_E$ such that

$$\|\hat{M} - M_a\|^2 + \|\hat{D} - D_a\|^2 + \|\hat{K} - K_a\|^2 = \min_{(M, D, K) \in \mathbf{S}_E} (\|M - M_a\|^2 + \|D - D_a\|^2 + \|K - K_a\|^2),$$

where

$$\mathbf{S}_E = \{(M, D, K) \in \mathbf{SR}^{n \times n} \times \mathbf{SR}^{n \times n} \times \mathbf{SR}^{n \times n} \mid M X \Lambda^2 + D X \Lambda + K X = 0\}.$$

In Section II, an efficient gradient based iterative method is presented to solve Problem P and the convergence properties are discussed. By using the proposed iterative algorithm, the unique minimum Frobenius norm symmetric solution can be obtained by choosing a special kind of initial matrices. Some concluding remarks are given in Section III.

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Throughout this paper, we shall adopt the following notation: $\mathbf{C}^{m \times n}$ and $\mathbf{R}^{m \times n}$ denote the set of all $m \times n$ complex and real matrices, and the set of all symmetric matrices in $\mathbf{R}^{n \times n}$ by $\mathbf{SR}^{n \times n}$. A^\top , $\text{tr}(A)$ and $R(A)$ stand for the transpose, the trace and the column space of the matrix A , respectively. $\lambda_{\max}(M^\top M)$ denotes the maximum eigenvalue of $M^\top M$, I_n represents the identity matrix of order n , $\bar{\alpha}$ denotes the conjugation of the complex number α and $\|\cdot\|$ represents the Frobenius norm. Given two matrices $A = [a_{ij}] \in \mathbf{R}^{m \times n}$ and $B \in \mathbf{R}^{p \times q}$, the Kronecker product of A and B is defined by $A \otimes B = [a_{ij}B] \in \mathbf{R}^{mp \times nq}$. Also, for an $m \times n$ matrix $A = [a_1, a_2, \dots, a_n]$, where $a_i, i = 1, \dots, n$, is the i -th column vector of A , the stretching function $\text{vec}(A)$ is defined as $\text{vec}(A) = [a_1^\top, a_2^\top, \dots, a_n^\top]^\top$.

II. THE SOLUTION OF PROBLEM P

Define a complex matrix T_p as

$$T_p = \text{diag} \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}, \dots, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}, I_{p-2l} \right\} \in \mathbf{C}^{p \times p},$$

where $i = \sqrt{-1}$. It is easy to verify that T_p is a unitary matrix, that is, $\bar{T}_p^\top T_p = I_p$. Using this transformation matrix, we have

$$\tilde{\Lambda} = \bar{T}_p^\top \Lambda T_p = \text{diag} \left\{ \begin{bmatrix} \zeta_1 & \eta_1 \\ -\eta_1 & \zeta_1 \end{bmatrix}, \dots, \begin{bmatrix} \zeta_{2l-1} & \eta_{2l-1} \\ -\eta_{2l-1} & \zeta_{2l-1} \end{bmatrix}, \lambda_{2l+1}, \dots, \lambda_p \right\} \in \mathbf{R}^{p \times p}, \quad (3)$$

$$\tilde{X} = X T_p = [\sqrt{2}y_1, \sqrt{2}z_1, \dots, \sqrt{2}y_{2l-1}, \sqrt{2}z_{2l-1}, x_{2l+1}, \dots, x_p] \in \mathbf{R}^{n \times p}, \quad (4)$$

where ζ_j and η_j are respectively the real part and the imaginary part of the complex number λ_j , and y_j and z_j are respectively the real part and the imaginary part of the complex vector x_j for $j = 1, 3, \dots, 2l-1$. It follows from (3) and (4) that the equation of $M\tilde{X}\tilde{\Lambda}^2 + D\tilde{X}\tilde{\Lambda} + K\tilde{X} = 0$ can be equivalently written as

$$M\tilde{X}\tilde{\Lambda}^2 + D\tilde{X}\tilde{\Lambda} + K\tilde{X} = 0, \\ \text{s. t. } \tilde{M} \in \mathbf{SR}^{n \times n}, \tilde{D} \in \mathbf{SR}^{n \times n}, \tilde{K} \in \mathbf{SR}^{n \times n}.$$

