

# An Optimal Algorithm for Finding $(r, Q)$ Policy in a Price-Dependent Order Quantity Inventory System with Soft Budget Constraint

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**Abstract**—This paper is concerned with the single-item continuous review inventory system in which demand is stochastic and discrete. The budget consumed for purchasing the ordered items is not restricted but it incurs extra cost when exceeding specific value. The unit purchasing price depends on the quantity ordered under the all-units discounts cost structure. In many actual systems, the budget as a resource which is occupied by the purchased items is limited and the system is able to confront the resource shortage by charging more costs. Thus, considering the resource shortage costs as a part of system costs, especially when the amount of resource occupied by the purchased item is influenced by quantity discounts, is well motivated by practical concerns. In this paper, an optimization problem is formulated for finding the optimal  $(r, Q)$  policy, when the system is influenced by the budget limitation and a discount pricing simultaneously. Properties of the cost function are investigated and then an algorithm based on a one-dimensional search procedure is proposed for finding an optimal  $(r, Q)$  policy which minimizes the expected system costs.

**Keywords**— $(r, Q)$  policy, Stochastic demand, backorders, limited resource, quantity discounts.

## I. INTRODUCTION

THE  $(r, Q)$  policy also known as reorder-point/order-quantity policy is one of the most common practical policy in inventory control systems. In an  $(r, Q)$  inventory system, the inventory position of the item is reviewed continuously and a fixed quantity  $Q$  is ordered when the inventory position drops to the reorder point  $r$  or lower. The purpose of an  $(r, Q)$  inventory control system is to determine the values of  $r$  and  $Q$  which minimize system costs. Such  $(r, Q)$  inventory models under various practical assumptions and constraints have been extensively studied in the literature. The basic  $(r, Q)$  policy was first introduced by Galliher et al. [1]. In spite of a large number of heuristics, the first efficient algorithm for finding optimal  $(r, Q)$  policy in a discrete inventory system with backorders for shortages, was proposed by Federgruen and Zheng [2] after almost 30 years. This algorithm is based on the unimodality of the cost function. For lost sales inventory models, studies can be referred to [3] and the references cited therein. The last study for properties of optimal policy in  $(r, Q)$  inventory systems were derived by

Federgruen and Wang [4]. They have provided general conditions for monotocity of the optimal policy parameters as a function of various model primitives and extend the results both to standard inventory models and to the models with general shelf age and delay dependent inventory costs.  $(r, Q)$  policies continue to be studied under various important extension of the base model. Olsson [5] considered an inventory system with continuous review  $(r, Q)$  policy under the assumption of Poisson demand and perishable items with fixed lead times and lifetimes. The resource constraint is one of the important and practical extensions of  $(r, Q)$  inventory models. In practice, the resource available in inventory system is usually limited. So, making a decision about a replenishment policy under the resource constraint is one of the most important issues in real inventory systems. As described by [6], "Practical applications are budget constraints, where the total amount of capital tied up in inventories at any time is limited by corporate strategy in industry or by law/regulation in government/military applications". Thus, in regard to practical concerns, considering a constraint on the budget used by all items is one of the most significant issues in inventory management. Nevertheless, as mentioned by [7], yet studies on inventory systems with limited and sharable-common resource (like budget) are limited. Moreover the literature on stochastic continuous review  $(r, Q)$  models can be separated into two categories: discrete and continuous demand model. There are major differences between the properties of discrete model cost function and the continuous demand version. Ang et al. [8] have made a comprehensive study on the behavior of these two types of inventory models. Based on the discrete convexity concept, the non-convexity of discrete  $(r, Q)$  system is proved and also they show that the significant properties which are applied to analyze the continuous models no longer hold in discrete counterparts. Hence, the analysis of the various extensions of inventory models with discrete demand is more complicated than the continuous demand models. This can partly explain why most of existing studies on  $(r, Q)$  models in the literature are focused on continuous demand. On the other hand, it was shown that the cost function of continuous demand model is an adequate approximation of the cost function in discrete demand model only when the order quantity is large enough [9]. So, considering continuous cost function instead of discrete cost function is not efficient for all cases in systems with discrete demand. Now we refer to literature on continuous review system with limited resource

