International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:9, No:10, 2015

Flow of a Second Order Fluid through Constricted Tube with Slip Velocity at Wall Using Integral Method

Nosheen Zareen Khan, Abdul Majeed Siddiqui, Muhammad Afzal Rana

***Abstract*---**The steady flow of a second order fluid through constricted tube with slip velocity at wall is modeled and analyzed theoretically. The governing equations are simplified by implying no slip in radial direction. Based on Karman Pohlhausen procedure polynomial solution for axial velocity profile is presented. Expressions for pressure gradient, shear stress, separation and reattachment points, and radial velocity are also calculated. The effect of slip and no slip velocity on magnitude velocity, shear stress, and pressure gradient are discussed and depicted graphically. It is noted that when Reynolds number increases magnitude velocity of the fluid decreases in both slip and no slip conditions. It is also found that the wall shear stress, separation, and reattachment points are strongly affected by Reynolds number.

***Keywords*---**Approximate solution, constricted tube, non-Newtonian fluids, Reynolds number.

I. Introduction

A STENOSIS, localized narrowing in an arterial system of mammals, disturb the normal pattern of blood flow through the artery and causes arterial disease. Smoking, high cholesterol levels, high blood pressure and diabetes play a momentous character in the progression and development of this disorder. Flow properties of blood such as pressure, wall shear stress, vortices, and turbulence may have potential medical significance. In the vicinity of a stenosis, a brief knowledge of rheological and dynamical characteristics of the fluid flow may help to understand the complications of constrictions [1], [2]. These include the progression of a thrombus, the hardening, weakening, swelling, and tissues growth into the arteries. Once an arterial abrasion has developed, a significant change in fluid characteristics can be seen [3].

Many researchers have pointed out that the major cause of intravascular plaques is faulty lipid metabolism. As intravascular abrasions are commonly found in the segments of curved arteries, at the entrance of branching vessels, or generally at locations of abrupt changes in geometry which should take into account the flow characteristics of the blood. Due to the flow separation from the artery wall, static zones occur in the arterial system [4]. They [4] indicate that the fluid flows with points of inflexion in the velocity profiles and reserved flow in some cases. The considerable complexity of prosthetic heart valve is the formation of thrombus. Numerous researchers have anticipated that the formation of thrombosis near the valve is the contribution of the static zone. Therefore, it seems objective to venture that if static zone occur near constriction, they may well contribute to the problems of arthrosclerosis.

In the branch of biomechanics, the study of fluid flow in obstructed tube is always a challenging problem though the importance of hydrodynamic has been reported for many years. Young [5] has studied in detail the flow in a mildly constricted tube. Forrester and Young [6] extended this work to embrace the effects of flow separation on a mild constriction. The study of flow in constricted tubes was analyzed numerically by [7]. Many physicians, researchers, and scientists have made their efforts to understand the mechanics of fluid flow in constricted arteries considering the blood as Newtonian fluid. The blood, however, only under certain conditions behaves like a Newtonian fluid, of course, at low shear stress it becomes as a non-Newtonian fluid [8]. The blood, in the larger arteries where the constriction commonly occurs, treated as incompressible Newtonian fluid. They [4] proposed that the atherosclerosis plaque (constriction) is caused by intravascular clotting. For the first time Fry [9] reported the endothelial changes by inserting a plug in the thoratic aorta of mongrel dogs, which abrupt the blood flow. Further, he obtained theoretical results for unsteady, axisymmetric, incompressible Newtonian fluid flow numerically and compared with the experimental one. Young [5] reported the time dependent constriction in tube for viscous flow. Forrester and Young [6] developed theoretical and experimental results for the blood flow through constricted tube. A primary goal of their research was to predict analytically the separation of flow at Reynolds number in constricted tube. For analytical results, they use an integral method. An experiment was performed to check the theory which was valid for water and glycerol-water solution (viscous fluid) but not valid for the whole blood, as at low shear rate blood behaves as a non-Newtonian fluid. Further results were obtained for pressure drop, separation, and reattachment regions and compared with the theoretical one. Morgan and Young [10] investigated a simple and approximate solution of the fluid flowing through an

N. Z. Khan is Ph D Scholar at Riphah International University I-14 Islamabad, 44000, Pakistan (phone: 0092-332-5549737; e-mail: noshikhan14 @yahoo.com).

A. M. Siddiqui is a Professor of Mathematics in Department of Mathematics, Penn State York, 1031 Edgecomb Avenue, York, PA 1740., USA (e-mail: ams5r@psu.edu).

M. A. Rana is a Professor of Mathematics at Riphah International University, Sector I-14, Islamabad, Pakistan (e-mail: Muhammad.afzal @)riphah.edu.pk.

axisymmetric artery having cosine shape constriction, valid for both mild and sever constriction. Their general approach was an extension and modification of the work done by Forrester and Young [6] and made use of integral-momentum and energy equation for viscous, steady, and incompressible fluid. Haldar [11] has analyzed the blood flow treating it as a Newtonian fluid flowing through an axisymmetric artery having constriction. References [12], [13] explored various information related to the fluid flowing through an axisymmetric artery having cosine shape constriction. Reference [14] analyzed steady and incompressible Newtonian fluid flowing through axisymmetric constricted tube taking constant volume flow rate.

In the aforementioned articles, the usual no-slip condition at the uniform and constricted artery walls has been taken. Experiments on blood flow shows that under certain conditions there exist slip velocity at wall. Therefore, it is necessary to study the blood flow in a constricted artery with axial velocity slip at the wall. Bennett [15] carried out the experiments on blood flow in arteries and suggested the possibility of slip velocity at wall under certain conditions. Several other investigators [16]-[19] also indicate the axial velocity slip at the inner surface of the artery wall. In the view above theoretical and experimental work, it is improper to ignore slip in the constricted walls.

