An Approximate Lateral-Torsional Buckling Mode Function for Cantilever I-Beams

H. Ozbasaran

Abstract—Lateral torsional buckling is a global buckling mode which should be considered in design of slender structural members under flexure about their strong axis. It is possible to compute the load which causes lateral torsional buckling of a beam by finite element analysis, however, closed form equations are needed in engineering practice for calculation ease which can be obtained by using energy method. In lateral torsional buckling applications of energy method, a proper function for the critical lateral torsional buckling mode should be chosen which can be thought as the variation of twisting angle along the buckled beam. Accuracy of the results depends on how close is the chosen function to the exact mode. Since critical lateral torsional buckling mode of the cantilever I-beams varies due to material properties, section properties and loading case, the hardest step is to determine a proper mode function in application of energy method.

This paper presents an approximate function for critical lateral torsional buckling mode of doubly symmetric cantilever I-beams. Coefficient matrices are calculated for concentrated load at free end, uniformly distributed load and constant moment along the beam cases. Critical lateral torsional buckling modes obtained by presented function and exact solutions are compared. It is found that the modes obtained by presented function coincide with differential equation solutions for considered loading cases.

Keywords—Buckling mode, cantilever, lateral-torsional buckling, I-beam.

I. Introduction

LATERAL TORSIONAL BUCKLING (LTB) is a global buckling mode of slender structural members in which the beam experiences non uniform twisting and buckling about its weak axis. There are three main methods to obtain critical LTB load of beams, which is defined as the smallest load that causes LTB. First one is the solution of equilibrium equation of LTB [1]. It should be remembered that equilibrium equation of LTB is only valid if the load is acting at the shear center of the section. Solution of the mentioned equation usually involves numerical methods which cannot be applied easily without a mathematical software [2]. The second one is finite element analysis (FEA). In FEA, structural member, boundary conditions and loading case should be modeled properly and solved with a FEA software. These two methods are not practical and require programming and finite element modeling skills. The last one, which is the concern of this study, is energy method. Although application of energy method is not practical itself, the main motivation to use this method in LTB problems is that closed form and/or parametric equations can be presented [3].

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By energy method, critical LTB load can be calculated by equalizing the work done by internal and external forces at critical state. Instead of considering all work terms, it is sufficient to take into consideration the works of twisting. For a doubly symmetric cantilever I-beam which is loaded from its shear center, (1) can be written at the moment of instability [4].

$$\frac{1}{2}\int_0^L GI_t \left(\frac{d\phi}{dz}\right)^2 dz + \frac{1}{2}\int_0^L EI_w \left(\frac{d^2\phi}{dz^2}\right)^2 dz = \frac{1}{2}\int_0^L \frac{(M_x\phi)^2}{EI_y} dz \qquad (1)$$

In (1), L is cantilever length, E is elasticity modulus, G is shear modulus, I_t is torsional constant, I_w is warping coefficient, I_y is moment of inertia about weak axis, M_x is bending moment about strong axis, z is the distance from fixed end, and finally ϕ is the twisting angle.

The left side of (1) includes work terms of internal forces. The first and second terms are the works done by torsion and warping, respectively. The only term at the right side of (1) is for the work done by the bending moment component about weak axis.

As all methods do, energy method has a drawback. In energy method, a function should be chosen which is in the form of the considered deformation along the member [3]. Twisting angle along the buckled beam is the considered deformation for LTB.

Since the ratio of twisting angle to LTB mode is constant at any point on the span of the buckled beam, twisting angle (ϕ) introduced in (1) can be safely replaced with LTB mode. Then critical LTB load of the beam can be calculated. The point is, there are more than one LTB mode of a beam. Each one leads to a different buckling load. The LTB mode that leads to smallest LTB load, which is called critical LTB mode, should be used in energy equations.