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and continuous demand. Minner and Silver [6] considered a continuous review inventory system with zero lead times and common budget or space limitation, in which backorders are not permitted. Thus, when the inventory level of a product drops to zero, the product is replenished. With such assumptions, the on-hand inventory equals the inventory position. They derived a semi-Markov decision formulation for the problem under Poisson demand, which can be solved optimally for small instances of the problem and they proposed heuristics to solve the problem by referring to the economic ordering quantity (EOQ). Ghalebsaz-Jeddi et al. [10] considered a stochastic inventory system under continuous review with backordered shortages and marginal shortage costs, in which the purchasing costs are paid upon order arrival and budget/storage space is limited. Due to considering continuous demand instead of discrete quantities, the cost function in their proposed model is approximate. They use a Lagrange multiplier technique to solve the problem. Hariga [11] investigate a single-item continuous review inventory system with the stochastic continuous demand and a space restriction, in which the over-ordered quantity is returned to the supplier at a certain cost. They consider the approximate expression of the expected holding cost, which is affected by the storage space limitation. Betts and Johnston [12] proposed a multi-item  $(r, Q)$  inventory model under the capital constraint with an approximate objective function. They develop a new approach to determine replenishment policy by obtaining an optimal trade-off between maximizing profit and reducing risk of failure. An approximate solution approach is presented to solve the model. Kundu and Chakrabarti [13] considered a stochastic continuous review inventory model with budget constraint and mixture of backorders and lost sales, in which the distribution of demand and lead time are unknown and purchasing costs are paid upon order arrival. The cost function is approximate and a Lagrange multiplier technique is applied to solve the problem.

For the discrete demand systems, only a few studies have dealt with these systems under continuous review and resource constraint. Zhao et al. [14] studied an  $(r, Q)$  model with discrete stochastic demand and a constraint on storage-space for on-hand inventory. The solution approaches for finding an optimal policy in single-item system and undominated solution in multi-item case have been developed. Zhao et al. [7] studied an  $(r, Q)$  model with discrete stochastic demand and limited sharable-common resource for both single-item and multi-item cases. They proved that an existing algorithm can be applied to find the optimal solution of single-item case. Also they developed an algorithm for finding an optimal/near optimal replenishment policy for the multi-item case. We consider the same assumption as [7] did for the resource constraint in our model. Because of the resource limitation, most of the actual inventory systems may be encounter resource shortage. Additionally, one of the common strategies to confronting the resource shortage in practice is that the inventory systems usually rent extra resource temporarily to provide the extra resource requirement more than the available resource. In this case the resource constraint is applied as a soft constraint and

the corresponding shortage cost should be considered in the cost function. Studies on systems with soft resource constraint are limited while their use in real inventory systems is widespread. Applying the resource limitation as a soft constraint and modeling the resource shortage cost as a part of the cost function can make the model more practical than models with hard resource constraint which must be fulfilled. Moreover, the first case can cover the second one by changing the cost parameters. On the other hand, in many actual inventory systems (especially the ones with competitive supply market) usually several quantity discount opportunities are offered by suppliers. Katehakis and Smit [15] consider a single-item inventory system with continuous review  $(r, Q)$  policy and stochastic discrete demand under both all-units and incremental quantity discounts. They proposed sufficient algorithms for computing the optimal policy. Feng and Sun [16] considered a single-item continuous review inventory system with stochastic demand and discount opportunities. They assume that discount opportunities occur according to a Poisson process. They proposed an algorithm based on a bisection search procedure to find the optimal replenishment policy. For other literature on inventory management with quantity discount, we can refer to [17] and [18]

Though limited resource and quantity discounts are two common characteristics in inventory management, very few works consider both in a continuous review model. Most of the existing literature on inventory systems which faces both a limited resource and quantity discount, have been devoted to inventory models with constant demand (see, for example [19]-[22])

Since the available budget is spent on purchasing items, the amount of budget occupied by a unit of goods is equal to the unit purchasing price which is a function of the quantity ordered when a quantity discount is available. In the other words, the resource shortage cost depends on quantity discount parameters and they could effect on the system policy conversely. Therefore, the consideration of the limited budget and a quantity discount opportunity simultaneously in the model, in spite of increasing its complexity to find an optimal policy, makes the model more practical. Thus, the major issue in such inventory systems is the determination of  $r$  and  $Q$  values which lead to a trade-off between all types of costs consist of holding, resource shortage, ordering, purchasing and inventory shortage costs.