For a given matrix triplet (M_a, D_a, K_a) , we have

$$M\tilde{X}\tilde{\Lambda}^2 + D\tilde{X}\tilde{\Lambda} + K\tilde{X} = 0 \\ \Leftrightarrow (M - M_a)\tilde{X}\tilde{\Lambda}^2 + (D - D_a)\tilde{X}\tilde{\Lambda} + (K - K_a)\tilde{X} \\ = -M_a\tilde{X}\tilde{\Lambda}^2 - D_a\tilde{X}\tilde{\Lambda} - K_a\tilde{X}.$$

Let

$$\tilde{M} = M - M_a, \\ \tilde{D} = D - D_a, \\ \tilde{K} = K - K_a, \\ F = -M_a\tilde{X}\tilde{\Lambda}^2 - D_a\tilde{X}\tilde{\Lambda} - K_a\tilde{X},$$

then solving Problem P is equivalent to finding the minimum Frobenius norm solution of the matrix equation

$$M\tilde{X}\tilde{\Lambda}^2 + D\tilde{X}\tilde{\Lambda} + K\tilde{X} = F, \\ \text{s. t. } \tilde{M} \in \mathbf{SR}^{n \times n}, \tilde{D} \in \mathbf{SR}^{n \times n}, \tilde{K} \in \mathbf{SR}^{n \times n}. \quad (5)$$

Once the minimum Frobenius norm solution $(\tilde{M}^*, \tilde{D}^*, \tilde{K}^*)$ of (5) is obtained, the solution of the matrix optimal approximation Problem P can be computed. In this case, can be expressed as

$$\hat{M} = M_a + \tilde{M}^*, \\ \hat{D} = D_a + \tilde{D}^*, \\ \hat{K} = K_a + \tilde{K}^*. \quad (6)$$

Lemma 1: [11], [12]. If the linear equation system $Hx = b$, where $H \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$, has a unique solution x^* , then the gradient based iterative algorithm

$$x_k = x_{k-1} + \mu H^\top (b - Hx_{k-1}), \quad 0 < \mu < \frac{2}{\lambda_{\max}(H^\top H)},$$

yields $\lim_{k \rightarrow \infty} x_k = x^*$.

Lemma 2: [13]. Suppose that the consistent linear equation $Ax = b$, where $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$, has a solution $x \in R(A^\top)$, then x is the unique minimum Frobenius norm solution of the linear equation.

Lemma 3: Equation (5) has a symmetric solution triplet $(\tilde{M}, \tilde{D}, \tilde{K})$ if and only if the matrix equations

$$\tilde{M}\tilde{X}\tilde{\Lambda}^2 + \tilde{D}\tilde{X}\tilde{\Lambda} + \tilde{K}\tilde{X} = F, \\ \tilde{\Lambda}^{2\top} \tilde{X}^\top \tilde{M} + \tilde{\Lambda}^\top \tilde{X}^\top \tilde{D} + \tilde{X}^\top \tilde{K} = F^\top, \quad (7)$$

are consistent.

Proof. If (5) has a symmetric solution triplet $(\tilde{M}^*, \tilde{D}^*, \tilde{K}^*)$, then $\tilde{M}^*\tilde{X}\tilde{\Lambda}^2 + \tilde{D}^*\tilde{X}\tilde{\Lambda} + \tilde{K}^*\tilde{X} = F$, and $(\tilde{M}^*\tilde{X}\tilde{\Lambda}^2 + \tilde{D}^*\tilde{X}\tilde{\Lambda} + \tilde{K}^*\tilde{X})^\top = \tilde{\Lambda}^{2\top} \tilde{X}^\top \tilde{M}^* + \tilde{\Lambda}^\top \tilde{X}^\top \tilde{D}^* + \tilde{X}^\top \tilde{K}^* = F^\top$. That is to say, $(\tilde{M}^*, \tilde{D}^*, \tilde{K}^*)$ is a solution of (7). Conversely, if (7) has a solution, say, $\tilde{M} = U$, $\tilde{D} = V$, $\tilde{K} = W$. Let $\tilde{M}^* = \frac{1}{2}(U + U^\top)$, $\tilde{D}^* = \frac{1}{2}(V + V^\top)$, $\tilde{K}^* = \frac{1}{2}(W + W^\top)$, then \tilde{M}^*, \tilde{D}^* and \tilde{K}^* are symmetric matrices, and

