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Coefficients of Some Double Trigonometric Cosine and Sine Series

Jatinderdeep Kaur

Abstract—In this paper, the results of Kano from one dimensional cosine and sine series are extended to two dimensional cosine and sine series. To extend these results, some classes of coefficient sequences such as class of semi convexity and class R are extended from one dimension to two dimensions. Further, the function $f(x,y)$ is two dimensional Fourier Cosine and Sine series or equivalently it represents an integrable function or not, has been studied. Moreover, some results are obtained which are generalization of Moricz's results.

Keywords—Conjugate Dirichlet kernel, conjugate Fejer kernel, Fourier series, Semi-convexity.

I. Introduction

THE study of special trigonometric series in one dimensional cosine and sine series as well as in two dimensional ones characterize some properties from their coefficients has been kept up by many authors ([6], [7], [10], [8], [2], [3], [4], [5]). Their central topics are mostly concerned with *Fourier Series Problem* and *Integrability Problem,* which as is shown later, are equivalent in our cases.

Let $\{a_{jk}\}$ be double sequence of real numbers such that

$$a_{jk} \to 0 \ \ as \ \ j+k \to \infty, \tag{1}$$

and

$$\sum_j \sum_k |\triangle_{11} a_{jk}| < \infty. \tag{2}$$

where j,k run over either $0,1,2,...$ or $1,2,3,...$ independently of one another. Here $\triangle_{11} a_{jk} = a_{jk} - a_{j+1,k} - a_{j,k+1} + a_{j+1,k+1}$.

The following notations are used:

$$\triangle_{10} a_{jk} = a_{jk} - a_{j+1,k}$$

$$\triangle_{01} a_{jk} = a_{jk} - a_{j,k+1}$$

Conditions (1) and (2) express the fact that $\{a_{jk}\}$ is a double null sequence of *bounded variation*.

Under conditions (1) and (2), the double cosine series

$$f(x,y) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \lambda_j \lambda_k \quad a_{jk} \cos jx \cos ky \tag{3}$$

on the positive quadrant $T^2 = [0,\pi] \times [0,\pi]$ of the two dimensional torus, where $\lambda_0 = \frac{1}{2}$ and $\lambda_j = 1$ for $j = 1,2,3,...$ converges for all $0 < x,y \le \pi$ while the double sine series

$$g(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} \sin jx \sin ky \tag{4}$$

Jatinderdeep Kaur is with School of Mathematics, Thapar University, Patiala, INDIA (e-mail: jkaur@thapar.edu).

converges for all x,y. Let $S_{mn}(x,y) = \sum_{j=0}^{m}\sum_{k=0}^{n} \lambda_j \lambda_k \quad a_{jk} \cos jx \cos ky \quad (m,n \ge 0)$ be the rectangular partial sum of the series (3) and $f(x,y) = \lim_{m+n\to\infty} S_{mn}$.

It can be seen that in the most cases, double cosine series (3) is more prickly than the double sine series (4). When (1) and $\triangle_{11} a_{jk} \ge 0$ are satisfied, then (4) is integrable on T^2 if and only if $\sum_j \sum_k \frac{a_{jk}}{jk} < \infty$. However, for (3) if the condition (1) and (2) are satisfied, then (3) is integrable on T^2 and it is the Fourier series of f.

A double sequence $\{a_{jk}\}$ is said to be *quasi-convex* if

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (j+1)(k+1)|\triangle_{22} a_{jk}| < \infty \tag{5}$$

and *convex* if

$$\triangle_{22} a_{jk} \ge 0 \quad (j,k \ge 0) \tag{6}$$

It can be easily noted that every *bounded convex* sequence is also *quasi-convex*.