The aim of this theoretical work is to study the effects of slip at the constricted tube on the flow variables, magnitude of velocity profile, pressure drop, wall shear stress and separation and reattachment points for non-Newtonian model. The flow is assumed to be steady just to make the problem mathematically manageable. However, it is anticipated that the result for steady flow will be responsible for useful information. As arterial flow is pulsatile, this assumption, of course, cannot be justified totally. Moreover, it is believed that blood flow, except in the ascending aorta or under pathological circumstances, is laminar. In addition, it is well established that for knowledge of the relationship between the constriction and blood flow in arteries, a better understanding the flow characteristics in constriction is a necessary prerequisite. In this work, therefore, velocity, pressure, shear stress, separation and reattachment points of fluid flowing through constricted tube are analyzed.

II. Governing Equations

$$\boldsymbol{\nabla}.\mathbf{V} = 0 \tag{1}$$

$$\rho\left(\frac{\partial\widetilde{V}}{\partial t}+\frac{1}{2}\nabla(\widetilde{\boldsymbol{V}})^2-\widetilde{\boldsymbol{V}}\times\left(\nabla\times\widetilde{\boldsymbol{V}}\right)\right)=-\nabla p+\mu div\boldsymbol{A}_1+\alpha_1 div\left(\frac{\partial \boldsymbol{A}_1}{\partial t}\right)+\alpha_1\begin{pmatrix}(\widetilde{\boldsymbol{V}}.\nabla)div\boldsymbol{A}_1+(\widetilde{\nabla}.\widetilde{\boldsymbol{V}})^T div\boldsymbol{A}_1\\ +\boldsymbol{A}_1.\left(\boldsymbol{\nabla}(\widetilde{\nabla}.\widetilde{\boldsymbol{V}})^T\right)\end{pmatrix}+(\alpha_1+\alpha_2)div\boldsymbol{A}_1^2+\rho\tilde{f}, \tag{2}$$

where $\widetilde{\boldsymbol{V}}$ is the velocity vector, ρ the constant density, μ is the dynamic viscosity, α_1 and α_2 are the material constants, $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are the first and second Rivlin-Ericksen tensors defined as:

$$\boldsymbol{A}_1=\widetilde{\nabla}.\widetilde{\boldsymbol{V}}+\left(\widetilde{\nabla}.\widetilde{\boldsymbol{V}}\right)^T, \tag{3}$$

and,

$$\boldsymbol{A}_2=\frac{dA_1}{dt}+\boldsymbol{A}_1\left(\widetilde{\nabla}.\widetilde{\boldsymbol{V}}\right)+\left(\boldsymbol{A}_1\left(\widetilde{\nabla}.\widetilde{\boldsymbol{V}}\right)\right)^T \tag{4}$$

For the model (2) required to be compatible with thermodynamics in the sense that all motions satisfy the Clasius-Duhem inequality and assumption that the specific Helmholtz free energy is a minimum in equilibrium, then the material parameters must meet the following conditions [20]:

$$\mu\geq 0, \alpha_1\leq 0, \text{and } \alpha_1+\alpha_2\geq 0. \tag{5}$$

III. Problem Formulation

We consider an incompressible steady and laminar flow of a second order fluid in a constricted tube of an infinite length having cosine shaped symmetric constriction of height δ. The radius of the unobstructed tube is R_0 and $R(\tilde{z})$ is the variable radius of the obstructed tube. The $\tilde{z}-$ axis is taken along the flow direction and $\tilde{r}-$ axis normal to it. Following [6], the dimensional boundary form for the constriction is taken as:

$$R(\tilde{z})=\begin{cases}R_0-\frac{\delta}{2}\left(1+\cos\left(\frac{\pi\tilde{z}}{z_0}\right)\right), & -\tilde{z}_0<\tilde{z}<\tilde{z}_0,\\ R_0 & \text{otherwise.}\end{cases} \tag{6}$$

In (9), δ is the maximum height of the constriction and $\tilde{z}_0$ is the length of the constricted region, R_0 is the radius of the unobstructed tube as shown in Fig. 1.

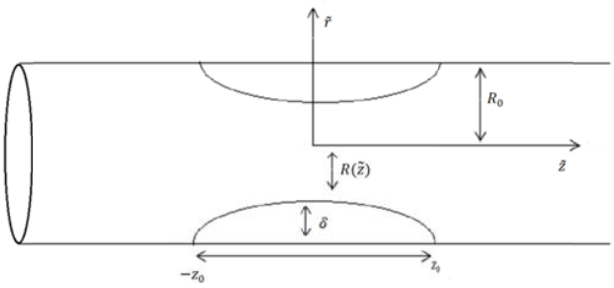

Fig. 1 Geometry of the problem

For steady axisymmetric flow of blood in tube, the velocity vector $\widetilde{\boldsymbol{V}}$ is assumed to be of the form

$$\widetilde{\boldsymbol{V}}=[\tilde{u}(\tilde{r},\tilde{z}),0,\widetilde{w}(\tilde{r},\tilde{z})]. \tag{7}$$

where $\tilde{u}$ and $\widetilde{w}$ are the velocity components in$\tilde{r}-$, $\tilde{z}-$ directions respectively.

