International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:9, No:11, 2015

Numerical Computation of Sturm-Liouville Problem with Robin Boundary Condition

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***Abstract*—**The modelling of physical phenomena, such as the earth's free oscillations, the vibration of strings, the interaction of atomic particles, or the steady state flow in a bar give rise to Sturm-Liouville (SL) eigenvalue problems. The boundary applications of some systems like the convection-diffusion equation, electromagnetic and heat transfer problems requires the combination of Dirichlet and Neumann boundary conditions. Hence, the incorporation of Robin boundary condition in the analyses of Sturm-Liouville problem. This paper deals with the computation of the eigenvalues and eigenfunction of generalized Sturm-Liouville problems with Robin boundary condition using the finite element method. Numerical solution of classical Sturm–Liouville problem is presented. The results show an agreement with the exact solution. High results precision is achieved with higher number of elements.

***Keywords*—**Sturm-Liouville problem, Robin boundary condition, finite element method, eigenvalue problems.

I. Introduction

STURM-LIOUVILLE boundary value problem or eigenvalue problem is an important theory, which essentially is an extension of the spectral theorem from discretised vector spaces into continuous function spaces. They have continued to provide new ideas and major advances in the field of spectral Analysis solutions to separable partial differential equations and various applications in the field of physics.

Sturm-Liouville problem is a second-order ordinary differential equations problem where two boundary conditions are specified, but where no unique solution exists. These problems may be regular or singular at each endpoint of the underlying interval [1]. They arise throughout the field of applied mathematics, for example, they are used to describe the vibrational modes of various systems, such as the vibrations of a string. These equations are common, both in the field of classical physics (example is thermal conduction) and quantum mechanics (example is Schrodinger Equation), where they are used to describe processes where some boundary value is held constant, while the system is in operation.

The Classical Sturm-Liouville theory consists of finding the eigensolutions (and eigenvalues) for second-order ordinary differential equations on a finite interval with no singularities. They commonly arise from linear partial differential equations (PDEs) in several space dimensions when the equations are separable, in some coordinate systems such as cylindrical or spherical coordinates. Some examples of these equations and their applications are the Bessel, Legendre, and Laguerre equations. Bessel equations arise when solving the Laplace and Helmholtz equations by separation of variables in cylindrical polar coordinates. Legendre equation arises in solving Laplace equations in spherical polar coordinates, and they give expressions for the spherical harmonic functions. While the Laguerre equation arises in solutions of 3-dimensional Schrodinger equation with an inverse-square potential and in Gaussian integration.

In 1836-1837, Sturm and Liouville published a series of papers on second order linear ordinary differential equations including boundary value problems [2]. The influence of their work was such that this subject became known as Sturm-Liouville theory. Many thousands of papers, by mathematicians, physicists, engineers, and others, relating to this area have been written since then. Yet, remarkably, this subject is an intensely active field of research today. Dozens of papers are published on Sturm-Liouville Problems (SLP) every year.

Reference [3] studied a procedure for the automatic computation of the eigenvalues and the eigenfunctions of one-dimensional linear Sturm-Liouville boundary value Eigen problems, for both singular and nonsingular. The continuous coefficients of a regular Sturm-Liouville problem were approximated by a finite number of step functions. Reference [4] obtained the asymptotic formulas for eigenvalues, eigen functions, and the reciprocals of the eigenfunction norms for eigenvalue problems associated with the general Sturm-Liouville equation having regular endpoint. Reference [5] found an expression for the derivative of an eigenvalue with respect to a given parameter: an endpoint, a boundary condition, a coefficient or the weight function of a Sturm-Liouville problem. The Homotopy Analysis Method (HAM) was applied to numerically approximate the eigenvalues of the second and fourth-order Sturm–Liouville problems in [6]. The eigenvalues were calculated by starting the HAM algorithm with one initial guess.