$$\begin{aligned} & \tilde{M}^*\tilde{X}\tilde{\Lambda}^2 + \tilde{D}^*\tilde{X}\tilde{\Lambda} + \tilde{K}^*\tilde{X} \\ &= \frac{1}{2}(U\tilde{X}\tilde{\Lambda}^2 + V\tilde{X}\tilde{\Lambda} + W\tilde{X}) \\ &+ \frac{1}{2}(U^\top\tilde{X}\tilde{\Lambda}^2 + V^\top\tilde{X}\tilde{\Lambda} + W^\top\tilde{X}) \\ &= \frac{1}{2}F + \frac{1}{2}(F^\top)^\top = F. \end{aligned}$$

Hence, $(\tilde{M}^*, \tilde{D}^*, \tilde{K}^*)$ is a symmetric solution triplet of (5).

Using the Kronecker product and the stretching function, we know that (7) is equivalent to

$$\begin{bmatrix} \tilde{\Lambda}^{2\top} \tilde{X}^\top \otimes I_n & \tilde{\Lambda}^\top \tilde{X}^\top \otimes I_n & \tilde{X}^\top \otimes I_n \\ I_n \otimes \tilde{\Lambda}^{2\top} \tilde{X}^\top & I_n \otimes \tilde{\Lambda}^\top \tilde{X}^\top & I_n \otimes \tilde{X}^\top \end{bmatrix} \begin{bmatrix} \text{vec}(\tilde{M}) \\ \text{vec}(\tilde{D}) \\ \text{vec}(\tilde{K}) \end{bmatrix} \\ = \begin{bmatrix} \text{vec}(F) \\ \text{vec}(F^\top) \end{bmatrix}.$$

Let

$$H = \begin{bmatrix} \tilde{\Lambda}^{2\top} \tilde{X}^\top \otimes I_n & \tilde{\Lambda}^\top \tilde{X}^\top \otimes I_n & \tilde{X}^\top \otimes I_n \\ I_n \otimes \tilde{\Lambda}^{2\top} \tilde{X}^\top & I_n \otimes \tilde{\Lambda}^\top \tilde{X}^\top & I_n \otimes \tilde{X}^\top \end{bmatrix}.$$

According to Lemma 1, we have the gradient based iterative algorithm for (7) described as follows.

$$\begin{bmatrix} \text{vec}(\tilde{M}_s) \\ \text{vec}(\tilde{D}_s) \\ \text{vec}(\tilde{K}_s) \end{bmatrix} = \begin{bmatrix} \text{vec}(\tilde{M}_{s-1}) \\ \text{vec}(\tilde{D}_{s-1}) \\ \text{vec}(\tilde{K}_{s-1}) \end{bmatrix} + \mu H^\top \left(\begin{bmatrix} \text{vec}(F) \\ \text{vec}(F^\top) \end{bmatrix} - H \begin{bmatrix} \text{vec}(\tilde{M}_{s-1}) \\ \text{vec}(\tilde{D}_{s-1}) \\ \text{vec}(\tilde{K}_{s-1}) \end{bmatrix} \right). \quad (8)$$