In view of (7), (1) and (2) become;

$$\frac{\partial\tilde{u}}{\partial\tilde{r}}+\frac{\tilde{u}}{\tilde{r}}+\frac{\partial\widetilde{w}}{\partial\tilde{z}}, \tag{8}$$

$$\frac{\partial\tilde{h}}{\partial\tilde{r}}-\rho\widetilde{w}\Omega=-\mu\frac{\partial\Omega}{\partial\tilde{z}}-\alpha_1\widetilde{w}\left(\nabla^2\Omega-\frac{\Omega}{\tilde{r}^2}\right)+(\alpha_1+\alpha_2)\left(\frac{2}{\tilde{r}}\frac{\partial(\tilde{u}\Omega)}{\partial\tilde{z}}+\frac{\Omega^2}{\tilde{r}}\right), \tag{9}$$

$$\frac{\partial\tilde{h}}{\partial\tilde{z}}+\rho\tilde{u}\Omega=-\mu\left(\frac{\partial\Omega}{\partial\tilde{r}}+\frac{\Omega}{\tilde{r}}\right)+\alpha_1\tilde{u}\left(\nabla^2\Omega-\frac{\Omega}{\tilde{r}^2}\right)-2\frac{(\alpha_1+\alpha_2)}{\tilde{r}}\frac{\partial(\tilde{u}\Omega)}{\partial\tilde{r}} \tag{10}$$

where;

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:9, No:10, 2015

$$\Omega = \frac{\partial \tilde{w}}{\partial \tilde{r}} - \frac{\partial \tilde{u}}{\partial \tilde{z}}, \tag{11}$$

$$\tilde{h} = \frac{\rho}{2}(\tilde{u}^2 + \tilde{w}^2) - \alpha_1\left(\tilde{u}\nabla^2\left(\tilde{u} - \frac{\tilde{u}}{\tilde{r}^2}\right) + \tilde{w}\nabla^2\tilde{w}\right) - \frac{3\alpha_1 + 2\alpha_2}{4}|\boldsymbol{A}_1|^2 + \tilde{p}, \tag{12}$$

$$|\boldsymbol{A}_1|^2 = 4\left(\frac{\partial \tilde{u}}{\partial \tilde{r}}\right)^2 + 4\left(\frac{\tilde{u}}{\tilde{r}}\right)^2 + 4\left(\frac{\partial \tilde{w}}{\partial \tilde{z}}\right)^2 + 2\left(\frac{\partial \tilde{u}}{\partial \tilde{z}} + \frac{\partial \tilde{w}}{\partial \tilde{r}}\right), \tag{13}$$

and ∇^2 is the Laplacian parameter, $\tilde{h}$ is the generalized pressure.

According to the geometry of the problem the boundary conditions are:

$$\tilde{u} = \tilde{w} = 0 \quad at \quad \tilde{r} = R(\tilde{z}),$$

$$\frac{\partial \tilde{w}}{\partial \tilde{r}} = 0 \quad at \quad \tilde{r} = 0. \tag{14}$$

For convenience introducing the dimensionless variables;

$$r = \frac{\tilde{r}}{R_0}, z = \frac{\tilde{z}}{z_0}, w = \frac{\tilde{w}}{U_0}, u = \frac{\tilde{u} z_0}{U_0 \delta}, h = \frac{\tilde{h}}{\rho U_0^2}, p = \frac{\tilde{p}}{\rho U_0^2}, R_e = \frac{U_0 R_0 \rho}{\mu}, \tag{15}$$

Now, (8) becomes;

$$\frac{\partial w}{\partial z} + \frac{\delta}{R_0}\frac{1}{r}\frac{\partial}{\partial r}(ru) = 0. \tag{16}$$

From (16) it is noted that $\frac{\partial w}{\partial z}$ is an order of $\frac{\delta}{R_0}$ i.e $\frac{\partial w}{\partial z} \sim O\left(\frac{\delta}{R_0}\right)$. Forrester and Young [6] assumed that for mild constriction if ${}^1/_{R_e}\, {}^{\delta}/_{R_e} \ll 1$, ${}^{\delta}/_{z_0} \ll 1$ and ${}^{R_0}/_{z_0} \approx 1$ then axial normal stress gradient $\frac{\partial^2 w}{\partial z^2}$ is negligible as compared to the gradient of shear component. So, (10) and (11) become

$$\frac{\partial h}{\partial r} = 0, \tag{17}$$

$$\frac{\partial h}{\partial z} = \frac{1}{R_e}\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right], \tag{18}$$

$$h = \frac{1}{2}w^2 - \alpha^*\begin{pmatrix} w\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \\ +\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 \end{pmatrix} + \beta^*\left(\frac{\partial w}{\partial r}\right)^2 + p, \tag{19}$$

where $\alpha^* = \frac{\alpha_1}{\rho U_0^2}$ and $\beta^* = \frac{\alpha_1 + \alpha_2}{\rho U_0^2}$. From (18) integrating it over the tube cross-sections and using the boundary conditions, slip in the axial velocity $w = v_s$ at $r = R$ we obtain;

$$\int_0^R r\,\frac{\partial h}{\partial z}dr = \frac{1}{R_e}\left(\frac{\partial w}{\partial r}\right)_R. \tag{20}$$

The non-dimensional form of cosine shape constriction profile is:

$$R(z) = \begin{cases} 1 - \frac{\delta^*}{2}(1 + \cos(\pi z)) & -1 < z < 1, \\ 1 & otherwise \end{cases} \tag{21}$$

where $\delta^* = {}^{\delta}/_{R_0}$. Exact solution of (19) cannot be obtained. In order to find the approximate solution we assume fourth order polynomial which is called Karman-Pohlhausen approach [21]. Therefore;

$$\frac{w}{U} = A + B\left(1 - \frac{r}{R}\right) + C\left(1 - \frac{r}{R}\right)^2(10) + D\left(1 - \frac{r}{R}\right)^3 + E\left(1 - \frac{r}{R}\right)^4 \tag{22}$$

where U is the centerline velocity and A, B, C, D and E are undetermined coefficients which can be evaluated from the following five conditions:

$$w = v_s \quad \text{at } r = R, \tag{23}$$

$$w = U \quad \text{at } r = 0, \tag{24}$$

$$\frac{\partial w}{\partial r} = 0 \quad \text{at } r = 0, \tag{25}$$

$$\frac{dh}{dz} = \frac{1}{R_e}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \quad \text{at } r = R \tag{26}$$

$$\frac{\partial^2 w}{\partial r^2} = -2\frac{U}{R^2} \quad \text{at } r = 0. \tag{27}$$