Reference [7] considered the *n*th eigenvalue as a function on the space of self-adjoint regular Sturm–Liouville problems with positive leading coefficient and weight functions. Reference [8] derived a method of computing accurate approximations to the eigenvalues and Eigen functions of regular Sturm–Liouville differential equations. The method consists of replacing the coefficient functions of the given problem by piecewise polynomial functions and then solving

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the resulting simplified problem. References [9]-[12] studied the spectral of perturbed Sturm-Liouville problem and considered the boundary-value problem which consists of the integro-differential equation.

Robin boundary conditions, also called impedance boundary conditions, from their application in electromagnetic problems, or convective boundary conditions, from their application in heat transfer problems [13] are a weighted combination of Dirichlet boundary and Neumann boundary conditions. The Robin boundary conditions take the form

$$\alpha T + \beta(\nabla T \cdot n) = h \qquad on\ \Gamma \tag{1}$$

Robin boundary conditions are applicable to the solution of Sturm-Liouville problems. In this work, a finite element method (FEM) for computing the eigenvalues and eigenfunctions of a Sturm-Lioville problem with Robin Boundary conditions is presented and analysed.

II. The Problem Formulation

Consider a Sturm-Liouville boundary value problem with Robin boundary conditions

$$-\frac{d}{dx}\left(P(x)\frac{du}{dx}\right) + q(x)u = \lambda R(x)v(x)\ ;\ a < x < b \tag{2}$$

$$B_1 u(a) + B_2 \frac{du}{dx}(a) = 0 \tag{3}$$

$$\alpha_1 u(b) + \alpha_2 \frac{du}{dx}(b) = 0 \tag{4}$$

We can apply the finite element method to this problem in the usual way by first constructing a weak form for the equation;

$$\begin{aligned} P(a)\frac{du}{dx}(a)v(a) - P(b)\frac{du}{dx}(b)v(b) + \int_a^b P(x)\frac{du}{dx}(x)\frac{dv}{dx}dx \\ + \int_a^b q(x)u(x)\,v(x)dx \\ = \int_a^b \lambda R(x)w(x)v(x)dx \end{aligned} \tag{5}$$

The boundary conditions imply

$$\frac{du}{dx}(a) = \frac{\beta_1}{\beta_2}u(a) \tag{6}$$

$$\frac{du}{dx}(b) = -\frac{\alpha_1}{\alpha_2}u(b) \tag{7}$$

Therefore, the weak form can be written in the form:

$$\begin{aligned} \frac{\beta_1}{\beta_2}P(a)u(a)v(a) + \frac{\alpha_1}{\alpha_2}P(b)u(b)v(b) + \int_a^b P(x)\frac{du}{dx}(x)\frac{dv}{dx}(x) \\ + \int_a^b q(x)u(x)\,v(x)dx \\ = \int_a^b R(x)w(x)v(x)dx \end{aligned} \tag{8}$$

Assume that an approximate solution can be written in the form:

$$V_n(x) = \sum_{j=1}^{n} u_d \gamma_d(x) \tag{9}$$

where γ_d is the spike function.

Substituting (6) and (7) into (8) gives the generalized eigenvalue problem

$$Au = \lambda Ru \tag{10}$$

where;

$$A = K + M + G \tag{11}$$

K, M, and F are the stiffness matrix, mass matrix and load vector respectively corresponding to Neumann conditions

$$M_{ij} = \int_a^b q(x)\phi_j(x)\phi_i(x)dx \tag{12}$$

$$R_{ij} = \int_a^b R(x)\phi_j(x)\phi_i(x)dx \tag{13}$$

The boundary term can be written as Gu. We have

$$\begin{aligned} \frac{\beta_1}{\beta_2}P(a)u(a)\phi_2(a) + \frac{\alpha_1}{\alpha_2}P(b)u(b)\phi_1(b) \\ = \frac{\beta_1}{\beta_2}P(a)\sum_{j=1}^{n} U_j\phi_j(a)\phi_i(a) \\ + \frac{\alpha_1}{\alpha_2}P(b)\sum_{j=1}^{n} U_j\phi_j(b)\phi_i(b) \end{aligned} \tag{14}$$

$$\begin{cases} \frac{\beta_1}{\beta_2}P(a)V_n & i = 0 \\ \frac{\alpha_1}{\alpha_2}P(b)V_n & i = n \\ 0 & otherwise \end{cases} \tag{15}$$

This implies that:

$$G_{ij} = \begin{cases} \frac{\beta_1}{\beta_2}P(a) & i = j = 0 \\ \frac{\alpha_1}{\alpha_2}P(b) & i = j = n \\ 0 & otherwise \end{cases} \tag{16}$$

A. Heat Transfer Problem

Consider the heat transfer problem that models the temperature distribution in a rectangular fin of length, L and thickness, a.