After some algebra manipulations this results in

$$\begin{aligned} \tilde{M}_s &= \tilde{M}_{s-1} + \mu \left[F \tilde{\Lambda}^{2\top} \tilde{X}^\top + \tilde{X} \tilde{\Lambda}^2 F^\top \right. \\ &\quad - \tilde{M}_{s-1} \tilde{X} \tilde{\Lambda}^2 \tilde{\Lambda}^{2\top} \tilde{X}^\top - \tilde{X} \tilde{\Lambda}^2 \tilde{\Lambda}^{2\top} \tilde{X}^\top \tilde{M}_{s-1} \\ &\quad - \tilde{D}_{s-1} \tilde{X} \tilde{\Lambda} \tilde{\Lambda}^{2\top} \tilde{X}^\top - \tilde{X} \tilde{\Lambda} \tilde{\Lambda}^{2\top} \tilde{X}^\top \tilde{D}_{s-1} \\ &\quad \left. - \tilde{K}_{s-1} \tilde{X} \tilde{\Lambda}^{2\top} \tilde{X}^\top - \tilde{X} \tilde{\Lambda}^2 \tilde{X}^\top \tilde{K}_{s-1} \right], \quad (9) \end{aligned}$$

$$\begin{aligned} \tilde{D}_s &= \tilde{D}_{s-1} + \mu \left[F \tilde{\Lambda}^\top \tilde{X}^\top + \tilde{X} \tilde{\Lambda} F^\top \right. \\ &\quad - \tilde{M}_{s-1} \tilde{X} \tilde{\Lambda}^2 \tilde{\Lambda}^\top \tilde{X}^\top - \tilde{X} \tilde{\Lambda} \tilde{\Lambda}^{2\top} \tilde{X}^\top \tilde{M}_{s-1} \\ &\quad - \tilde{D}_{s-1} \tilde{X} \tilde{\Lambda} \tilde{\Lambda}^\top \tilde{X}^\top - \tilde{X} \tilde{\Lambda} \tilde{\Lambda}^\top \tilde{X}^\top \tilde{D}_{s-1} \\ &\quad \left. - \tilde{K}_{s-1} \tilde{X} \tilde{\Lambda}^\top \tilde{X}^\top - \tilde{X} \tilde{\Lambda} \tilde{X}^\top \tilde{K}_{s-1} \right], \quad (10) \end{aligned}$$

$$\begin{aligned} \tilde{K}_s &= \tilde{K}_{s-1} + \mu \left[F \tilde{X}^\top + \tilde{X} F^\top \right. \\ &\quad - \tilde{M}_{s-1} \tilde{X} \tilde{\Lambda}^2 \tilde{X}^\top - \tilde{X} \tilde{\Lambda}^{2\top} \tilde{X}^\top \tilde{M}_{s-1} \\ &\quad - \tilde{D}_{s-1} \tilde{X} \tilde{\Lambda} \tilde{X}^\top - \tilde{X} \tilde{\Lambda}^\top \tilde{X}^\top \tilde{D}_{s-1} \\ &\quad \left. - \tilde{K}_{s-1} \tilde{X} \tilde{X}^\top - \tilde{X} \tilde{X}^\top \tilde{K}_{s-1} \right]. \quad (11) \end{aligned}$$

From (9), (10) and (11) we can easily see that if the initial matrices $\tilde{M}_0, \tilde{D}_0, \tilde{K}_0 \in \mathbf{SR}^{n \times n}$, then $\tilde{M}_s \in \mathbf{SR}^{n \times n}$, $\tilde{D}_s \in \mathbf{SR}^{n \times n}$ and $\tilde{K}_s \in \mathbf{SR}^{n \times n}$ for $s = 1, 2, \dots$.

Theorem 1: Suppose that (5) has a unique symmetric solution $(\tilde{M}^*, \tilde{D}^*, \tilde{K}^*)$. If we choose the convergence factor as

$$0 < \mu < \mu_0, \quad (12)$$

where $\mu_0 = \frac{1}{\lambda_{\max}(\tilde{\Lambda}^{2\top} \tilde{X}^\top \tilde{X} \tilde{\Lambda}^2) + \lambda_{\max}(\tilde{\Lambda}^\top \tilde{X}^\top \tilde{X} \tilde{\Lambda}) + \lambda_{\max}(\tilde{X}^\top \tilde{X})}$. then the sequences $\{\tilde{M}_i\}$, $\{\tilde{D}_i\}$ and $\{\tilde{K}_i\}$ generated by (9), (10) and (11) satisfy

$$\begin{aligned} \lim_{s \rightarrow \infty} \tilde{M}_s &= \tilde{M}^*, \\ \lim_{s \rightarrow \infty} \tilde{D}_s &= \tilde{D}^*, \\ \lim_{s \rightarrow \infty} \tilde{K}_s &= \tilde{K}^* \end{aligned} \quad (13)$$

for any arbitrary initial matrix triplet $(\tilde{M}_0, \tilde{D}_0, \tilde{K}_0)$ with $\tilde{M}_0, \tilde{D}_0, \tilde{K}_0 \in \mathbf{SR}^{n \times n}$.