Better results are obtained by energy method if a function is chosen which matches the exact critical LTB mode. Otherwise, results obtained by energy method diverge from exact solution. A simple function which can take the form of critical LTB mode of cantilever I-beams for any loading case is not encountered in the literature. Besides, such function does not seem to be exist. Related to LTB of steel beams, Hodges and Peters investigated the LTB of cantilever strip and I-beams. They proposed an approximate function for the elastic critical LTB mode of a cantilever I-beam which is subjected to a concentrated load acting at its free end [5]. Andrade and Camotim presented an approach to LTB analysis of doubly symmetric prismatic and tapered thin-walled beams. [6]. Andrade and Camotim introduced a general formulation

for the elastic LTB behavior of singly symmetric thin-walled tapered beams [7]. Andrade et al. extended the application domain of 3-factor formula, which is commonly employed to determine elastic LTB loads of beams, to I-section cantilever beams [8]. Andrade et al. discussed the difference in LTB loads between 1D model and shell FEA [9]. Challamel and Wang presented exact stability criteria for LTB of cantilever strip beams and introduced closed-form solutions in terms of Bessel functions. They underlined that no such solution is obtained for cantilever I-beams by that moment [10]. Goncalves presented a geometrically exact beam formulation and finite element implementation, aiming at calculating LTB loads of thin-walled beams with deformable cross-section [11]. Benyamina et al. investigated the elastic LTB behavior of doubly symmetric web-tapered thin-walled I-beams and proposed an accurate analytical formula to calculate LTB loads in function of stiffness terms, load height position and tapering parameter [12]. Ozbasaran presented a parametric formula to calculate elastic critical LTB loads of European IPN and IPE section cantilever beams [13]. Recently, Sonck and Belis investigated the lateral–torsional buckling behavior of cellular beams [14]. Ibrahim et al. studied the inelastic LTB of coped beams with simply supported ends experimentally [15]. Kucukler et al. presented a stiffness reduction approach for the LTB assessment of steel beams [16]. Nguyen et al. introduced a design procedure for elastic LTB of PFRP beams [17].

This paper presents a function for critical LTB mode of doubly symmetric cantilever I-beams in bivariate polynomial form. Coefficient matrices of the polynomial are calculated for three simple loading cases; Concentrated load at free end, uniformly distributed load and constant moment along the beam. Results obtained by presented function are compared to exact numerical solutions.

II. METHOD

Let us consider a doubly symmetric cantilever I-beam with a concentrated load acting at its free end. By assuming vertical load position is at shear center, (2) can be written [1].

$$\frac{d^4\phi}{dz^4} - \frac{C}{C_1}\frac{d^2\phi}{dz^2} - \frac{P^2(L-z)^2}{EI_yC_1}\phi = 0 \qquad (2)$$

In (2), $C = GI_t$ is the torsional rigidity of the section. $C_1 = EI_w$ is the warping rigidity of the section and P is the magnitude of the concentrated load. Solution of ϕ is dependent to a dimensionless slenderness which is given in (3):

$$\psi = \frac{L^2C}{C_1} \qquad (3)$$

Although dependence of ϕ to ψ and loading case is known, yet a closed form solution of ϕ which includes ψ and loading case parameters is not encountered in literature. To visualize variation of twisting angle related to ψ, three-dimensional plots are used. The solution is performed by finite differences method with a precision of 100 finite elements for each beam

[2], [4]. First, critical LTB load of the beam is determined by solving the eigenvalue problem. Then, critical LTB mode of the beam is obtained numerically by substituting critical LTB load into node equilibrium equations.

Fig. 1 presents the variation of critical LTB mode of a cantilever I-beam subjected to a concentrated load acting at its free end for the ψ values between 0.1 and 150.
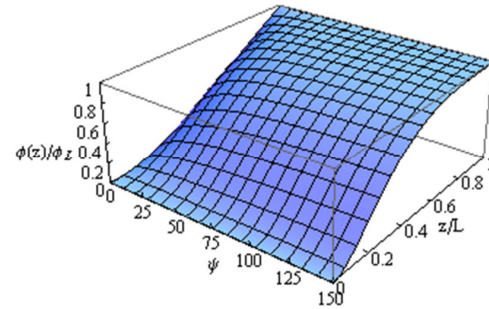

Fig. 1 Critical LTB mode surface for concentrated load at free end case

In Fig. 1, $\phi(z)$ is the twisting angle at point z, ϕ_L is the twisting angle at free end. $\phi(z)/\phi_L$ is the relative twisting angle at point z. ψ is the dimensionless slenderness introduced in (3). Finally, z/L is the relative position on the beam length which is 0 at fixed end and 1 at free end.