In this paper, we consider a single-item  $(r, Q)$  inventory systems with stochastic discrete demand, backorder, constant lead time, all-units discount and limited budget in which the resource shortage can be offset via charging extra costs. Based on the properties of the cost function an algorithm for finding the optimal  $(r, Q)$  policy is proposed.

The remainder of the paper is organized as follows. In Section III, we review the single-item  $(r, Q)$  inventory systems with limited resource and provide some existing results which are the foundation for analyzing the presented models in this paper. Section IIIII, analyzes the single-item  $(r, Q)$  system with limited budget under all-units discount. Then based on the cost function properties, an algorithm with one-dimensional

search procedure is presented for finding the optimal solution in Section IV. Section V contains concluding remarks and future directions.

## II. SINGLE-ITEM SYSTEM WITH LIMITED RESOURCE

In this section a single-item system with limited resource which was introduced by [7] is reviewed first. Then, we provide some existing properties on the system cost function and then based on these properties the problem with both budget constraint and all-units discount will be analyzed in the subsequent sections.

### A. System Description

The single-item inventory system, which consists of a supplier and customers, is controlled by applying an  $(r, Q)$  policy, where  $r$  is a finite integer and  $Q$  is a finite positive integer. Customer demands arrivals occur based on a renewal process with a mean  $\lambda$ , the expected value of arrivals per time unit. Demands arise discretely on a unit-by-unit basis. The demands that cannot be satisfied immediately are backordered. Inventory level, denoted by  $I_t$ , is equal to the amount of on-hand inventory minus the number of backorders. More than one outstanding order can exist at a time. The inventory position, denoted by  $I_p$ , is defined as the inventory level plus all outstanding orders. The inventory position of the item is reviewed continuously and a fixed quantity  $Q$  is ordered to the manufacturer when the inventory position drops to the reorder point  $r$  or lower. After a constant lead time  $L$  since an order is placed, the replenishment goods are received from the supplier. Because of the Poisson demand arrival, the inventory position process  $(I_p)$  can be described by a continuous time Markov Chain with state space  $\{r+1, r+2, \dots, r+Q\}$ . Since all transition rates of this Markov Chain are equal, for the inventory position at time  $t$ , denoted by  $I_p(t)$ , we have  $\lim_{t \rightarrow \infty} \text{prob}\{I_p(t) = y\} = 1/Q$ , for all  $y = r+1, r+2, \dots, r+Q$ . Consequently, in the steady state, the inventory position  $I_p$ , is uniformly distributed on  $\{r+1, r+2, \dots, r+Q\}$ . (See e.g., [2], [23])

Since the resource is occupied when an order is placed, the amount of resource occupied depends on the inventory position not the on-hand inventory. It is assumed that a customer pays for the order when the customer can be satisfied immediately by a unit of goods in on-hand inventory or when a unit of goods in outstanding orders is assigned to the customer. When the customer pays for the goods the corresponding resource occupied by the goods is released. Note that a unit of outstanding orders can only be assigned to one customer. Based on the customer payment strategy, the resource is occupied only by the unassigned goods in on-hand inventory and outstanding orders that is given by  $I_p^+ = \max\{0, I_p\}$ . It means that when the inventory position  $I_p$  is non-positive, no unassigned good exists in inventory system and in consequence the resource is entirely available and there is no resource occupied. Assume that the amount of resource occupied by each unit of goods is denoted by  $c'$ . For a given  $(r, Q)$  policy the maximum amount of resource that can be