Proof. Define the error matrices \tilde{M}_s^* , \tilde{D}_s^* and \tilde{K}_s^* as

$$\begin{aligned} \tilde{M}_s^* &= \tilde{M}_s - \tilde{M}^*, \\ \tilde{D}_s^* &= \tilde{D}_s - \tilde{D}^*, \\ \tilde{K}_s^* &= \tilde{K}_s - \tilde{K}^*. \end{aligned}$$

Using (9)-(11) and (7), we have

$$\begin{aligned} \tilde{M}_s^* &= \tilde{M}_{s-1}^* \\ &\quad - \mu \left(\tilde{M}_{s-1}^* \tilde{X} \tilde{\Lambda}^2 \tilde{\Lambda}^{2\top} \tilde{X}^\top + \tilde{X} \tilde{\Lambda}^2 \tilde{\Lambda}^{2\top} \tilde{X}^\top \tilde{M}_{s-1}^* \right. \\ &\quad + \tilde{D}_{s-1}^* \tilde{X} \tilde{\Lambda} \tilde{\Lambda}^{2\top} \tilde{X}^\top + \tilde{X} \tilde{\Lambda} \tilde{\Lambda}^{2\top} \tilde{X}^\top \tilde{D}_{s-1}^* \\ &\quad \left. + \tilde{K}_{s-1}^* \tilde{X} \tilde{\Lambda}^{2\top} \tilde{X}^\top + \tilde{X} \tilde{\Lambda}^2 \tilde{X}^\top \tilde{K}_{s-1}^* \right), \quad (14) \end{aligned}$$

$$\begin{aligned} \tilde{D}_s^* &= \tilde{D}_{s-1}^* \\ &\quad - \mu \left(\tilde{M}_{s-1}^* \tilde{X} \tilde{\Lambda}^2 \tilde{\Lambda}^\top \tilde{X}^\top + \tilde{X} \tilde{\Lambda} \tilde{\Lambda}^{2\top} \tilde{X}^\top \tilde{M}_{s-1}^* \right. \\ &\quad + \tilde{D}_{s-1}^* \tilde{X} \tilde{\Lambda} \tilde{\Lambda}^\top \tilde{X}^\top + \tilde{X} \tilde{\Lambda} \tilde{\Lambda}^\top \tilde{X}^\top \tilde{D}_{s-1}^* \\ &\quad \left. + \tilde{K}_{s-1}^* \tilde{X} \tilde{\Lambda}^\top \tilde{X}^\top + \tilde{X} \tilde{\Lambda} \tilde{X}^\top \tilde{K}_{s-1}^* \right), \quad (15) \end{aligned}$$

$$\begin{aligned} \tilde{K}_s^* &= \tilde{K}_{s-1}^* \\ &\quad - \mu \left(\tilde{M}_{s-1}^* \tilde{X} \tilde{\Lambda}^2 \tilde{X}^\top + \tilde{X} \tilde{\Lambda}^{2\top} \tilde{X}^\top \tilde{M}_{s-1}^* \right. \\ &\quad + \tilde{D}_{s-1}^* \tilde{X} \tilde{\Lambda} \tilde{X}^\top + \tilde{X} \tilde{\Lambda}^\top \tilde{X}^\top \tilde{D}_{s-1}^* \\ &\quad \left. + \tilde{K}_{s-1}^* \tilde{X} \tilde{X}^\top + \tilde{X} \tilde{X}^\top \tilde{K}_{s-1}^* \right). \quad (16) \end{aligned}$$

