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Characterization of Monoids by a Generalization of Flatness Property

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Abstract—It is well-known that, using principal weak flatness property, some important monoids are characterized, such as regular monoids, left almost regular monoids, and so on. In this article, we define a generalization of principal weak flatness called GP-Flatness, and will characterize monoids by this property of their right (Rees factor) acts. Also we investigate new classes of monoids called generally regular monoids and generally left almost regular monoids.

Keywords—G-left stabilizing, GP-flatness, generally regular, principal weak flatness.

I. Introduction

THROUGHOUT this paper S will denote a monoid. We refer the reader to [5] for basic definitions and terminology relating to semigroups and acts over monoids.

Recall that a monoid S is called *right (left) reversible* if for every $s, t \in S$, there exist $u, v \in S$ such that $us = vt (su = tv)$. A right ideal K_S of a monoid S is called *left stabilizing* if for every $k \in K_S$, there exists $l \in K_S$ such that $lk = k$.

A nonempty set A is called a right S-act, usually denoted A_S, if S acts on A unitarily from the right, that is, there exists a mapping $A \times S \to A$, $(a, s) \mapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A$, and all $s, t \in S$.

An act A_S is called flat if the functor $A \otimes -$ preserves embeddings of left S-acts. If this functor preserves embeddings of (principal) left ideals of S into S, then A_S is called (principally) weakly flat. Hence a right S-act A_S is principally weakly flat if and only if for every $s \in S$, any $a, a' \in A_S, a \otimes s = a' \otimes s$ in $A \otimes S$ implies $a \otimes s = a' \otimes s$ in $A_S \otimes Ss$.

II. General properties

In this section we introduce a generalization of principal weak flatness, called GP-flatness and will give some general properties.

Definition 1. A right S-act A_S is called $GP - Flat$ if for every $S \in S$, any $a, a' \in A, a \otimes s = a' \otimes s$ in $A_S \otimes S$ implies that there exists $n \in N$ such that $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes Ss^n$.

It is clear that every principally weakly flat S-act is GP-Flat, but Example 1 will show that the converse is not true in general.

Lemma 1. The following statements are easy consequences of the definition:

(1) The right S-act S_S is GP-Flat.

(2) The one element right S-act Θ_S is GP-Flat.

(3) $A = \coprod_{i \in I} A_i$ is GP-Flat, if and only if every $A_i, i \in I$, is GP-Flat.

Theorem 1. Let S be an idempotent monoid. Then every GP-Flat right S-act is principally weakly flat.

Theorem 2. Let S be a cancellative monoid. Then every GP-Flat right S-act is principally weakly flat.

Theorem 3. Every GP-Flat S-act is torsion free.

III. Main Results

At first, we recall the following two lemmas from [5].

Lemma 2. Let ρ be a right and λ a left congruence on a monoid S. Then $[u]_\rho \otimes [s]_\lambda = [v]_\rho \otimes [t]_\lambda$ in $S/\rho \otimes S/\lambda$ for $u, v, s, t \in S$, if and only if $us(\rho \vee \lambda)vt$.

Lemma 3. Let S_A be a left S-act and $a \in S_A$. Then $g : S/ker\rho_a \to S_a$ with $g([t]) = ta$ for every $t \in S$ is an S-isomorphism.

Lemma 4. Let ρ be a right congruence on a monoid S and $s \in S$. Then $[u]_\rho \otimes s^n = [v]_\rho \otimes s^n$ in $S/\rho \otimes Ss^n$ for $u, v \in S$ and $n \in N$, if and only if $u(\rho \vee ker\rho_{s^n})v$.

Proof: Necessity. Let $[u]_\rho \otimes s^n = [v]_\rho \otimes s^n$ in $S/\rho \otimes Ss^n$ for $u, v, s \in S$ and $n \in N$. Using Lemma 3, we get $[u]_\rho \otimes [1]_{ker\rho_{s^n}} = [v]_\rho \otimes [1]_{ker\rho_{s^n}}$ in $S/\rho \otimes S/ker\rho_{s^n}$. Now it follows from Lemma 2, that $u(\rho \vee ker\rho_{s^n})v$.