The three dimensional plot given in Fig. 1 (in further text: critical LTB mode surface) can be represented by a bivariate polynomial as (4):

$$\phi(z)/\phi_L = L\sum_{i=1}^n\sum_{j=1}^n K_{ij}\psi^{i-1}\left(\frac{z}{L}\right)^{j-1} \qquad (4)$$

K_{ij} terms for the values of i and j from 1 to n produce a matrix which is called "coefficient matrix" in the further text. For each loading case, a different coefficient matrix is found.

The degree of the bivariate polynomial is limited to $n = 4$. If a higher degree polynomial template is used, accuracy increases as expected. However, a proper template should be constructed for optimal function complexity and precision. The coefficient matrix, which is composed of 16 members, for concentrated load at free end case is as;

$$K_P = 10^{-5}\begin{bmatrix} -156.30 & 3013.06 & 16823.90 & -9839.09 \\ -23.61 & 531.94 & -435.69 & -85.54 \\ 2.38*10^{-1} & -4.76 & 1.86 & 2.84 \\ -7.89*10^{-4} & 1.53*10^{-2} & -2.80*10^{-3} & -1.23*10^{-2} \end{bmatrix} \quad (5)$$

Coefficient matrix given in (5) is calculated by fitting the bivariate polynomial given in (4) to numerical data.

By following the same steps, critical LTB mode surface and coefficient matrix of any loading case can be obtained. Differential equation of LTB for uniformly distributed load along the beam case [2], [4] is given in (6):

$$\frac{d^4\phi}{dz^4} - \frac{C}{C_1}\frac{d^2\phi}{dz^2} - \frac{q^2(L-z)^4}{4EI_yC_1}\phi = 0 \qquad (6)$$

In (6), q is magnitude of uniformly distributed load. Critical LTB mode surface for uniformly distributed load along the beam case is given in Fig. 2.
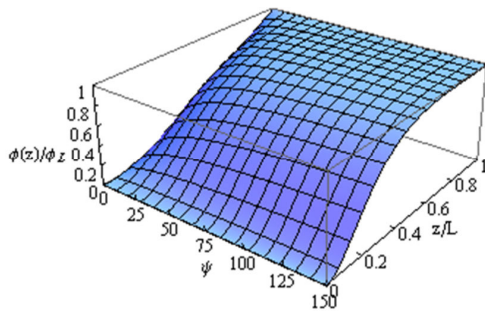

Fig. 2 Critical LTB mode surface for uniformly distributed load along the beam case

Coefficient matrix for the surface presented in Fig. 2 is:

$$K_q = 10^{-5} \begin{bmatrix} -245.36 & 5076.15 & 14090.68 & -9111.57 \\ -33.76 & 804.48 & -947.29 & 164.25 \\ 3.79*10^{-1} & -7.89 & 6.60 & 1.10 \\ -1.34*10^{-3} & 2.68*10^{-2} & -1.82*10^{-2} & -8.01*10^{-3} \end{bmatrix} \quad (7)$$

Differential equation of LTB for constant moment along the beam case [2], [4] is given in (8). It is easier to solve if compared to other considered loading cases.

$$\frac{d^4\phi}{dz^4} - \frac{C}{C_1}\frac{d^2\phi}{dz^2} - \frac{M^2}{EI_yC_1}\phi = 0 \quad (8)$$

In (8), M is magnitude of the bending moment about strong axis. Finally, Fig. 3 presents critical LTB mode surface for constant moment along the beam case.
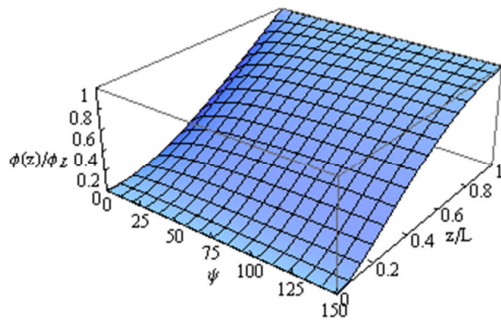