Concerning the L^1-convergence of the double cosine series Moricz [5] proved the following result:

Theorem [5] *If a double sequence $\{a_{jk}\}$ satisfies (1) and (5) then the sum $f(x,y)$ of series (3) is integrable and (3) is a Fourier series of $f(x,y)$. If, in addition,*

$$\triangle_{20} a_{jk} \ge 0, \qquad \triangle_{02} a_{jk} \ge 0, \qquad (j,k \ge 0) \tag{7}$$

then

$$||S_{mn} - f|| \to 0, \ \ as \ \ \min(m,n) \to \infty \tag{8}$$

if and only if

$$a_{mn} ln(m+2) ln(n+2) \to 0 \qquad as \quad \max(m,n) \to \infty. \tag{9}$$

The aim of this paper is to present more results concerning these problems. In this concern, some results have been extended by extending the semi-convexity of coefficient sequences from one dimension to two dimension as:

Definition *A double sequence $\{a_{jk}\}$ is said to be a semi-convex null sequence, if (1) and*

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (j+1)(k+1) \left| \begin{array}{l} \triangle_{22} a_{j,k} + \triangle_{22} a_{j,k+1} \\ \quad + \triangle_{22} a_{j+1,k} + \triangle_{22} a_{j+1,k+1} \end{array} \right| < +\infty, \quad (a_{jk} = 0; \quad for \ \ either \ j = 0 \ or \ k = 0)$$

holds.

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II. Semi-Convex Null Sequences

The first main result of this paper which is slight generalization of that of Moricz [5] reads as follows:

Theorem 1 *If a double sequence $\{a_{jk}\}$ is a semi-convex null sequence, i.e., if (1) and (10) then (3) is a Fourier Series, or equivalently, it represents an integrable function.*

Proof Consider, for $x, y \not\equiv 0(\mod \pi)$,

$$\sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}\cos jx \cos ky$$
$$= \frac{1}{4\sin x \sin y}$$
$$\sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}(2\cos jx \sin x)(2\cos ky \sin y)$$
$$= \frac{1}{4\sin x\sin y}\sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}[\sin(j+1)x - \sin(j-1)x]$$
$$[\sin(k+1)y - \sin(k-1)y]$$
$$= \frac{1}{4\sin x\sin y}\sum_{j=1}^{m}[\, a_{j,1}(\sin 2y - \sin 0y) + a_{j,2}(\sin 3y - \sin 1y) + a_{j,3}(\sin 4y - \sin 2y) + ... + a_{j,n}(\sin(n+1)y - \sin(n-1)y)]$$
$$(\sin(j+1)x - \sin(j-1)x)$$
$$= \frac{1}{4\sin x\sin y}\sum_{j=1}^{m}[\, \sin y(a_{j,0} - a_{j,2}) + \sin 2y(a_{j,1} - a_{j,3}) + \sin 3y(a_{j,2} - a_{j,4}) + ... + \sin ny(a_{j,n-1} - a_{j,n+1})$$
$$a_{j,n}\sin(n+1)y + a_{j,n+1}\sin ny]$$
$$(\sin(j+1)x - \sin(j-1)x)$$
$$= \frac{1}{4\sin x\sin y}\sum_{j=1}^{m}\left[\sum_{k=1}^{n}(a_{j,k-1} - a_{j,k+1})\sin ky\right]$$
$$(\sin(j+1)x - \sin(j-1)x)$$
$$= \frac{1}{4\sin x\sin y}\sum_{k=1}^{n}[\, (a_{1,k-1} - a_{1,k+1})(\sin 2x - \sin 0x) + (a_{2,k-1} - a_{2,k+1})(\sin 3x - \sin x) + (a_{3,k-1} - a_{3,k+1})(\sin 4x - \sin 2x) + ... + (a_{m,k-1} - a_{m,k+1})(\sin(m+1)x - \sin(m-1)x)]\sin ky + I_2$$
$$= \frac{1}{4\sin x\sin y}\sum_{k=1}^{n}[\, \triangle_{01}(a_{1,k-1} + a_{1,k})(\sin 2x - \sin 0x) + \triangle_{01}(a_{2,k-1} + a_{2,k})(\sin 3x - \sin x) + \triangle_{01}(a_{3,k-1} + a_{3,k})(\sin 4x - \sin 2x) + ... + \triangle_{01}(a_{m,k-1} + a_{m,k})$$
$$(\sin(m+1)x - \sin(m-1)x)]\sin ky + I_2$$
$$= \frac{1}{4\sin x\sin y}\sum_{k=1}^{n}[\, \triangle_{01}(a_{0,k-1} + a_{0,k} - a_{2,k-1} - a_{2,k})\sin x + \triangle_{01}(a_{1,k-1} + a_{1,k} - a_{3,k-1} - a_{3,k})\sin 2x + \triangle_{01}(a_{2,k-1} + a_{2,k} - a_{4,k-1} - a_{4,k})\sin 3x + ... + \triangle_{01}(a_{m-1,k-1} + a_{m-1,k} - a_{m+1,k-1} - a_{m+1,k})\sin mx + \triangle_{01}(a_{m,k-1} + a_{m,k})\sin(m+1)x + \triangle_{01}(a_{m+1,k-1} + a_{m+1,k})$$
$$\sin mx]\sin ky + I_2$$
$$= \frac{1}{4\sin x\sin y}\sum_{j=1}^{m}\sum_{k=1}^{n}[\triangle_{11}(a_{j-1,k-1} + a_{j-1,k} + a_{j,k-1} + a_{j,k})$$
$$\sin jx \sin ky] + I_1 + I_2$$
$$= I_0 + I_1 + I_2$$