Let

$$\begin{aligned} P_{s-1} &= \tilde{M}_{s-1}^* \tilde{X} \tilde{\Lambda}^2, \\ Q_{s-1} &= \tilde{D}_{s-1}^* \tilde{X} \tilde{\Lambda}, \\ W_{s-1} &= \tilde{K}_{s-1}^* \tilde{X}. \end{aligned}$$

By (14) we have

$$\begin{aligned} \tilde{M}_s^* &= \tilde{M}_{s-1}^* - \mu (P_{s-1} + Q_{s-1} + W_{s-1}) \tilde{\Lambda}^{2\top} \tilde{X}^\top \\ &\quad - \mu \tilde{X} \tilde{\Lambda}^2 (P_{s-1} + Q_{s-1} + W_{s-1})^\top. \quad (17) \end{aligned}$$

Using the relation of (17) and noting that the symmetry of \tilde{M}_s^* , $i = 0, 1, \dots$, we obtain

$$\begin{aligned} \|\tilde{M}_s^*\|^2 &= \|\tilde{M}_{s-1}^*\|^2 - 4\mu \text{tr}(P_{s-1}^\top G_{s-1}) \\ &\quad + \mu^2 \|G_{s-1} \tilde{\Lambda}^{2\top} \tilde{X}^\top + \tilde{X} \tilde{\Lambda}^2 G_{s-1}^\top\|^2, \quad (18) \end{aligned}$$

where $G_{s-1} = P_{s-1} + Q_{s-1} + W_{s-1}$. Observe that

$$\begin{aligned} &\|G_{s-1} \tilde{\Lambda}^{2\top} \tilde{X}^\top + \tilde{X} \tilde{\Lambda}^2 G_{s-1}^\top\|^2 \\ &\leq \left(\|G_{s-1} \tilde{\Lambda}^{2\top} \tilde{X}^\top\| + \|\tilde{X} \tilde{\Lambda}^2 G_{s-1}^\top\| \right)^2 \\ &= 4 \|G_{s-1} \tilde{\Lambda}^{2\top} \tilde{X}^\top\|^2 \leq 4 \lambda_{\max}(\tilde{\Lambda}^{2\top} \tilde{X}^\top \tilde{X} \tilde{\Lambda}^2) \|G_{s-1}\|^2. \end{aligned}$$

Thus, it follows from (18) that

$$\begin{aligned} \|\tilde{M}_s^*\|^2 &\leq \|\tilde{M}_{s-1}^*\|^2 - 4\mu \text{tr}(P_{s-1}^\top G_{s-1}) \\ &\quad + 4\mu^2 \lambda_{\max}(\tilde{\Lambda}^{2\top} \tilde{X}^\top \tilde{X} \tilde{\Lambda}^2) \|G_{s-1}\|^2. \quad (19) \end{aligned}$$

Similarly, by (15) and (16) we can obtain

$$\begin{aligned} \|\tilde{D}_s^*\|^2 &\leq \|\tilde{D}_{s-1}^*\|^2 - 4\mu \text{tr}(Q_{s-1}^\top G_{s-1}) \\ &\quad + 4\mu^2 \lambda_{\max}(\tilde{\Lambda}^\top \tilde{X}^\top \tilde{X} \tilde{\Lambda}) \|G_{s-1}\|^2, \quad (20) \end{aligned}$$

$$\begin{aligned} \|\tilde{K}_s^*\|^2 &\leq \|\tilde{K}_{s-1}^*\|^2 - 4\mu \text{tr}(W_{s-1}^\top G_{s-1}) \\ &\quad + 4\mu^2 \lambda_{\max}(\tilde{X}^\top \tilde{X}) \|G_{s-1}\|^2. \quad (21) \end{aligned}$$