Fig. 3 Critical LTB mode surface for constant moment along the beam case

Coefficient matrix is as follows for the surface given in Fig. 3.

$$K_M = 10^{-5} \begin{bmatrix} -48.21 & 862.03 & 17200.83 & -8081.84 \\ -10.10 & 227.95 & 11.55 & -236.97 \\ 1.01*10^{-1} & -1.97 & -1.32 & 3.28 \\ -3.34*10^{-4} & 6.23*10^{-3} & 5.96*10^{-3} & -1.22*10^{-2} \end{bmatrix} \quad (9)$$

Critical LTB mode of the beam can be obtained by substituting the coefficient matrix of the considered loading case into (4).

III. VALIDATION

Critical LTB modes of various slenderness values for concentrated load at free end case are given in Fig. 4.
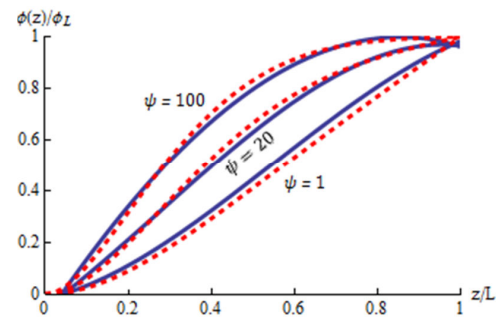

Fig. 4 LTB mode comparison for concentrated load at free end case

In Fig. 4, solid lines show presented function and dashed lines show numerical solution of differential equation of LTB.

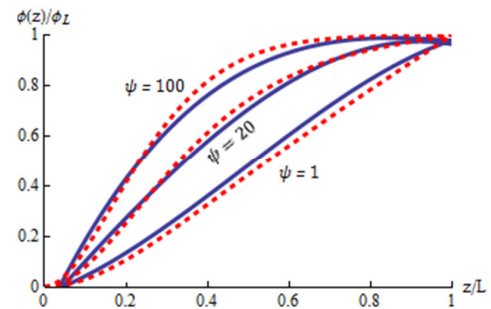Fig. 5 presents LTB modes for uniformly distributed load along the beam case.

Fig. 5 LTB mode comparison for uniformly distributed load along the beam case

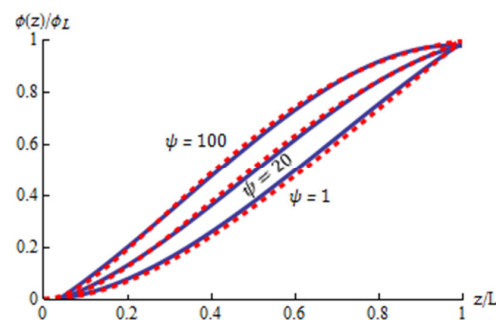Finally, Fig. 6 is given for constant moment along the beam case.

Fig. 6 LTB mode comparison for constant moment along the beam case

It can be seen from Figs. 4-6 that the modes obtained by presented critical LTB mode function are fairly close to exact solutions for considered loading cases.

To increase the accuracy of presented mode function is possible by adding higher degree terms. However, precision of

its introduced state is sufficient for practical applications.

IV. CONCLUSIONS

This paper presents a function for critical lateral torsional buckling mode of doubly symmetric cantilever I-beams. First, differential equation of lateral torsional buckling is solved numerically for various loading cases and critical lateral torsional buckling modes of each considered loading case are determined for the slenderness values between 0.1 and 150. Then, a fourth degree bivariate polynomial template is used for general representation of critical lateral torsional buckling mode and parameters are calculated for each considered loading case. Finally, the modes obtained by differential equation solution and presented function are compared. It is seen that the results obtained by presented study coincide with numerical solution of differential equation. Presented function can be safely used in energy equations to calculate the critical lateral torsional buckling load of doubly symmetric cantilever I-beams.

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