occupied is equal to  $c'(r+Q)^+$ . When the available resource is not enough to provide the resource requirement, it is possible to offset the resource shortage via renting extra resource. Due to this act a resource shortage cost that is proportional to the amount of the rented resource is incurred. Let,  $B'$  and  $a'$ , denote respectively the amount of available resource and the resource shortage cost of one unit of extra resource. A resource shortage occurs when  $c'I_p^+ > B'$ . Thus, the amount of resource shortage is given by  $(c'I_p^+ - B')^+$ . Consequently, the expected resource shortage cost per time unit is given by  $a'.E(c'I_p^+ - B')^+$ . Without loss of generality, we rewrite  $a'.E(c'I_p^+ - B')^+$  as  $E(cI_p^+ - B)^+$  by rewriting  $a.c'$  as  $c$  and  $a'.B'$  as  $B$  as Zhao et al [7] did. Since the inventory position  $I_p$ , is uniformly distributed on  $\{r+1, r+2, \dots, r+Q\}$ , we have:

$$E(cI_p^+ - B)^+ = \frac{1}{Q} \sum_{y=r+1}^{r+Q} (cy^+ - B)^+ \quad (1)$$

Four types of costs are considered in the model: ordering cost, holding cost, inventory shortage cost and resource shortage cost. Let,

- $D_L$  = demand during a lead time with a mean  $\lambda L$
- $K$  = ordering (set-up) cost per order ( $K > 0$ )
- $h$  = holding cost per unit of goods per time unit
- $p$  = penalty cost per demand backordered per time unit
- $g(y)$  = sum of expected holding and shortage costs per time unit

Consequently, for the given  $(r, Q)$  policy the long-run average system cost per time unit can be expressed as

$$C(r, Q) = \frac{K\lambda}{Q} + \frac{1}{Q} \sum_{y=r+1}^{r+Q} [g(y) + (cy^+ - B)^+] \quad (2)$$

where,

$$\begin{aligned} g(y) &= h \sum_{i=0}^y (y-i) \cdot \text{Pr}_i + p \sum_{i=y+1}^{\infty} (i-y) \cdot \text{Pr}_i \\ &= (h+p) \sum_{i=0}^{y-1} \text{prob}\{D_L \leq i\} + p(\lambda L - y) \end{aligned} \quad (3)$$

and,

$$\text{Pr}_i = \text{prob}\{D_L = i\} = \frac{e^{-\lambda L} \cdot (\lambda L)^i}{i!} \quad (4)$$

Let,  $f$  denote the set of all feasible  $(r, Q)$  policies defined as  $f = \{(r, Q) | -\infty < r < \infty, 1 \leq Q < \infty\}$ . The problem is as follows:

**Problem 1.** Find  $(r, Q)$  in  $f$  to minimize the long-run average cost  $C(r, Q)$ .

### B. Properties of the Cost Function $C(r, Q)$

This subsection explains some existing and structural properties of the cost function  $C(r, Q)$  which are used to

analyze the  $(r, Q)$  inventory model with both resource limitation and all-unit discounts in the subsequent sections.

Let,  $G(y) = g(y) + (cy^+ - B)^+$ . Zhao et al. [7] have shown that all of the desired properties of  $g(y)$  hold for the developed function  $G(y)$ [7]. Hence the following results in the literature can be extended to the cost function  $C(r, Q)$ .

**Property I.**  $G(y)$  is convex with respect to  $y$  and  $-G(y)$  is a unimodal function with  $\lim_{|y| \rightarrow \infty} G(y) = \infty$  [7]. (See Fig. 1)

**Definition I.**

- a)  $y^*$  is refer to the maximal minimizing point of  $G(y)$  i.e.,  $y^* = \max\{y : G(y) < G(y-1)\}$ .
- b) Let  $G(y_1), G(y_2), \dots, G(y_Q)$  be the  $Q$  smallest values of the  $G(y)$ , then based on the unimodality of  $G(y)$ ,  $\tau_G^Q = \{y_1, y_2, \dots, y_Q\}$  contains  $Q$  successive integer with  $y_1 \leq y^* \leq y_Q$ .
- c) Let  $G^Q$  denote the  $Q$ -th smallest value of  $G(y)$  function over all integers for  $y$ .
- d)  $r(Q)$  is refer to the optimal reorder point for given order quantity  $Q$  with corresponding cost function  $C(Q) = \min_r C(r, Q)$ .