Therefore, from (19), (20) and (21) we have

$$\begin{aligned}
& \|\tilde{M}_s^*\|^2 + \|\tilde{D}_s^*\|^2 + \|\tilde{K}_s^*\|^2 \\
& \leq \|\tilde{M}_{s-1}^*\|^2 + \|\tilde{D}_{s-1}^*\|^2 + \|\tilde{K}_{s-1}^*\|^2 \\
& - 4\mu\|G_{s-1}\|^2 + 4\mu^2\lambda_{\max}(\tilde{\Lambda}^{2\top}\tilde{X}^\top\tilde{X}\tilde{\Lambda}^2)\|G_{s-1}\|^2 \\
& + 4\mu^2\lambda_{\max}(\tilde{\Lambda}^\top\tilde{X}^\top\tilde{X}\tilde{\Lambda})\|G_{s-1}\|^2 \\
& + 4\mu^2\lambda_{\max}(\tilde{X}^\top\tilde{X})\|G_{s-1}\|^2 \\
& = \|\tilde{M}_{s-1}^*\|^2 + \|\tilde{D}_{s-1}^*\|^2 + \|\tilde{K}_{s-1}^*\|^2 \\
& - 4\mu\left[1 - \mu\left(\lambda_{\max}(\tilde{\Lambda}^{2\top}\tilde{X}^\top\tilde{X}\tilde{\Lambda}^2) + \lambda_{\max}(\tilde{\Lambda}^\top\tilde{X}^\top\tilde{X}\tilde{\Lambda})\right.\right. \\
& \left. + \lambda_{\max}(\tilde{X}^\top\tilde{X})\right)\|G_{s-1}\|^2 \\
& = \|\tilde{M}_0^*\|^2 + \|\tilde{D}_0^*\|^2 + \|\tilde{K}_0^*\|^2 \\
& - 4\mu\left[1 - \mu\left(\lambda_{\max}(\tilde{\Lambda}^{2\top}\tilde{X}^\top\tilde{X}\tilde{\Lambda}^2) + \lambda_{\max}(\tilde{\Lambda}^\top\tilde{X}^\top\tilde{X}\tilde{\Lambda})\right.\right. \\
& \left. + \lambda_{\max}(\tilde{X}^\top\tilde{X})\right)\sum_{l=0}^{s-1}\|G_l\|^2.
\end{aligned} \tag{22}$$

If the convergence factor μ is chosen to satisfy $0 < \mu < \mu_0$, then the inequality of (22) implies that

$$\begin{aligned}
& 4\mu\left[1 - \mu\left(\lambda_{\max}(\tilde{\Lambda}^{2\top}\tilde{X}^\top\tilde{X}\tilde{\Lambda}^2) + \lambda_{\max}(\tilde{\Lambda}^\top\tilde{X}^\top\tilde{X}\tilde{\Lambda})\right.\right. \\
& \left. + \lambda_{\max}(\tilde{X}^\top\tilde{X})\right)\sum_{l=0}^{s-1}\|G_l\|^2 \\
& \leq \|\tilde{M}_0^*\|^2 + \|\tilde{D}_0^*\|^2 + \|\tilde{K}_0^*\|^2 < \infty,
\end{aligned}$$

which implies that $\sum_{s=0}^{\infty}\|G_s\|^2 < \infty$. Thus, we can conclude that $G_s \rightarrow 0$, as $s \rightarrow \infty$, or equivalently,

$$\tilde{M}_{s-1}^*\tilde{X}\tilde{\Lambda}^2 + \tilde{D}_{s-1}^*\tilde{X}\tilde{\Lambda} + \tilde{K}_{s-1}^*\tilde{X} \rightarrow 0, \text{ as } s \rightarrow \infty.$$

Under the condition that the solution to (5) is unique, we can deduce that $\tilde{M}_s^* \rightarrow 0$, $\tilde{D}_s^* \rightarrow 0$ and $\tilde{K}_s^* \rightarrow 0$ as $s \rightarrow \infty$. This proves Theorem 1.