The slip boundary conditions of velocity v_s at the wall and centerline velocity U are given by (23) and (24), condition (25) is a simple definition, and condition (26) is obtained from (18). It is assumed that at $r = 0$ the velocity profile at the center of the tube is parabolic (negligible effect of the slip velocity v_s) $w = U\left[1 - \frac{r^2}{R^2}\right]$, so that the second derivative of w with respect to r, we get (27). Thus (22) becomes;

$$w = U\left[\left(\frac{-\lambda + 10 - 12v_s}{7}\right)\left(1 - \frac{r}{R}\right) + \left(\frac{3\lambda + 5 - 6v_s}{7}\right)\left(1 - \frac{r}{R}\right)^2 + \left(\frac{-3\lambda - 12 + 20v_s}{7}\right)\left(1 - \frac{r}{R}\right)^3 + \left(\frac{\lambda + 4 - 9v_s}{7}\right)\left(1 - \frac{r}{R}\right)^4\right], \tag{28}$$

where;

$$\lambda = \frac{R^2\,R_e}{U}\frac{dh}{dz}. \tag{29}$$

We note that λ is the function of z only, since R, U and h depend only on z. In (29), U and $h(z)$ are unknowns. If Q is the flux through the tube, then

$$Q = \int_0^R 2\pi r w dr = \pi R^2 U. \tag{30}$$

Using (28) in (30) we obtain:

$$Q = \frac{\pi R^2 U}{210}(97 - 2\lambda) + \frac{17\pi R^2 v_s}{35}, \tag{31}$$

and centerline velocity U can also be written as

$$U = \frac{210}{97}\cdot\frac{1}{\pi R^2}\left[Q + \frac{R^4 \pi R_e}{105}\frac{dh}{dz} - \frac{17\pi R^2 v_s}{35}\right]. \tag{32}$$

For constant volume flux, i.e. $Q = \pi$, we have

$$U = \frac{210}{97}\cdot\frac{1}{\pi R^2}\left[Q + \frac{R^4 \pi R_e}{105}\frac{dh}{dz} - \frac{17\pi R^2 v_s}{35}\right]. \tag{33}$$

In order to obtain a closed solution one more approximation is taken into account that the velocity profile is parabolic.

$$w = U\left[1 - \frac{r^2}{R^2}\right], \quad (34)$$

as discussed by [6]. If we neglect the non-linear terms, the flow will be a Poiseuille flow through the constriction [5]. Substitution of (34) into (19) yield and differentiating w.r.t z, we get;

$$\frac{dh}{dz} = 4\frac{v_s}{R^3}\frac{dR}{dz} - \alpha^*\frac{1}{R^5}\frac{dR}{dz}\left(32v_s - \frac{48}{R^2}\right) + 96\frac{\beta^*}{R^7}\frac{dR}{dz} + \frac{dp}{dz}. \quad (35)$$

Using (28) and (35) in (20), we obtain;

$$\frac{dp}{dz} = \frac{388}{225}\frac{1}{R^5}\frac{dR}{dz} - \frac{8}{R^4 R_e} + \frac{\alpha^*}{R^5}\frac{dR}{dz}\left(\frac{848}{75}v_s + \frac{2608}{75}\frac{1}{R^2}\right) + \frac{1}{R^3}\frac{dR}{dz}\left(\frac{2336}{15}\beta^* + \frac{88}{75}v_s\right), \quad (36)$$

In order to obtain velocity w we substitute (29), (33) and (35) in (28), to get

$$w = \frac{2}{R^2}[2\eta - \eta^2] + \frac{R_e}{R}\frac{dR}{dz}\left(\frac{4v_s}{75} + \frac{4}{225}\frac{1}{R^2}\right)[117\eta - 281\eta^2 + 267\eta^3 - 89\eta^4] + \frac{2}{50925}\frac{R_e v_s}{R^3}\frac{dR}{dz}[59722\eta - 233591\eta^2 + 244476\eta^3 - 81492\eta^4] + \frac{16}{50925}\frac{R_e\alpha^*}{R^5}\frac{dR}{dz}[-33626\eta + 114913\eta^2 - 117720\eta^3 + 39240\eta^4] + \frac{32}{10185}\frac{R_e\beta^*}{R^5}\frac{dR}{dz}[-17941\eta + 24931\eta^2 - 73395\eta^3 + 34465\eta^4] + v_s\left[1 + \frac{1}{7}(12\eta - 6\eta^2 + 20\eta^3 - 9\eta^4)\right], \quad (37)$$

where $\eta = 1 - {}^{r}/_{R}$, is the axial velocity component as a function of r and z through constricted tube. We can get velocity of unobstructed tube by taking R as constant or unity. The axial velocity profile for [6] can readily be recovered as a special case by setting $\alpha^* = \beta^* = 0$ and $v_s = 0$ in (37). On the other hand by setting $\alpha^* = \beta^* = 0$, in (37) the axial velocity component for [19] can readily be obtained.