$$f(x,y) = \lim_{(m,n)\to\infty}[I_0 + I_1 + I_2]$$

By applying double summation by parts

$$\lim_{(m,n)\to\infty} I_0$$
$$= \frac{1}{4\sin x\sin y}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}[\triangle_{11}(a_{j-1,k-1} + a_{j-1,k} + a_{j,k-1} + a_{j,k})$$
$$\sin jx \sin ky]$$
$$= \frac{1}{4\sin x\sin y}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}[\triangle_{22}(a_{j-1,k-1} + a_{j-1,k} + a_{j,k-1} + a_{j,k})$$
$$\tilde{D}_j(x)\tilde{D}_k(y)]$$

where $\tilde{D}_n(x) = \sum_{k=1}^{n}\sin kx$, the conjugate Dirichlet kernel and since,

$$\int_{-\pi}^{\pi}\left|\frac{\tilde{D}_n(x)}{2\sin x}\right|dx = \begin{pmatrix} \int_{-\pi}^{\pi}\left|\frac{\sin n/2x}{2\sin x/2}\right|\left|\frac{\sin (n+1)/2x}{2\sin x}\right|dx & (n \quad even) \\ \int_{-\pi}^{\pi}\left|\frac{\sin (n+1)/2x}{2\sin x/2}\right|\left|\frac{\sin n/2x}{2\sin x}\right|dx & (n \quad odd) \end{pmatrix}$$

$$= \left(\int_{-\pi}^{\pi}\left|\left(\frac{\sin(n+1)x/2}{\sin x/2}\right)^2\right|dx\right)^{1/2}\left(\int_{-\pi}^{\pi}\left|\left(\frac{\sin n/2x}{\sin x/2}\right)^2\right|dx\right)^{1/2}$$
$$= O(n)$$

Therefore,

$$\lim_{(m,n)\to\infty} I_0 = O\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}[jk|\triangle_{22}(a_{j-1,k-1} + a_{j-1,k} + a_{j,k-1} + a_{j,k})|]\right)$$
$$= O(1)$$

Further,

$$\lim_{(m,n)\to\infty} I_1 = \frac{1}{4\sin x\sin y}\sum_{j=m+1}^{\infty}\sum_{k=1}^{\infty}[\triangle_{01}(a_{j-1,k-1} + a_{j-1,k} + a_{j,k-1}$$
$$+ a_{j,k})\sin jx \sin ky]$$