**Property II.** For any given integer  $Q \geq 1$ :

- a)  $C(r, Q)$  is convex with respect to  $r$ , therefore there is a minimizing point  $r(Q)$ , such that  $r(Q) = \min\{r : C(r, Q) = C(Q)\}$  [14].
- b)  $C(r, Q)$  is minimized with respect to  $r$  when  $r(Q) = (\min \tau_G^Q) - 1$  [2].
- c)  $r(Q)$  is decreasing in  $Q$  with  $\lim_{Q \rightarrow \infty} r(Q) = -\infty$  [4].
- d)  $-C(Q)$  is a unimodal function with respect to  $Q$  where  $C(Q) = C(r(Q), Q)$  [2].
- e) The optimal order quantity denoted by  $Q^*$  is the smallest integer value of  $Q$  for which  $C(Q) \leq G(y_{Q+1})$  [2].
- f)  $Q < Q^*$  if and only if  $G(y_{Q+1}) < C(Q)$  [4].

As a result we have  $\min_{(r, Q) \in f} \{C(r, Q)\} = C(r(Q^*), Q^*)$  which refers to the optimal solution of Problem I. Based on the above properties an efficient algorithm has been proposed to find the optimal solution of the Problem I by Federgruen and Zheng [2] which can be summarized as follows:

**Algorithm I.**

- Step1. Find  $y^*$  which minimizes  $G(y)$ .
- Step2. Set  $Q=1$  and  $r = y^* - 1$ .
- Step3. If  $\min\{G(r), G(r+Q+1)\} \geq C(r, Q)$ , then stop. Otherwise go to step 4.
- Step4. If  $G(r) \leq G(r+Q+1)$ , then  $r := r - 1$ .
- Step5.  $Q := Q + 1$ , go to step 3.

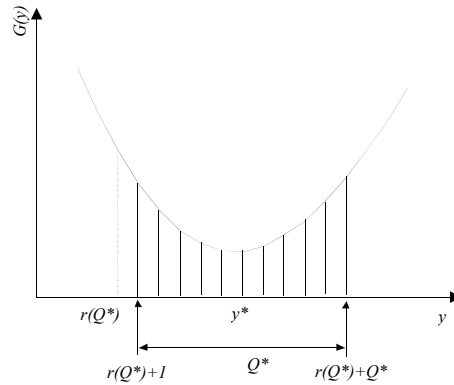


Fig. 1 The graphical view of  $G(y)$  and related notations

**III. SINGLE-ITEM SYSTEM WITH LIMITED BUDGET UNDER ALL-UNITS DISCOUNT**

In this section, we introduce the single-item system faces both a budget constraint and all-units discounts offered by the supplier. Consider a single-item inventory system which operates in the same way as the limited resource system explained in Section III but with a major difference, which is that the unit purchasing cost (unit price) depends on the order quantity  $Q$ . When the quantity discount in purchasing is offered by the supplier, the unit purchasing price depends on the order size  $Q$ . Since the available budget as a resource is spent on purchasing items, the amount of budget occupied by each unit of goods is equal to the unit purchasing price which is depends on the order quantity  $Q$ . Consequently, the resource shortage costs is affected by discount pricing structure While this assumption makes the model more practical, it increases the complexity of analysis. In such case, resource shortage cost and purchasing cost depends on the quantity discount parameters and could exert influence conversely on the optimal value of  $r$  and  $Q$ . Let,

- $j(Q)$  = the unique discount level  $j$  for which  $Q \in [q_j, q_{j+1})$ ,  $j = 1, \dots, n$
- $n$  = number of discount levels with  $n \geq 1$ , ( $q_1 = 0$  and  $q_{n+1} = \infty$ )
- $c_j$  = unit purchasing price when  $Q$  is in the discount level  $j$ , for  $j = 1, \dots, n$

Similar to (2), for the given  $(r, Q)$  policy the long-run average system cost per time unit with limited budget under all-units discount can be expressed as

$$C_A(r, Q) = \frac{\lambda K}{Q} + \frac{\sum_{y=r+1}^{r+Q} [g(y) + (c_{j(Q)} y^+ - B)^+]}{Q} + \lambda c_{j(Q)} \tag{5}$$

where,  $g(y)$  is defined by (3). In this case we have an optimization problem as follows:

**Problem II.** Find  $(r, Q)$  in  $f$  to minimize  $C_A(r, Q)$ .

Let  $(r^*, Q^*)$  denote the optimal solution of Problem II. In remainder of this section the properties of the expected system

cost function,  $C_A(r, Q)$  are analyzed and based on them, a solution approach is developed for finding  $(r^*, Q^*)$  in the subsequent section.