The boundary value problem is given as:

$$-\nabla \cdot (k \cdot \nabla T) + \rho\, c_v(T - T_\infty)\frac{\partial T}{\partial t} = 0\,, \quad a < x < L \tag{17}$$

Subjected to the boundary conditions:

$$T(0,t) = T_\infty\,;\ \left(\frac{k}{\rho\, c_v}\cdot\nabla T + \beta\,(T - T_\infty)\right)\Big|_{x=L} = 0 \tag{18}$$

where k is the thermal conductivity; ρ is the density; c_v is the specific heat capacity, and T_∞ is the ambient temperature. Using the dimensionless parameters,

$$\alpha = \frac{k}{\rho\, c_v}\,; \bar{x} = \frac{x}{L}\,;\ \bar{t} = \frac{\alpha t}{L^2}\,;\ \Theta = \frac{T - T_\infty}{T_0 - T_\infty} \tag{19}$$

The normalized form of (17) and (18) becomes;

$$-\alpha\frac{\partial^2\Theta}{\partial\bar{x}^2} + \frac{\partial\Theta}{\partial t} = 0 \tag{20}$$

$$\Theta(0,\bar{t}) = 0\,;\ \left(\frac{\partial\Theta}{\partial\bar{x}} + \frac{\beta\, L\,\Theta}{k}\right)\Big|_L = 0 \tag{21}$$

If a periodic solution of the form $\Theta = \Theta\, e^{-\lambda\bar{t}}$ is assumed, (20) and (21) becomes;

$$-\alpha\frac{\partial^2\Theta}{\partial\bar{x}^2} + \lambda\Theta = 0 \tag{22}$$

$$\Theta(0) = 0\,;\ \left(\frac{\partial\Theta}{\partial\bar{x}} + \frac{\beta\, L\,\Theta}{k}\right)\Big|_L = 0 \tag{23}$$

Equation (22) is the Sturm-Liouville eigenvalue problem with Robin boundary conditions in (23).

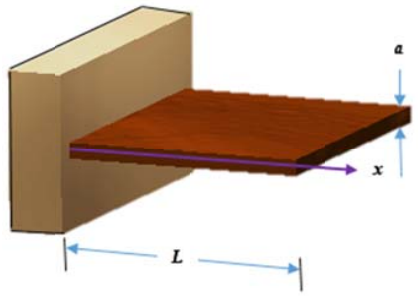

Fig. 1 Rectangular Fin

B. Structural Mechanics

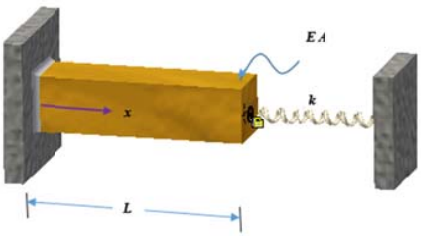

Fig. 2 Cantilever with Fixed Spring

Consider the cantilever with the fixed spring set in Fig. 2. The governing differential equation is given by:

$$-\nabla\cdot(EA\cdot\nabla u) + \rho A\frac{\partial^2 u}{\partial x^2} = 0\,, \tag{24}$$

$$a < x < L$$

$$u(0,t) = 0\ ;\quad \left(EA\frac{\partial u}{\partial x} + k\,u\right)\Big|_{x=L} = 0 \tag{25}$$

If a periodic solution of the form $U = U\, e^{-\lambda\bar{t}}$, the Sturm-Liouville eigenvalue problem becomes;

$$-\frac{E}{\rho}\frac{\partial^2 U}{\partial\bar{x}^2} + \lambda^2 U = 0 \tag{26}$$