By making use of abel's transformation twice

$$\lim_{(m,n)\to\infty} I_1$$
$$= \frac{1}{2\sin y}\sum_{j=m+1}^{\infty}\sum_{k=1}^{\infty}[\triangle_{11}(a_{j-1,k-1} + a_{j-1,k} + a_{j,k-1} + a_{j,k})$$
$$\tilde{D}_j(x)\sin ky]$$
$$= \sum_{j=m+1}^{\infty}\sum_{k=1}^{\infty}[j\triangle_{22}(a_{j-1,k-1} + a_{j-1,k} + a_{j,k-1} + a_{j,k})\tilde{F}_j(x)\tilde{D}_j(y)]$$
$$= O\left(\sum_{j=m+1}^{\infty}\sum_{k=1}^{\infty}[jk|\triangle_{22}(a_{j-1,k-1} + a_{j-1,k} + a_{j,k-1} + a_{j,k})|]\right)$$
$$= O(1)$$

where $\tilde{F}_n(x)$ is the conjugate Fejer's kernel. Similarly,

$$\lim_{(m,n)\to\infty} I_2 = \left(\sum_{j=1}^{\infty}\sum_{k=n+1}^{\infty}[jk|\triangle_{22}(a_{j-1,k-1} + a_{j-1,k} + a_{j,k-1} + a_{j,k})|]\right) = O(1)$$

This concludes that $f \in L$ by Lebesgue's theorem. Thus, (3) should converge to f every where apart from $x, y \cong 0(\mod \pi)$. Hence, (3) should be the Fourier series by virtue of generalized du Bois-Reymond theorem [9], [1].

III. Double Cosine and Sine Series

S. A. Tejakovski [11] has proved the following result for one dimensional trigonometric sine series:

Theorem 3 [11] *If $\{b_n\}$ is a quasi-convex null sequence, $\sum_{k=1}^{\infty} b_k \sin kx$ is a Fourier series if and only if*

$$\sum_{k=1}^{\infty} \frac{|b_k|}{k} < +\infty, \tag{10}$$

and this cannot be replaced by the mere convergence of $\sum_{k=1}^{\infty} \frac{b_k}{k}$.

In connection with the above theorem Kano [12] proved the following result:

Theorem 4 [12] *If $\{b_n\}$ is a null sequence such that*

$$\sum_{k=1}^{\infty} k^2 \left| \triangle^2 \left(\frac{b_k}{k} \right) \right| < +\infty, \tag{11}$$

then $\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$ or $\sum_{k=1}^{\infty} b_k \sin kx$ is a Fourier series, or equivalently, it represents an integrable function.

In this paper, the Theorem 4 has also been extended from one dimensional trigonometric series to two dimensional trigonometric series and it reads as follows:

Theorem 5 *If $\{a_{m,n}\}$ is a double null sequence such that*

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} j^2 k^2 \left| \triangle_{22} \left(\frac{a_{j,k}}{jk} \right) \right| < +\infty, \tag{12}$$

then (3) or (4) is a Fourier series, or equivalently, it represents an integrable function.

Proof Here, the proof is given for (3) (cosine series) only since it goes in quite the same way for (4) (sine series).

Let

$$\begin{aligned} S_{mn}(x,y) &= \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} \cos jx \cos ky \\ &= \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{a_{jk}}{jk} [\sin jx]' [\sin ky]' \end{aligned}$$

where prime means derivative.

Performing double summation by parts twicer,

$$\begin{aligned} S_{mn}(x,y) = & \sum_{j=1}^{m-2} \sum_{k=1}^{n-2} (j+1)(k+1) \triangle_{22} \left(\frac{a_{jk}}{jk} \right) \tilde{F}'_j(x) \tilde{F}'_k(y) \\ & - \sum_{j=1}^{m-1} n \triangle_{21} \left(\frac{a_{j,n-1}}{j(n-1)} \right) \tilde{F}'_j(x) \tilde{F}'_{n-1}(y) \\ & - \sum_{k=1}^{n-1} m \triangle_{12} \left(\frac{a_{m-1,k}}{(m-1)k} \right) \tilde{F}'_{m-1}(x) \tilde{F}'_k(y) \\ & + mn \triangle_{11} \left(\frac{a_{m-1,n-1}}{(m-1)(n-1)} \right) \tilde{F}'_{m-1}(x) \tilde{F}'_{n-1}(y) \\ & - \sum_{j=1}^{m} n \triangle_{10} \left(\frac{a_{jn}}{jn} \right) \tilde{D}'_j(x) \tilde{D}'_n(y) \\ & - \sum_{k=1}^{n} m \triangle_{01} \left(\frac{a_{mk}}{mk} \right) \tilde{D}'_m(x) \tilde{D}'_k(y) \\ & + mn \left(\frac{a_{mn}}{mn} \right) \tilde{D}'_m(x) \tilde{D}'_n(y) \end{aligned}$$