Sufficiency. Let $u(\rho \vee ker\rho_{s^n})v$, for $u, v, s \in S$ and $n \in N$. Hence there exist $z_1, z_2, ...z_m \in S$ such that

$$u \ \rho \ z_1 \ (ker\rho_{s^n}) \ z_2 \ \rho \ ... \ \rho z_m \ (ker\rho_{s^n}) \ v.$$

Then we have in $S/\rho \otimes Ss^n$ the following equalities:

$$[u]_\rho \otimes s^n = [z_1]_\rho \otimes s^n = [1]_\rho \otimes z_1 s^n =$$

$$[1]_\rho \otimes z_2 s^n = [z_2]_\rho \otimes s^n = ... = [z_m]_\rho \otimes s^n$$

$$= [1]_\rho \otimes z_m s^n = [1]_\rho \otimes v s^n = [v]_\rho \otimes s^n.$$

Theorem 4. Let S be a monoid and ρ be a right congruence on S. Then the right S-act S/ρ is GP-Flat if, and only if, for all $u, v, s \in S$ with $(us)\rho(vs)$, there exists $n \in N$, such that $u(\rho \vee ker\rho_{s^n})v$.

Proof: Necessity.Suppose that the right S-act S/ρ is GP-Flat and let $(us)\rho(vs)$ for $u, v, s \in S$. That is $[u]_\rho s =$

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$[v]_\rho s$, and so by hypothesis there exists $n \in N$, such that $[u]_\rho \otimes s^n = [v]_\rho \otimes s^n$ in $S/\rho \otimes Ss^n$. Hence $u(\rho \vee ker\rho_{s^n})v$, by Lemma 4.

Sufficiency. Let $[u]_\rho s = [v]_\rho s$, for $u, v, s \in S$ and $n \in N$. This means that $(us)\rho(vs)$, and so by hypothesis there exists $n \in N$, such that $u(\rho \vee ker\rho_{s^n})v$. Now $[u]_\rho \otimes s^n = [v]_\rho \otimes s^n$, and so the right S-act S/ρ is GP-Flat.

Corollary 1. The right ideal zS, is GP-Flat if and only if, for all $x, y, s \in S, zxs = zys$ implies that there exists $n \in N$ such that $x(ker\lambda_z \vee ker\rho_{s^n})y$.

Definition 2. Let S be a monoid. The right ideal K_S of S is called *G-left stabilizing* if

$$(\forall s \in S)(\forall z \in S \setminus K_S)$$
$$(zs \in K_S \Rightarrow \exists n \in N, k \in K_S : zs^n = ks^n)$$

Theorem 5. Let S be a monoid and K_S be a proper right ideal of S. Then, S/K_S is GP-Flat if, and only if, K_S is a G-left stabilizing right ideal.

Proof: Necessity. Suppose that S/K_S is GP-Flat for the proper right ideal K_S of S, and let $s \in S$. If there exists $z \in S \setminus K_S$ such that $zs \in K_S$, then for every $j \in K_S$, we have $[z] \otimes s = [j] \otimes s$, and so by assumption there exist $n \in N$ such that $[z] \otimes s^n = [j] \otimes s^n$in $S/K_S \otimes Ss^n$. By [5] there exist $m \in N, u_1, ..., u_m \in S$, and $s_1, ..., s_m, t_1, ..., t_m \in S$ such that

$$[z] = [u_1]s_1$$
$$[u_1]t_1 = [u_2]s_2 \qquad s_1s^n = t_1s^n$$
$$\dots \qquad \dots$$
$$[u_m]t_m = [j] \qquad s_ms^n = t_ms^n.$$

Since $j \in K_S$, we have $u_mt_m \in K_S$. Let p be the least number such that $p \in \{1, 2, ..., m\}$ and $u_pt_p \in K_S$. Let $k = u_pt_p$, then $u_{p-1}t_{k-1} \in S \setminus K_S$. Since $[u_{p-1}]t_{p-1} = [u_p]s_p$, we have $u_{p-1}t_{p-1} = u_ps_p$, and

$$zs^n = u_1s_1s^n = u_1t_1s^n = u_2s_2s^n =$$
$$u_2t_2s^n = ... = u_{k_1}tk - 1s^n =$$
$$u_ks_ks^n = u_kt_ks^n = ks^n$$

sufficiency. Suppose that $[p] \otimes s = [q] \otimes s$ for $p, q, s \in S$. We have the following cases to consider:

Case 1. $p, q \in K_S$. Then it is clear that $[p] = [q]$, and so for every $n \in N$, $[p] \otimes s^n = [q] \otimes s^n$ in $S/K_S \otimes Ss^n$.