We can rewrite  $C_A(r, Q)$  as follows:

$$C_A(r, Q) = C(r, Q | j(Q)) + \lambda c_{j(Q)} \quad (6)$$

where,

$$C(r, Q | j) = \frac{K\lambda}{Q} + \frac{1}{Q} \sum_{y=r+1}^{r+Q} G_j(y), j=1,2,\dots,n. \quad (7)$$

and,

$$G_j(y) = g(y) + (c_j y^+ - B)^+, j=1,2,\dots,n \quad (8)$$

The idea behind these functions, is that for a given  $j$ , the cost function  $C(r, Q | j)$  is treated as  $C(r, Q)$  defined in (2). Therefore, this representation of  $C_A(r, Q)$  breaks Problem II down into at most  $n$  several instances of Problem I in which the parameter  $c$  in cost function  $C(r, Q)$  is converted to  $c_j$  for  $j=1,2,\dots,n$ . Now, the important issue is how to obtain the optimal policy  $(r^*, Q^*)$  by solving these  $n$  sub problems. Let  $(r_j(Q_j^*), Q_j^*)$  denote the optimal policy under cost function  $C(r, Q | j)$  and  $r_j(Q)$  be the optimal reorder point for a given order quantity  $Q$  under cost function  $C(r, Q | j)$  for  $j=1,2,\dots,n$ . Also, let  $\tilde{j}$  be the highest discount level for which  $Q_j^* \geq q_{\tilde{j}}$  i.e.  $\tilde{j} = \max\{j: Q_j^* \geq q_j\}$ . The following lemma obtains some lower bounds for  $C_A(r, Q)$  in different discount levels. Based on these lower bounds a solution approach is constructed to find the optimal solution of Problem II.

**Lemma I.**

- a) If  $Q_j^* < q_{\tilde{j}+1}$  then  $C_A(r, Q) \geq C_A(r_{\tilde{j}}(Q_j^*), Q_j^*)$  for  $Q \leq Q_j^*$
- b) If  $Q_j^* \geq q_{\tilde{j}+1}$  the  $C_A(r, Q) \geq C_A(r_{\tilde{j}+1}(q_{\tilde{j}+1}), q_{\tilde{j}+1})$  for  $Q \leq q_{\tilde{j}+1}$
- c)  $C_A(r, Q) \geq C_A(r_j(q_j), q_j)$  for  $j = \tilde{j}+1, \tilde{j}+2, \dots, n$  and  $q_j \leq Q < q_{j+1}$

**Proof.** Let  $Q_j^* < q_{\tilde{j}+1}$ . By definitions, for a given  $(r, Q)$  we have:

$$C_A(r, Q) = C(r, Q | j(Q)) + \lambda c_{j(Q)} \geq C(r_{j(Q)}(Q), Q | j(Q)) + \lambda c_{j(Q)} \quad (9)$$

Since  $C(r, Q | j)$  is non-increasing on  $j, j=1,2,\dots,n$ , and  $j(Q) \leq \tilde{j}$  therefore,

$$\begin{aligned} C(r_{j(Q)}(Q), Q | j(Q)) + \lambda c_{j(Q)} &\geq C(r_{j(Q)}(Q), Q | \tilde{j}) + \lambda c_{\tilde{j}} \\ &\geq C(r_{\tilde{j}}(Q), Q | \tilde{j}) + \lambda c_{\tilde{j}} \end{aligned} \quad (10)$$

The last term is larger than or equal to  $C(r_{\tilde{j}}(Q_j^*), Q_j^* | \tilde{j}) + \lambda c_{\tilde{j}}$ .