With the Robin boundary conditions:

$$U(0) = 0\ ;\quad \left(\frac{\partial U}{\partial\bar{x}} + \frac{k}{EA}\,U\right)\Big|_L = 0 \tag{27}$$

Using the Lagrange interpolation function, the condense equations are given by:

$$\left(\frac{E}{\rho\, L}\begin{bmatrix}2 & -1\\ -1 & \frac{\rho}{E}k\end{bmatrix} - \lambda^2\,\frac{L}{6}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}\right)\begin{Bmatrix}U_2\\ U_3\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix} \tag{28}$$

The natural frequencies are given as:

$$\lambda_i = \frac{1}{L}\sqrt{\frac{6\,E}{\rho}}\,\omega_i\ ,\quad i = 1,2 \tag{29}$$

III. Analysis of Results

The free vibration equation of the heat problem has been derived by using the Sturm-Liouville Equation. The problem is analysed and presented. In order to analyse the mode shape, the discretization is done at different number of element to ascertain the accuracy of the methodology. Here, an isotropic material of density $\rho = 7.88$ kg/m^3 is assumed. The specific heat capacity is $c_v = 0.437$; the thermal conductivity, $k = 0.836$ kW/mK and the heat transfer coefficient, $\beta = 0.005$ kW/m^{2}K .

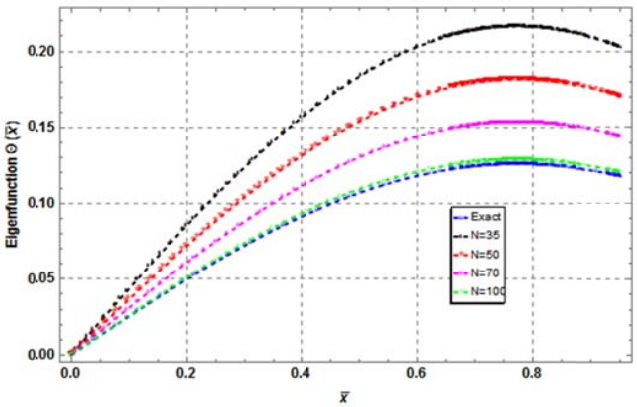

Fig. 3 The Plot of Eigenfunction against the Dimensionless Displacement for the First Mode at Different Number of Elements, $N = 30$, $N = 50$, $N = 70$, $N = 100$

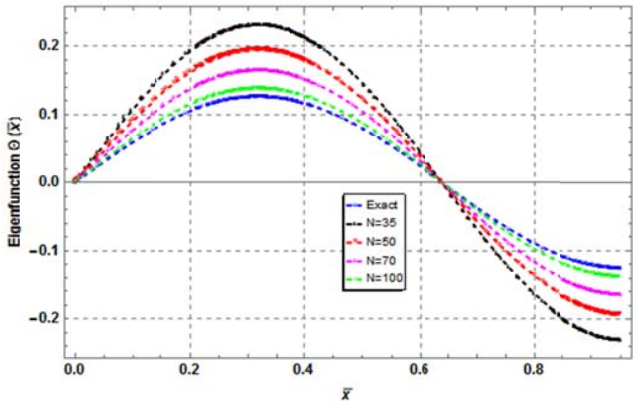

Fig. 4 The Plot of Eigenfunction against the Dimensionless Displacement for the Second Mode at Different Number of Elements, N=30, N=50, N=70, N=100

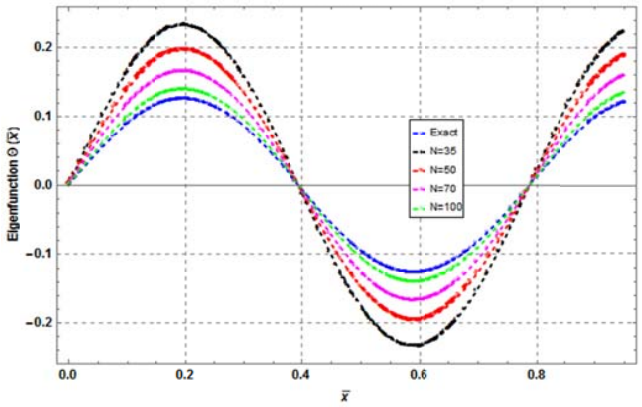

Fig. 5 The Plot of Eigenfunction against the Dimensionless Displacement for the Third Mode at Different Number of Elements, N=30, N=50, N=70, N=100