Case 2. $p \in K_S, q \in S \setminus K_S$. By assumption there exist $n \in N$ and $k \in K_S$ such that $qs^n = ks^n$. So

$$[p] \otimes s^n = [k] \otimes s^n = [1] \otimes ks^n$$
$$= [1] \otimes qs^n = [q] \otimes s^n.$$

Case 3. $q \in K_S, p \in S \setminus K_S$. It is similar to Case 2.

Case 4. $p, q \in S \setminus K_S$. Since $[p] \otimes s = [q] \otimes s$, by definition of Rees congruence we have $ps = qs$ or $ps, qs \in K_S$. If $ps = qs$ the statement is obvious. Let $ps, qs \in K_S$, by assumption there exist $n, m \in N$ and $k, l \in K_S$ such that $ps^n = ks^n$ and $qs^m = ls^m$. Set $\alpha = max\{n, m\}$. Thus

$$[p] \otimes s^\alpha = [1] \otimes ps^\alpha = [1] \otimes ks^\alpha$$
$$= [k] \otimes s^\alpha = [l] \otimes s^\alpha = [1] \otimes ls^\alpha$$
$$= [1] \otimes qs^\alpha = [q] \otimes s^\alpha$$

Example 1. Let $S = \{1, x, 0\}$ with $x^2 = 0$, and let $K_S = \{x, 0\}$. It is easy to check that K_S is G-left stabilizing and so the right Rees factor S-act S/K_S is GP-Flat, but it is not principally weakly flat.

A. Characterization Of Monoids By GP-Flatness Property Of Right Rees Factor Acts

In this subsection we give a characterization of monoids by GP-Flatness property of right Rees factor acts.

Theorem 6. Let S be a monoid. Then all right GP-Flat Rees factor S-acts are principally weakly flat if, and only if, every G-left stabilizing proper right ideal of S is left stabilizing.

Proof: Suppose that all right GP-Flat Rees factor S-acts are principally weakly flat and let K_S be a G-left stabilizing proper right ideal of S. Then by Theorem 5, S/K_S is GP-Flat, and so by assumption S/K_S is principally weakly flat. Hence by [5], K_S is left stabilizing.

Conversely, suppose that for the right ideal K_S of S, S/K_S is GP-Flat. Then there are two cases:

Case 1. $K_S = S$. Then $S/K_S \cong \Theta_S$ is principally weakly flat by [5].

Case 2. $K_S \neq S$. Then by Theorem 4, K_S is G-left stabilizing. Thus by assumption K_S is left stabilizing, and so S/K_S is principally weakly flat by [5].

The proof of the following theorem is similar to Theorem 6.

Theorem 7. Let S be a monoid. Then all right GP-Flat Rees factor S-acts are (weakly) flat if, and only if, the existence of a G-left stabilizing proper right ideal K_S of S implies that K_S is a left stabilizing ideal, and S is right reversible.

Theorem 8. Let S be a monoid. Then all right GP-Flat Rees factor S-acts satisfy Condition (P) if, and only if, S is right reversible and there exist no G-left stabilizing proper right ideal K_S of S with $|K_S| \geq 2$.

Proof: Suppose first that all right GP-Flat Rees factor S-acts satisfy Condition (P) and let K_S be a G-left stabilizing proper right ideal of S. Then by Theorem 5, S/K_S is GP-Flat, and so by assumption S/K_S satisfies Condition (P). Hence by [5], $|K_S| = 1$. Since by Lemma 1, the one element right S-act Θ_S is GP-Flat, it satisfies Condition (P) by assumption, and so by [5] S is right reversible.

Conversely, suppose for the right ideal K_S of S, S/K_S is GP-Flat. Then there are two cases:

Case 1. $K_S = S$. Since S is right reversible, $S/K_S \cong \Theta_S$ satisfies Condition (P) by [5].

Case 2. $K_S \neq S$. Then by Theorem 5, K_S is G-left stabilizing, and so by assumption $|K_S| = 1$. Thus by [5], S/K_S satisfies Condition (P) as required.

We recall from [6] that a right $S-$act A_S is *weakly pullback flat* if, and only if it satisfies Conditions (P) and (E'). Also a submonoid P of a monoid S is *weakly left collapsible* if for every $s,t \in P, z \in S, sz = tz$ implies the existence of $u \in P$ such that $us = ut$.

The proof of following theorems are similar in nature as to that of Theorem 8.