IV. Pressure Drop Across the Constriction and Across the Whole Length of the Tube

We can get the pressure distribution across p at any cross section z along the constriction, (36) is integrated using boundary condition $p = p_0$ at $z = z_0$,

$$(\Delta p) = \frac{388}{225}\int_{R_0}^{R}\frac{1}{R^5}dR + \alpha^*\begin{pmatrix}\frac{848}{75}v_s\int_{R_0}^{R}\frac{1}{R^5}dR \\ +\frac{2608}{75}\int_{R_0}^{R}\frac{1}{R^7}dR\end{pmatrix} + \frac{2336}{15}\beta^*\int_{R_0}^{R}\frac{1}{R^7}dR + \frac{88}{75}\frac{v_s}{R_e}\int_{R_0}^{R}\frac{1}{R^3}dR - \frac{8}{R_e}\int_{-z_0}^{z_0}\frac{1}{R^4}dz, \quad (38)$$

or;

$$(\Delta p) = \frac{97}{225}\left(\frac{1}{R_0^4} - \frac{1}{R^4}\right) + \alpha^*\begin{pmatrix}\frac{212}{75}v_s\left(\frac{1}{R_0^4} - \frac{1}{R^4}\right) \\ +\frac{1304}{75}\left(\frac{1}{R_0^6} - \frac{1}{R^6}\right)\end{pmatrix} + \frac{1168}{45}\beta^*\left(\frac{1}{R_0^6} - \frac{1}{R^6}\right) + \frac{44}{75}\frac{v_s}{R_e}\left(\frac{1}{R_0^2} - \frac{1}{R^2}\right) - \frac{8}{R_e R_0^2}\int_0^{\pi}\frac{1}{(a - b\cos u)}du, \quad (39)$$

where;

$$a = 1 - \frac{\delta}{2R_0}, b = \frac{\delta}{2R_0}. \quad (40)$$

Now;

$$\int_0^{\pi}\frac{1}{a - b\cos u}du = \pi(a^2 - b^2)^{-1/2}. \quad (41)$$

Differentiating (41) thrice partially with respect to a, we get

$$\int_0^{\pi}\frac{1}{(a - b\cos u)^4}du = \pi a\left(a^2 + \frac{3}{2}b^2\right)(a^2 - b^2)^{-\frac{7}{2}} = \pi f\left(\frac{\delta}{R_0}\right), \quad (42)$$

where;

$$f\left(\frac{\delta}{R_0}\right) = \left(1 - \frac{\delta}{2R_0}\right)\left(1 - \frac{\delta}{R_0} + \frac{5}{8}\frac{\delta^2}{R_0^2}\right)\left(1 - \frac{\delta}{R_0}\right)^{-7/2}, \quad (43)$$

So that

$$(\Delta p) = \frac{97}{225}\left(\frac{1}{R_0^4} - \frac{1}{R^4}\right) + \alpha^*\begin{pmatrix}\frac{212}{75}v_s\left(\frac{1}{R_0^4} - \frac{1}{R^4}\right) \\ +\frac{1304}{75}\left(\frac{1}{R_0^6} - \frac{1}{R^6}\right)\end{pmatrix} + \frac{1168}{45}\beta^*\left(\frac{1}{R_0^6} - \frac{1}{R^6}\right) + \frac{44}{75}\frac{v_s}{R_e}\left(\frac{1}{R_0^2} - \frac{1}{R^2}\right) - \frac{8}{R_e R_0^2}f\left(\frac{\delta}{R_0}\right), \quad (44)$$

When there is no constriction i.e. $\delta = 0$ and $f\left(\frac{\delta}{R_0}\right) = 1$, the pressure drop across the normal tube is given by:

$$(\Delta p)_P = -\frac{8}{R_e R_0^2}. \quad (45)$$

In the absence of constriction, flow becomes Poiseuille and the subscript P denotes Poiseuille flow.

We note that (44) includes the results of [6] as a special case for $\alpha^* = \beta^* = 0$ and $v_s = 0$ in (44). On the other hand by setting $\alpha^* = \beta^* = 0$ in (44) the results for [19] can readily be obtained.

V. Shear Stress on Constricted Surface

The shear stress on the constricted surface is:

$$\tau_{\tilde{w}} = \mu\left(\frac{\partial\tilde{u}}{\partial\tilde{z}} + \frac{\partial\tilde{w}}{\partial\tilde{r}}\right)_R + \alpha_1\begin{bmatrix}\left(\tilde{u}\frac{\partial}{\partial\tilde{r}} + \tilde{w}\frac{\partial}{\partial\tilde{z}}\right)\left(\frac{\partial\tilde{u}}{\partial\tilde{z}} + \frac{\partial\tilde{w}}{\partial\tilde{r}}\right) \\ +2\frac{\partial\tilde{u}}{\partial\tilde{r}}\frac{\partial\tilde{w}}{\partial\tilde{r}} + 2\frac{\partial\tilde{u}}{\partial\tilde{z}}\frac{\partial\tilde{w}}{\partial\tilde{z}}\end{bmatrix}_R - \alpha_2\left[\frac{\tilde{u}}{\tilde{r}^2}\left(\frac{\partial\tilde{u}}{\partial\tilde{z}} + \frac{\partial\tilde{w}}{\partial\tilde{r}}\right)\right]_R. \quad (46)$$

We non-dimensionlize (46) and using order of magnitude analysis, wall shearing stress becomes;

$$\frac{\tau_{\tilde{w}}}{\rho U_0^2} = \frac{1}{R_e}\left(\frac{\partial w}{\partial r}\right)_R + \alpha^*\left(w\frac{\partial^2 w}{\partial r\partial z} + \frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\right)_R. \quad (47)$$

The result of (47) is very large and it is not possible to state the complete mathematical result so only the graphical results will be presented for the analysis of shear stress.