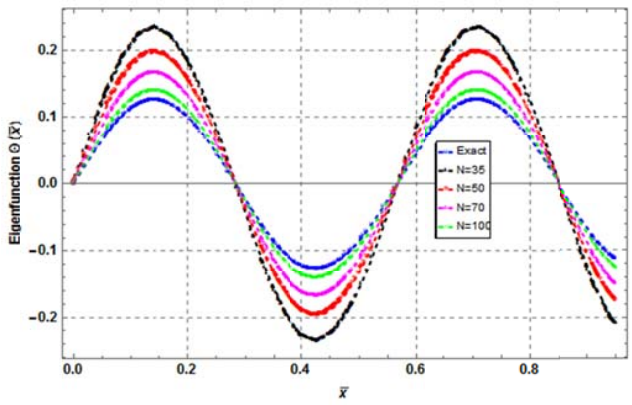

Fig. 6 The Plot of Eigenfunction against the Dimensionless Displacement for the Fourt Mode at Different Number of Elements, N=30, N=50, N=70, N=100

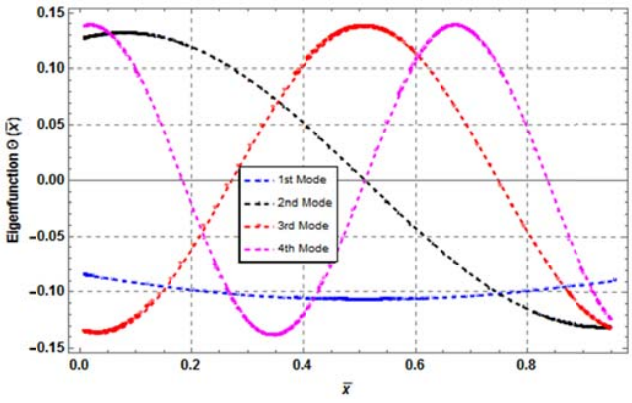

Fig. 7 The Plot of Eigenfunctions against the Dimensionless Displacement for the First Four Mode when the Number of Elements, N=100

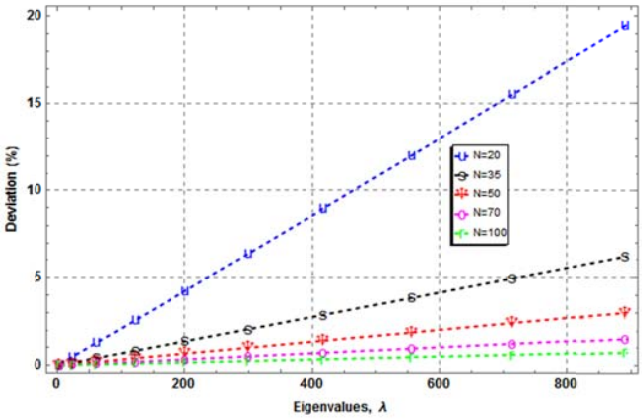

Fig. 8 Deviation of Eigenvalue from the Exact Solution at Different Number of Elements, N=30, N=50, N=70, N=100

The numerical results for the first four modes are shown in Figs. 3-6. The different modes results are presented together with the exact solution. Each mode has results at different number of elements from the finite element method. The two results converge when the number of discretised elements is relatively high.

IV. Conclusion

The results presented illustrate the effectiveness and advantage of the FEM in predicting the Eigen modes of physical systems through the formed Sturm-Liouville. Examples show that the method is very efficient when compared with the exact solution. Therefore, the formulation and solution of a Sturm-Liouville problem could be achieved with less computation rigor.

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