Theorem 9. Let S be a monoid. Then all right GP-Flat Rees factor S-acts are weakly pullback flat if, and only if, S is weakly left collapsible, and there exist no G-left stabilizing proper right ideal K_S of S with $|K_S| \geq 2$.

Theorem 10. Let S be a monoid. Then all GP-Flat right Rees factor are strongly flat if, and only if, S is left collapsible, and there exist no G-left stabilizing proper right ideal K_S of S with $|K_S| \geq 2$.

Theorem 11. Let S be a monoid. Then all right GP-Flat Rees factor S-acts are projective if and only if S contains a left zero, and there exist no G-left stabilizing proper right ideal K_S of S with $|K_S| \geq 2$.

Theorem 12. Let S be a monoid. Then all right GP-Flat Rees factor S-acts are free if and only if $S = \{1\}$.

Note that by [5], the above theorem is also valid for projective generators.

B. GP-Flatness Of Amalgamated Coproduct

Let J be a proper right ideal of a monoid S. If x, y and z denote elements not belonging to S, define

$$A(J) = (\{x, y\} \times (S \setminus J)) \cup (\{z\} \times J)$$

and define a right S-action on $A(J)$ by

$$(x,u)s = \{ \begin{array}{ll} (x, us) & us \notin J \\ (z, us) & us \in J \end{array}$$

$$(y,u)s = \{ \begin{array}{ll} (y, us) & us \notin J \\ (z, us) & us \in J \end{array}$$

$$(z,u)s = (z, us)$$

$A(J)$ is a right S-act, which usually denoted by $S_S \coprod^J S_S$, and we have;

Theorem 13. $A(J)$ is GP-Flat if and only if, the right ideal J is G-left stabilizing.

C. Classification Of Monoids By GP-Flatness

We recall that a monoid S is called regular, if for every $s \in S$, there exists $x \in S$ such that $s = sxs$. It is called eventually regular, if for ever $s \in S$, there exists $n \in N$ such that s^n is regular.

Definition 3. A monoid S is called a generally regular monoid, if for every $s \in S$, there exist $n \in N$ and $x \in S$ such that $s^n = sxs^n$.

It is clear that the class of generally regular monoids contains all regular monoids and all eventually regular monoids.

Theorem 14. Let $s \in S$. If the right S-act $S/\rho(s, s^2)$ is GP-Flat, then s is generally regular.

Theorem 15. If all right Rees factor S-acts of the form S/sS, are GP-Flat, then either s is a generally regular element or satisfies condition (tcu).

(tcu) : there exist $u, v, c \in S$, which c is a right cacellable element, such that $t \in sS, tc = su$

Theorem 16. For any monoid S the following statements are equivalent:

(1) All right S-acts are GP-Flat.

(2) All finitely generated right S-acts are GP-Flat.

(3) All cyclic right S-acts are GP-Flat.

(4) All monocyclic right S-acts are GP-Flat.

(5) All monocyclic right S-acts of the form $S/\rho(s, s^2)$, $s \in S$ are GP-Flat.

(6) All right Rees factor S-acts are GP-Flat.

(7) All right Rees factor S-acts of the form S/sS, $s \in S$ are GP-Flat.

(8) S is a generally regular monoid.

Proof: Implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5), (2) \Rightarrow (6) \Rightarrow (7)$ are obvious.

$(5) \Rightarrow (8)$. Let $s \in S$. Then by assumption the monocyclic right S-act $S/\rho(s, s^2)$ is GP-Flat, and so s is generally regular by Theorem 14.

$(7) \Rightarrow (8)$. Let $s \in S$. Then by assumption the right Rees factor S-act S/sS is GP-Flat, and so it is torsion free by Theorem 3. Thus every right cancellable element of S is right invertible, by [3]. We have also that every torsion free right Rees factor S-act of the form S/sS is GP-Flat, and so by Theorem 15, either s is generally regular or satisfies Condition (tcu). But that is there exist a right cancellable element $c \in S$ and elements $t, u \in S$ such that $tc = su$ and $t \notin sS$. Since c is right cancellable it has a right inverse and the equality $tc = su$ implies that $t = suc^{-1} \in sS$, a contradiction. Thus s is generally regular as required.