Hence, we have:  $C_A(r, Q) \geq C_A(r_{\tilde{j}}(Q_j^*), Q_j^*)$  and the proof of (a) is then completed. Now, let  $Q_j^* \geq q_{\tilde{j}+1}$ , therefore by

convexity of  $C(r_{\tilde{j}}(Q), Q | \tilde{j})$ , it is non-increasing over  $0 \leq Q \leq q_{\tilde{j}+1}$  and we have

$$C(r_{\tilde{j}}(Q), Q | \tilde{j}) \geq C(r_{\tilde{j}}(q_{\tilde{j}+1}), q_{\tilde{j}+1} | \tilde{j}) \quad (11)$$

By the definition of  $r_{\tilde{j}+1}(q_{\tilde{j}+1})$  and by the fact that  $C(r, Q | j)$  is non-increasing on  $j, j=1,2,\dots,n$ , we can write

$$\begin{aligned} C(r_{\tilde{j}}(q_{\tilde{j}+1}), q_{\tilde{j}+1} | \tilde{j}) + \lambda c_{\tilde{j}} &\geq C(r_{\tilde{j}+1}(q_{\tilde{j}+1}), q_{\tilde{j}+1} | \tilde{j}+1) \\ &\quad + \lambda c_{\tilde{j}+1} \end{aligned} \quad (12)$$

By (9), (10), (11) and (12) we can conclude that:

$$C_A(r, Q) \geq C(r_{\tilde{j}+1}(q_{\tilde{j}+1}), q_{\tilde{j}+1} | \tilde{j}+1) + \lambda c_{\tilde{j}+1} = C_A(r_{\tilde{j}+1}(q_{\tilde{j}+1}), q_{\tilde{j}+1})$$

then the proof of (b) is completed. For part (c), note that  $Q_j^* < q_j$  for  $j > \tilde{j}$  by definition of  $\tilde{j}$ , therefore, for  $j > \tilde{j}$  and  $q_j \leq Q < q_{j+1}$  we have  $C(r_j(Q), Q | j) \geq C(r_j(q_j), q_j | j)$  because by Property II-d  $C(r_j(Q), Q | j)$  is a unimodal function with respect to  $Q$ . The last inequality can easily be changed to  $C(r, Q | j) + \lambda c_j \geq C(r_j(q_j), q_j | j) + \lambda c_j$  by (9) which implies that  $C_A(r, Q) \geq C_A(r_j(q_j), q_j)$  and the proof of part (c) is completed.

**Theorem I.** If  $Q_j^* < q_{\tilde{j}+1}$  then  $Q^* \in \{Q_j^*, q_{\tilde{j}+1}, q_{\tilde{j}+2}, \dots, q_n\}$ ; otherwise  $Q^* \in \{q_{\tilde{j}+1}, q_{\tilde{j}+2}, \dots, q_n\}$ .

**Proof.** Let  $Q_f = \min\{q_{\tilde{j}+1}, Q_j^*\}$ .

$(r_{j(Q_f)}(Q_f), Q_f)$  is a feasible policy and by parts (a) and (b) of Lemma I its cost is lower than or equal to any other policy  $(r, Q)$  with  $Q \leq Q_f$ , therefore, by optimality of  $(r^*, Q^*)$ , we have:

$$Q^* \geq Q_f = \min\{q_{\tilde{j}+1}, Q_j^*\} \quad (13)$$

If  $Q_j^* < q_{\tilde{j}+1}$ , obviously  $C(r_{\tilde{j}}(Q), Q | \tilde{j})$  is non-decreasing over  $Q_j^* \leq Q < q_{\tilde{j}+1}$ , therefore,  $C_A(r, Q) \geq C_A(r_{\tilde{j}}(Q_j^*), Q_j^*)$  for  $Q_j^* \leq Q < q_{\tilde{j}+1}$ . Therefore, we have:

$$Q^* \notin (Q_j^*, q_{\tilde{j}+1}) \quad (14)$$

Lemma I-c states that the start point of each discount level  $j, j = \tilde{j}+1, \tilde{j}+2, \dots, n$ , has the lowest cost in the discount level, therefore, we can conclude that:

$$Q^* \notin (q_j, q_{j+1}), j = \tilde{j}+1, \tilde{j}+2, \dots, n \quad (15)$$

Considering (13), (14) and (15) the proof is completed.