Now, assume that $J \in \mathbf{R}^{n \times p}$ is an arbitrary matrix, then we have

$$\begin{aligned}
& \begin{bmatrix} \text{vec}(J\tilde{\Lambda}^{2\top}\tilde{X}^\top + \tilde{X}\tilde{\Lambda}^2J^\top) \\ \text{vec}(J\tilde{\Lambda}^\top\tilde{X}^\top + \tilde{X}\tilde{\Lambda}J^\top) \\ \text{vec}(J\tilde{X}^\top + \tilde{X}J^\top) \end{bmatrix} \\
& = \begin{bmatrix} \tilde{X}\tilde{\Lambda}^2 \otimes I_n & I_n \otimes \tilde{X}\tilde{\Lambda}^2 \\ \tilde{X}\tilde{\Lambda} \otimes I_n & I_n \otimes \tilde{X}\tilde{\Lambda} \\ \tilde{X} \otimes I_n & I_n \otimes \tilde{X} \end{bmatrix} \begin{bmatrix} \text{vec}(J) \\ \text{vec}(J^\top) \end{bmatrix} \\
& = H^\top \begin{bmatrix} \text{vec}(J) \\ \text{vec}(J^\top) \end{bmatrix} \in R(H^\top).
\end{aligned}$$

It is obvious that if we choose

$$\begin{aligned}
\tilde{M}_0 &= J\tilde{\Lambda}^{2\top}\tilde{X}^\top + \tilde{X}\tilde{\Lambda}^2J^\top, \\
\tilde{D}_0 &= J\tilde{\Lambda}^\top\tilde{X}^\top + \tilde{X}\tilde{\Lambda}J^\top, \\
\tilde{K}_0 &= J\tilde{X}^\top + \tilde{X}J^\top,
\end{aligned} \tag{23}$$

then all \tilde{M}_s , \tilde{D}_s and \tilde{K}_s generated by (9), (10) and (11) satisfy

$$\begin{aligned}
& \begin{bmatrix} \text{vec}(\tilde{M}_s) \\ \text{vec}(\tilde{D}_s) \\ \text{vec}(\tilde{K}_s) \end{bmatrix} \\
& \in R\left(\begin{bmatrix} \tilde{\Lambda}^{2\top}\tilde{X}^\top \otimes I_n & \tilde{\Lambda}^\top\tilde{X}^\top \otimes I_n & \tilde{X}^\top \otimes I_n \\ I_n \otimes \tilde{\Lambda}^{2\top}\tilde{X}^\top & I_n \otimes \tilde{\Lambda}^\top\tilde{X}^\top & I_n \otimes \tilde{X}^\top \end{bmatrix}\right)^\top \\
& = R(H^\top).
\end{aligned}$$

It follows from Lemma 2 that if we choose the initial symmetric matrix triplet by (23), then the iterative solution

triplet $(\tilde{M}_s, \tilde{D}_s, \tilde{K}_s)$ obtained by the gradient iterative algorithm (9), (10) and (11) converges to the unique minimum Frobenius norm symmetric solution triplet $(\tilde{M}^*, \tilde{D}^*, \tilde{K}^*)$.

In summary of above discussion, we have proved the following result.

Theorem 2: Suppose that the condition (12) is satisfied. If we choose the initial symmetric matrices by (23), where $J \in \mathbf{R}^{n \times p}$ is an arbitrary matrix, or especially, $\tilde{M}_0 = 0$, $\tilde{D}_0 = 0$ and $\tilde{K}_0 = 0$, then the iterative solution triplet $(\tilde{M}_s, \tilde{D}_s, \tilde{K}_s)$ obtained by the gradient iterative algorithm (9), (10) and (11) converges to the unique minimum Frobenius norm symmetric solution triplet $(\tilde{M}^*, \tilde{D}^*, \tilde{K}^*)$ of (5), and the unique solution of Problem P is achieved and given by (6).

III. CONCLUDING REMARKS

A gradient based iterative algorithm has been developed to incorporate measured experimental modal data into an analytical finite element model with nonproportional damping, such that the adjusted finite element model more closely matches the experimental results. However, we should point out that in all physical systems the matrices M, D and K are often structured or parameterized, that is, the parameters in the stiffness, damping and mass matrices are correlated, and updating one parameter requires that others be updated in a specific fashion to maintain the proper connectivities in the structure. However, the method proposed can't retain the physical configuration of the analytical model. Can the physical feasibility of the updated M, D and K be maintained? This is a question that might be worthy of further study.

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