VI. Separation and Reattachment

Prandtl [22] has explained the phenomena of separation in such a manner that the velocity of the fluid in the boundary layer drubbed towards the wall and inside the boundary layer the kinetic energy of the fluid particles appears to be less than that at the outer edge of the boundary layer. This means that the fluid particles inside the boundary layer may not be able to get the pressure which is applied in the outer layer. Even a small rise in pressure may trigger the fluid particles near at the wall to stop and turn back to form a recirculating flow region, which is the characteristic of the separated flows.

The separation and reattachment points can be calculated by taking negligible effects of shear stress at the wall, i.e. $\tau_w = 0$.

$$\frac{1}{R_e}\left(\frac{\partial w}{\partial r}\right)_R + \alpha^*\left(w\frac{\partial^2 w}{\partial r\partial z} + \frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\right)_R = 0, \tag{48}$$

Due to very large calculations only graphical results will be presented. It is noted that when $\alpha^* = 0$ and $v_s = 0$ the results of [6] for separation and reattachment are recovered.

VII. Graphs and Discussions

We are considering two-dimensional flow of a second order fluid as a blood flowing in a constricted tube of infinite length. This geometry, of course, is intended to simulate an arterial stenosis. The flow is assumed to be steady, laminar and an incompressible. An approximate method is used to get the solution for the velocity, pressure drop across the constriction length, across the whole length of the tube and shear stress on the constricted surface. Just to study the effect and influence of slip on the flow parameters, two values of $v_s = 0$ and 0.05 has been considered. The effect of different flow parameters on the fluid flow are simulated with the help of graphs and proper discussion related to each graph is also provided.

In Fig. 2, the variation of non-Newtonian parameter $\alpha^* = 0$ on the velocity profile is described at z = 0.435, for no slip $v_s = 0$ and slip$v_s = 0.05$, taking$R_e = 50$,$\delta^* = 0.083$. It is denoted that velocity decreases with an increase in non-Newtonian parameter which seems physically to be correct and on the other hand the magnitude of the velocity increases with wall slip.as compare to no slip.

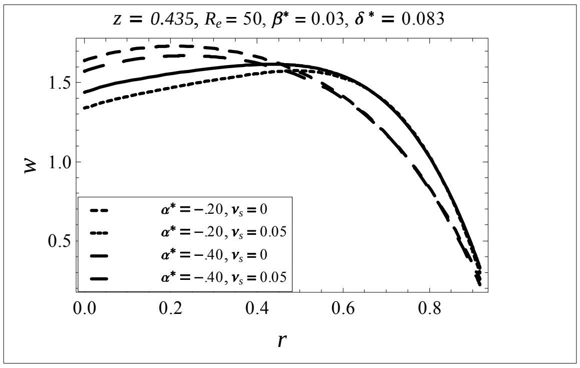

Fig. 2 Effect of non-Newtonian parameter $\alpha^*$and slip velocity v_s on velocity

It can be seen from Fig. 3 that with an increase in Reynolds number velocity of the fluid decreases near the throat of the stenosis, however, it increases in the diverging region and remains parabolic with slip or no slip. It can also be seen that velocity increases with slip effect. The effect of Reynolds number on dimensionless pressure gradient between $z = \pm 1$ takings$\alpha^* = -1.1$, $\delta^* = 0.083$ is shown in Fig. 4 it is well mentioned that the pressure gradient increases up to the throat of the constriction and then decreases in the diverging region. It is also observed that the value of pressure gradient at any point increases as Reynolds number increases in both converging and diverging region of constriction and pressure, on the mean while pressure decreases with slip Same behavior of $\alpha^*$and δ^*on the pressure gradient is observed in Figs. 5 and 6. The theoretical distribution of shearing stress along the wall is illustrated in Figs. 7-9. Fig. 7 depicts the influence of constriction height on wall shear stress in the presence of slip and no slip. It is noted that with an increase in constriction height δ^*wall shear stress increases, and its maximum value occurs at the middle of the constriction, which seems physically to be correct. It is also observed that slip causes an increase in wall shear. It is observed from Fig. 8 that for any Reynolds number in the presence of slip or no slip, the shearing stress reaches a maximum value on the throat and then rapidly decreases in the diverging region. It is also noted that shear stress decreases with an increase in Reynolds number. It means that Reynolds number provides a mechanism to control the wall shear stress. From Fig. 9, it is well mentioning, as expected, that as non-Newtonian parameter β^* increases wall shear stress also decreases. In the view of slip wall shear stress decreases more rapidly. Figs. 10 and 11 give the influence of Reynolds number on the separation and reattachment points respectively. It is observed, as naturally expected, that separation point moves upstream with an increase in Reynolds number while reattachment point moves downstream. It is also observed that the separation point moves upstream and the reattachment point moves downstream as the with slip condition.

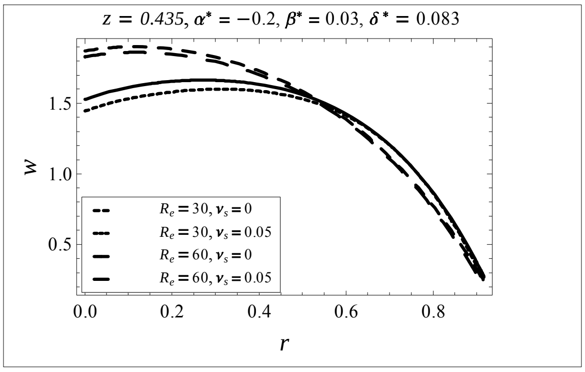

Fig. 3 Effect of Reynolds number R_e and slip velocity v_s on velocity

VIII. Conclusion

In the present study, an incompressible laminar and steady flow of a second order fluid with slip velocity at wall through constricted tube is modeled and analyzed theoretically. The fluid is assumed to be blood flowing through the artery. The expressions for magnitude of velocity field, pressure gradient, wall shear stress and separation phenomena for the geometry of the constriction are presented. An integral momentum method is applied for the solution of the problem. The summary of findings of the present work is as follows:

- Velocity decreases with an increase in non-Newtonian parameter.
- Velocity increases with slip velocity at wall.