where $\tilde{F}_n(x)$ the conjugate Fejer's kernel and by Zygmund's theorem

$$\int_{-\pi}^{\pi} \tilde{F}'_n(x) = O(n)$$

Moreover, for any fixed $x, y \ncong 0(\mod 2\pi)$

$$\begin{aligned} & n \sum_{j=1}^{m-1} \triangle_{21} \left(\frac{a_{j,n-1}}{j(n-1)} \right) \tilde{F}'_j(x) \tilde{F}'_{n-1}(y) \\ & \leq \sum_{j=1}^{m-1} \sum_{k=n-1}^{\infty} (j+1)(k+1) \triangle_{22} \left(\frac{a_{j,k}}{jk} \right) \tilde{F}'_j(x) \tilde{F}'_k(y) \\ & = O\left(\sum_{j=1}^{m-1} \sum_{k=n-1}^{\infty} (j+1)^2 (k+1)^2 \triangle_{22} \left(\frac{a_{j,k}}{jk} \right) \right) \\ & = O(1), \ m \to \infty. \end{aligned}$$

Similarly, for $n \to \infty$,

$$\sum_{k=1}^{n-1} m \triangle_{12} \left(\frac{a_{m-1,k}}{(m-1)k} \right) \tilde{F}'_{m-1}(x) \tilde{F}'_k(y) = O(1)$$

and for $m, n \to \infty$

$$\sum_{k=1}^{n-1} m \triangle_{12} \left(\frac{a_{m-1,k}}{(m-1)k} \right) \tilde{F}'_{m-1}(x) \tilde{F}'_k(y) = O(1)$$

and for $m, n \to \infty$

$$\begin{aligned} & mn \triangle_{11} \left(\frac{a_{m-1,n-1}}{(m-1)(n-1)} \right) \tilde{F}'_{m-1}(x) \tilde{F}'_{n-1}(y) \\ & = \sum_{j=m-1}^{\infty} \sum_{k=n-1}^{\infty} (j+1)(k+1) \triangle_{22} \left(\frac{a_{m-1,n-1}}{(m-1)(n-1)} \right) \tilde{F}'_{m-1}(x) \tilde{F}'_{n-1}(y) \\ & = O\left(\sum_{j=m-1}^{\infty} \sum_{k=n-1}^{\infty} (j+1)^2 (k+1)^2 \triangle_{22} \left(\frac{a_{m-1,n-1}}{(m-1)(n-1)} \right) \right) \\ & = O(1) \end{aligned}$$

also, it is known that $\tilde{D}'_n(x) = O(n/x^2)$, therefore

$$\begin{aligned} & \sum_{j=1}^{m} n \triangle_{10} \left(\frac{a_{jn}}{jn} \right) \tilde{D}'_j(x) \tilde{D}'_n(y) \\ & \leq \sum_{j=1}^{m} \sum_{k=n-1}^{\infty} (k+1) \triangle_{11} \left(\frac{a_{jk}}{jk} \right) \tilde{D}'_j(x) \tilde{D}'_k(y) = O(1). \end{aligned}$$

Similarly,

$$\sum_{k=1}^{n} m\triangle_{01}\left(\frac{a_{mk}}{mk}\right)\tilde{D}'_m(x)\tilde{D}'_k(y) \to 0 \quad m \to \infty$$

and

$$mn\left(\frac{a_{mn}}{mn}\right)\tilde{D}'_m(x)\tilde{D}'_n(y) \to 0 \quad \max(m,n) \to \infty.$$

For all x, y,

$$f(x,y) = \lim_{m,n\to\infty} S_{mn} = O\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(j+1)^2(k+1)^2\triangle_{22}\left(\frac{a_{jk}}{jk}\right)\right)$$

exists and $f(x,y) \in L$ by Lebesgue's theorem.

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