$(8) \Rightarrow (1)$. Let A_S be a right S-act and $as = a's$ for $a, a' \in A_S, s \in S$. Since S is generally regular there exist $x \in$

$S, n \in N$ such that $s^n = sxs^n$. Now we have the following equalities in $A_S \otimes Ss^n$

$$a \otimes s^n = a \otimes sxs^n = asx \otimes s^n = a'sx \otimes s^n = a' \otimes sxs^n = a' \otimes s^n.$$

Thus A_S is GP-Flat.

Corollary 2. Let S be a commutative monoid. The following statements are equivalent:

(1) All right S-acts are GP-flat.

(2) All finitely generated right S-acts are GP-Flat.

(3) S is a eventually regular monoid.

Theorem 17. Let S be a monoid. If all right S-act satisfying condition (E) are GP-Flat, then S is a generally regular monoid.

Proof: Let $s \in S$. If $sS = S$ then s is obviously regular. Suppose that $sS \notin S$. Then by [2] the right S-act $A(sS)$ satisfies Condition (E), and so by assumption it is GP-Flat. Now by Theorem 13, the right ideal sS is G-left stabilizing. That is, there exist $k \in sS,\ n \in N$ such that $s^n = ks^n$, but $k \in sS$ implies that there exists $x \in S$ such that $k = sx$, and so $s^n = sxs^n$. So s is generally regular as required.

From Theorem 16 and Theorem 18 we have:

Corollary 3. If all right S-acts which satisfies Condition (E), are GP-Flat, then all right S-acts are GP-Flat.

Definition 4. An element s of S is called generally left almost regular if there exist elements $r, r_1, ..., r_m, s_1, ..., s_m \in S$, right cancellable elements $c_1, ...c_m \in S$, and a natural number $n \in N$ such that

$$s_1c_1 = sr_1$$
$$s_2c_2 = s_1r_2$$
$$...$$
$$s_mc_m = s_{m-1}r_m$$
$$s^n = s_mrs^n.$$

A monoid S is called generally left almost regular if all its elements are generally left almost regular.

It is obvious that every left almost regular monoid is generally left almost regular.

Example 2. Let S be the monoid of all strictly upper triangular matrices in $M_{n\times n}(R)$ with the unit matrix adjoined. It is clear that S is generally left almost regular, since for every $s \in S$, we have

$$s1 = s1$$
$$s^n = s1s^n.$$

But it is not left almost regular.

Theorem 18. For any monoid S the following statements are equivalent:

(1) All torsion free right S-acts are GP-Flat.

(2) All cyclic torsion free right S-acts are GP-Flat.

(3) All torsion free right Rees factor S-acts are GP-Flat.

(4) S is a generally left almost regular monoid.

From [1] and Theorem 3, we have the following result:

Theorem 19. Let S be a commutative, cancellative monoid. Then every GP-Flat right S-act satisfies Condition (P) if, and only if, the principal ideals of S form a chain (under inclusion).

By a similar argument of [4], we have the following theorem:

Theorem 20. For any monoid S the following statements are equivalent:

(1) S is right cancellative.

(2) S is *left* PSF and all flat right S-acts satisfy Condition (PWP).

(3) S is *left* PSF and all weakly flat right S-acts satisfy Condition (PWP).

(4) S is *left* PSF and all principally weakly flat right S-acts satisfy Condition (PWP).

(5) S is *left* PSF and all GP-Flat right S-acts satisfy Condition (PWP).

IV. Conclusion

In this paper we introduced GP-Flatness as a generalization of principal weak flatness. We already knew that using principal weak flatness, some important monoids are characterized, such as regular monoids and left almost regular monoids. Here we generalized regular and left almost regular monoids, and defined new classes of monoids. By GP-Flatness property we give characterizations of those monoids, and many known results are generalized.

References

[1] S. Bulman-Fleming, *Flatness properties of acts over commutative, cancellative monoids*, Mathematica, Vol. 46, 1999.

[2] S. Bulman-Fleming, *Flat and strongly flat S-systems*, Commun.Algebra, vol. 20, 1992

[3] A. Golchin and J. Renshaw, *A flatness property of acts over monoids*, Conference on Semigroups, University of St Andrews, 1997.

[4] Q. Husheng, *Some new characterization of right cancellative monoids by Condition* (PWP), Semigroup Forum, Vol. 71, 2005.

[5] M. Kilp, U. Knauer and A. Mikhalev, *Monoids, Acts and Categories*, Walter de Gruyter, Berlin, New York, 2000.

[6] V. Laan, *Pullbacks and flatness properties of acts*, Ph.D Thesis, Tartu, 1999.

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