- Shear stress and pressure decreases with slip velocity at wall.
- Viscous forces are dominant over inertia forces near the throat of the constriction, however, opposite effect is observed in the diverging region
- Reynolds number and non-Newtonian parameter are economical parameters to control the wall shear stress.
- Reynolds number also provides a mechanism to control the separation and reattachment points.
- The present study includes the theoretical and experimental results for the velocity profile, pressure gradient and wall shear stress of [6] and [19]as a special case for $\alpha^* = \beta^* = 0$ and $v_s = 0$.

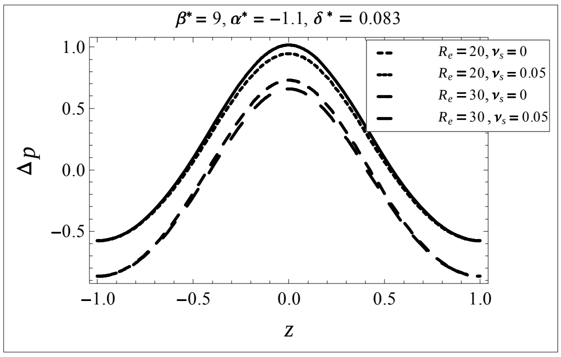

Fig. 4 Effect of Reynolds number R_e and slip velocity v_s on pressure distribution

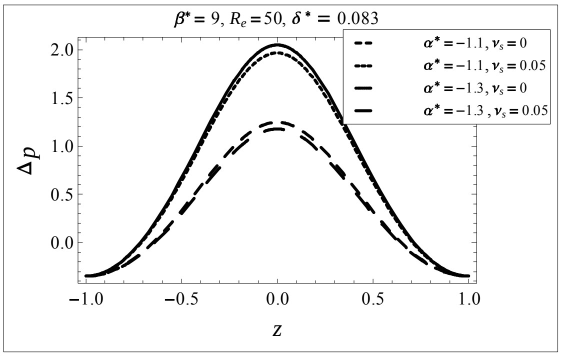

Fig. 5 Effect of non-Newtonian parameter α^* and slip velocity v_s on pressure distribution