By Theorem I, the search region for the optimal order quantity  $Q^*$  reduces to an enumerable finite set which has at

most  $n$  member. Based on Theorem I, in order to solve Problem II, it is sufficient to find the optimal policies  $(r_j(Q_j^*), Q_j^*)$  under cost function  $C(r, Q | j)$  until  $\tilde{j}$  is obtained. Since during each discount level  $j$ ,  $j=1,2,\dots,n$ , i.e.  $q_j \leq Q < q_{j+1}$ , the purchase price and the resource usage per unit of goods are constant,  $(r_j(Q_j^*), Q_j^*)$  can be obtained by using Algorithm I. After computing  $\tilde{j}$ , there are several candidates for optimal policy in which the one with minimum cost under  $C_A(r, Q)$  is  $(r^*, Q^*)$ . The details of this approach are described in subsequent sections.

#### IV. ALGORITHM

Now we can propose our algorithm for finding the optimal solution of Problem II. The procedure is as follows. Start from the last discount level  $j$ , i.e.  $j=n$ , utilize Algorithm I to find the optimal policy  $(r_j(Q_j^*), Q_j^*)$  under cost function  $C(r, Q | j)$ . If  $Q_j^* \geq q_j$  then the procedure is stopped and  $j$  is returned as  $\tilde{j}$ , otherwise, set  $Q = Q_j^*$  then find  $G_j^{Q+1}$  and let  $Q = Q + 1$ , continue the procedure until  $Q = q_j$  and so  $\tau_{G_j}^Q$  contains  $q_j$  points. The optimal reorder point  $r(q_j) = (\min \tau_{G_j}^{q_j}) - 1$  is obtained and the policy  $(r(q_j), q_j)$  is saved as a candidate of optimal solution. Then set  $j = j - 1$  and continue the procedure. After computing  $\tilde{j}$ , the algorithm is stopped, now if  $Q_j^* < q_{\tilde{j}+1}^*$  then the policy  $(r_{\tilde{j}}(Q_{\tilde{j}}^*), Q_{\tilde{j}}^*)$  is added to the candidate solution set. The optimal solution is the policy with minimum cost between all candidate solutions. The pseudo code of the presented algorithm is as follows.

##### Algorithm II.

```

j := n ; check := False ;
CAN =  $\emptyset$  (set of candidate policies for optimality)
While check := False Do
    Compute  $(r_j(Q_j^*), Q_j^*)$  using Algorithm I
    If  $Q_j^* \geq q_j$  then
        check := True ;  $\tilde{j} := j$ 
        If  $Q_j^* < q_{j+1}$  insert  $(r_j(Q_j^*), Q_j^*)$  to CAN
    Else
         $Q := Q_j^*$  ;  $r := r_j(Q_j^*)$ 
        While  $Q < q_j$  Do
            If  $\min\{G_j(r), G_j(r + Q + 1)\} = G_j(r)$ 
                then  $r := r - 1$ 
                 $Q := Q + 1$ 
            End While
            insert  $(r_j(q_j), q_j)$  to CAN ;  $j := j - 1$ 
        End If
    End While
 $(r^*, Q^*) = \text{Arg min}_{(r, Q) \in \text{CAN}} \{C_A(r, Q)\}$ 

```

End

#### V. CONCLUSION AND FUTURE STUDY

In this paper, we have investigated the  $(r, Q)$  inventory system under both all-units discount and a budget limitation which is considered as a soft constraint in the system cost function. It is assumed that the shortage of resource can be satisfied by renting the extra resource and so a resource shortage cost that is proportional to the amount of rented resource is incurred.

Some essential properties of the cost function were prove and based on these properties an algorithm with a one-dimensional search procedure is proposed to find the optimal solution of the problem. In this paper we analyze a single-item system under all-units discount. Considering the system with the incremental discount, deteriorating item or in multi-item case can be the other practical situations. Also, other than the continuous review  $(r, Q)$  policy, studies on periodic review models with resource constraint and discount in the same situation as this paper can be future research directions.

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