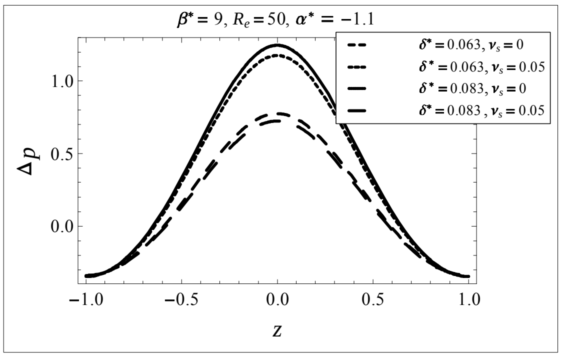

Fig. 6 Effect of δ^* and slip velocity v_s on pressure distribution

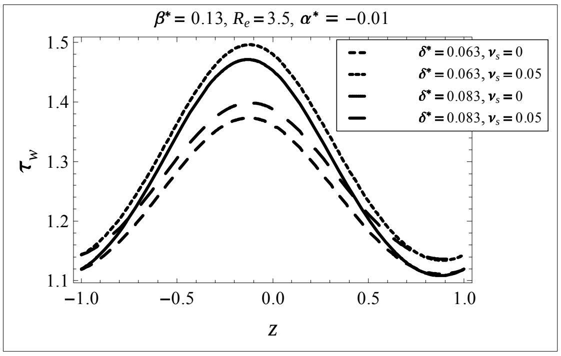

Fig. 7 Effect of δ^* and slip velocity v_s on wall shear

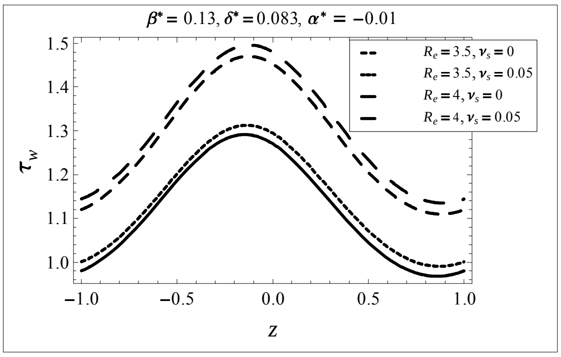

Fig. 8 Effect of Reynolds number R_e and slip velocity v_s on wall shear

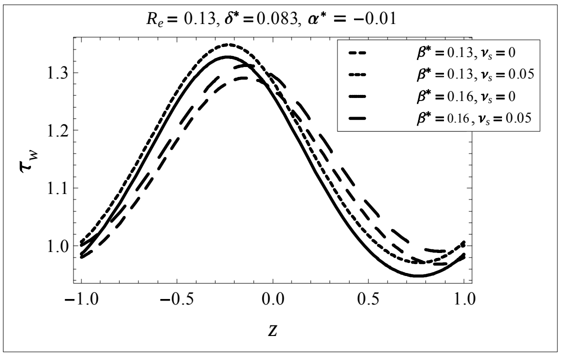

Fig. 9 Effect of non-Newtonian parameter β^* and slip velocity v_s wall shear

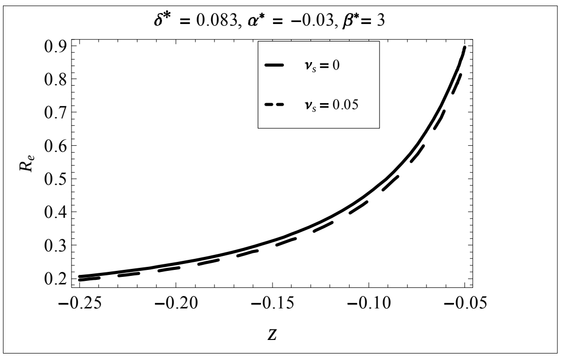

Fig. 10 Separation points in the converging region

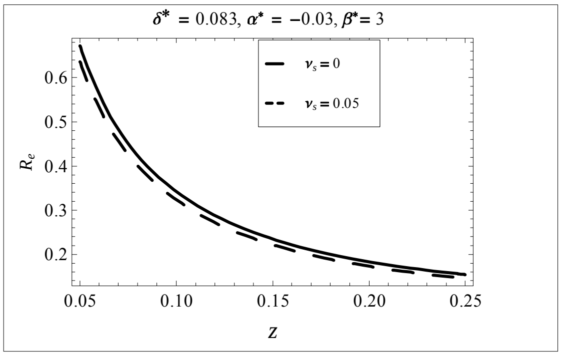

Fig. 11 Reattachment points in the diverging region

REFERENCES

[1] M. R. Roach, "An experimental study of the production and time course of post-stenotic dialation in the femoral and carotid arteries of adult dogs". *Circulation Res*, vol 13, 1963, pp. 537-551.
[2] S. Rodbard, "Dynamics of blood flow in stenotic vascular lesions". Am. Heart J. vol 72, 1966, pp. 698-704.
[3] J. H. Forrester, "Flow through a converging-diverging tube and its implications in occlusive vascular disease". PhD Thesis; Ames, Iowa, library, *Iowa State University,* 1968.
[4] J. A. Fox and A.E. Hugh, "Localization of atheroma a theory based on boundary layer separation". *British heart Journal*, vol 28, 1966, pp. 388-399.
[5] D. F. Young, "Effect of a Time-Dependent stenosis on flow through a tube". *J. Engg. Int., Trans. Am. Soc. Mech. Engrs*., vol 90, 1968, pp. 248-254.
[6] J. H. Forrester and D. F. Young, "Flow through a converging-diverging tube and its implications in occlusive disease". *J. Biomech*., vol 3, 1970, pp.297-316.
[7] J. S. Lee and Y. C. Fung, "Flow in locally constricted tubes at low Reynolds number". *J. Appl. Mech.*,vol 37, 1970, pp. 513-524.
[8] E. W. Merill, "Rheolgy of human blood and some speculations on its role in vascular homeostasis. In Biophysical mechanisms in vascular homeostasis and intravascular thrombosis", *Sawyer, P. N., ed. New York: Appleton-Century-Croft*. 1965.
[9] D. L. Fry, "Acute vascular Endothelial changes Associated with Increased Blood velocity Gradients". *Circulation Res*, vol 22, 1968, pp. 165-197.
[10] B. E. Morgan, and D. F. Young, "An Integral method for the analysis of flow in arterial stenosis". *Bulletin of Math Bio*. vol 36, 1974, pp. 39-46.
[11] K. Haldar, "Analysis of separation of blood flow in constricted arteries". *Archives of Mechanics*, vol 43, 1991, pp. 107-113.
[12] J. C. F. Chow and K. Soda, "Laminar flow in tubes with constriction". *Physics of fluids*, vol 15, issue 10, 1972, pp. 1700-1706.
[13] J. C. F. Chow, K. Soda and C. Dean, "On laminar flow in wavy channe"l. *Development in Mech*, Vol 6, Proceeding of the 12 th Midwestern Mechanics Conference, 1971.
[14] A. Mirza, A. R. Ansari, A. M. Siddiqui and T. Haroon, "On steady two-dimensional flow with heat transfer in the presence of a stenosis". *WEAS Transactions on fluids mechanics*, vol 8, issue 4, 2013,pp. 149-158.
[15] L. Bennett," Red cell slip at wall in vitro". *Sci*, vol 15, 1967, pp. 1554-1556.
[16] Y. Nubar, "Blood flow, slip and viscometer", *Biophysics J,* vol 11, 1971, pp. 252-264.
[17] P. Brumm, "The velocity slip for polar fluids", *Rheol. Acta,* vol 14, 1975,1039-1054.
[18] M. Sarma, "Analysis of blood flow through stenosed vessel under effect of magnetic field."*Int. J. for Basic Sci and Social sci,* vol 1, issue 3, 2012, pp. 78-88.
[19] J. C. Misra and B. K. Bar, "Momentum Integral method for studying flow characteristics of blood through a stenosed vessel", *Biorheology,* vol 26, issue 1, 1989, pp. 23-35.
[20] B. D. Coleman, & W. Noll, "An approximation theorem for functionals, with applications in continuum mechanics". *Arch. Rational Mech. Anal*. Vol 6, 1960, pp. 355-370
[21] Schlichting, H. Boundary Layer Theory, 6th Edn.,*McGraw-Hill, NewYork*, 1968. Pp. 122-192.
[22] L. Prandtl, "Motion of fluids with very little viscosity." ntrs.nasa.gov. 1928.

Nosheen Zareen Khan is the Ph D scholar at Riphah International University I-14 Islamabad Pakistan. She has done her M Phil in 2012 from Riphah International University I-14 Islamabad.

She has three publications. Her research interests include the mathematical modeling of Newtonian/non-Newtonian fluid flows in various geometries, analytical and numerical techniques for the solutions of linear/nonlinear boundary value problems.