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Combining Laws of Mechanics and Hydrostatics in Non-Inertial Reference Frames

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Abstract**—**Method of combined teaching laws of classical mechanics and hydrostatics in non-inertial reference frames for undergraduate students is proposed. Pressure distribution in a liquid (or gas) moving with acceleration is considered. Combined effect of hydrostatic force and force of inertia on a body immersed in a liquid can lead to paradoxical results, in a motion of pendulum in particular. The body motion under Stokes force influence and forces in rotating reference frames are investigated as well. Problems and difficulties in student perceptions are analyzed.

Keywords**—**Hydrodynamics, mechanics, non-inertial reference frames, teaching.

I. Internal Pressure in Accelerated Liquids

WHEN a liquid (or gas) is under the influence of external forces, the internal pressure in it is not a homogeneous one [1]. In order to demonstrate the fact, let us choose in the liquid an infinitesimal element of a cubic form whose volume is $dV = dxdydz$ and assume that the force $\vec{f}dV$ ($\vec{f}$ is called ***a force density***, i.e. force on a volume unit) acts on it (Fig. 1).

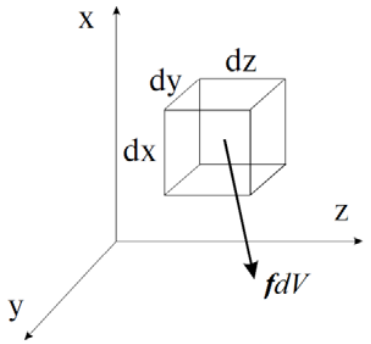

Fig. 1 An infinitesimal element in the liquid

As a result, the force $p(x,y,z)dydz$ acts in the positive *X* direction on a cubic side with *x* coordinate ($p(x,y,z)$ is a pressure function) and the force $p(x+dx,y,z)dydz$ acts in the same direction on the parallel side. If the liquid is in an equilibrium state (and then every volume inside is), the next equation must be fulfilled

$$p(x+dx,y,z)-p(x,y,z)=f_x dxdydz \tag{1}$$

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From this

$$\frac{\partial p}{\partial x}=f_x \tag{2}$$

Similar equation can be written in the two other *Y*, *Z* directions and then by means of a gradient ∇ it can be received

$$\nabla p=\vec{f} \tag{3}$$

When a body is immersed in a liquid, its surface points are under non-equal pressures and as a result the total force acts on it in the direction opposite to the gradient, i.e.

$$\vec{F}=-\int_V \nabla p dV=-\int_V \vec{f}dV \tag{4}$$

Here the integration is produced on the body volume that immersed in the liquid.

When the liquid is in the field of homogeneous gravity force, $\vec{f}=\rho\vec{g}$ (ρ is a liquid density), it follows from (4) that the force

$$\vec{F}_A=-\rho V\vec{g} \tag{5}$$

acts on the body in the direction opposite to the gravity. That is the ***buoyant*** or ***Archimedes force***. If z direction is chosen in the gravity direction, the change of the pressure with a depth (***hydrostatic pressure***) can be received from (3):

$$p(z)=p_0+\rho gz \tag{6}$$

where p_0 is a pressure on the liquid surface (atmospheric pressure, for example). Now let us suppose that a container with a liquid (or gas) moves with a uniform acceleration $\vec{a}$. As a result, its reference frame is non-inertial one [2]. In such reference frame a fictitious force $\rho\vec{a}$ acts on every unit of volume, so a total force that acts on it is

$$\vec{f}=\rho\vec{g}-\rho\vec{a} \tag{7}$$

Suppose the container moves in a horizontal direction normally to the gravity force and choose the direction of its motion as the *X* direction and the gravity as the *Z* direction.

Then one can receive for the internal two dimensional pressure in the liquid (or gas) similarly to (6)

$$p(x,z) = p_0 - \rho a x + \rho g z \tag{8}$$

where p_0 is a pressure in the origin $p_0 = p(0,0)$. It is seen from (8) that the internal pressure decreases in the direction of the liquid motion. As a result, a body of volume *V* that immersed in the liquid is acted by the force in the direction of its motion. The formula for the force can be received from (4):

$$\vec{F} = \rho V a \hat{x} \tag{9}$$

So the total force that acts on the body is

$$\vec{F} = \rho V a \hat{x} - \rho V g \hat{z} \tag{10}$$

It is interesting that the liquid surface is inclined at the angle of $\alpha = \tan^{-1}\left(\frac{a}{g}\right)$ relatively to the horizontal direction towards the motion (Fig. 2).

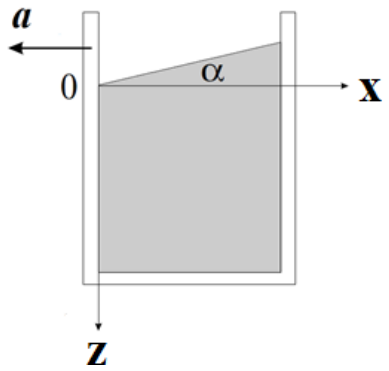

Fig. 2 A surface of an accelerated liquid

It follows from the fact that in the equilibrium state the liquid surface has to be normal to the total force direction.

II. Pendulum Motion in Accelerated Liquid

Let us imagine a simple pendulum hanging from the upper surface of a box filled with a liquid or gas and moving with a horizontal acceleration $\vec{a} = a\hat{x}$. So the box is a non-inertial reference frame. At the beginning, suppose that $\rho < \rho_0$ (ρ_0 is a density of the pendulum body).

Projecting the forces (Archimedes, inertia and tension *T*) into *X* and *Z* axes, we obtain the equations

$$\begin{cases} T\sin\alpha + \rho V a = \rho_0 V a \\ T\cos\alpha + \rho V g = \rho_0 V g \end{cases}, \tag{11}$$

where α is the angle of the pendulum inclination. From (11) one can see that the angle is

$$\alpha = \arctan\frac{a}{g} \tag{12}$$

and does not depend on a relationship between the body and liquid densities, unexpectedly. However, the direction of the inclination does depend on the relationship. In our case of $\rho < \rho_0$ the pendulum inclines backwards, like that in empty box [3].

Now let us imagine a simple pendulum attached to the bottom of a box filled with a liquid or gas (for example, a balloon filled with helium which string fixed to the bottom of a car) and moving with a horizontal acceleration $\vec{a} = a\hat{x}$. This time $\rho > \rho_0$ and the Archimedes force makes a string tension.

The equations of the pendulum motion are the same (11) and so the angle of the pendulum inclination is (12) as well. But in this case of $\rho > \rho_0$ (balloon in a car) the pendulum inclines **towards** the box motion, paradoxically (Fig. 3).

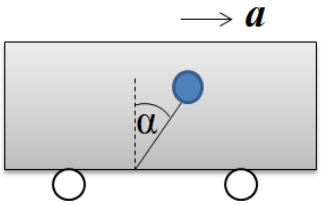

Fig. 3 Pendulum inclination in the case when the liquid density is greater than that of the pendulum body

The inclination above is the pendulum equilibrium state. A small deviation of the pendulum from this state causes its harmonic oscillations. In the first case of $\rho < \rho_0$ one can obtain the equation of pendulum motion by projecting the forces on the direction normal to the pendulum string (Fig. 4):

$$(\rho_0 - \rho)Va\cos(\alpha+\theta) - (\rho_0 - \rho)Vg\sin(\alpha+\theta) = \rho_0 VgL\ddot{\theta}, \tag{13}$$

where θ is the deviation angle and *L* is a pendulum string length. A damping is neglected, obviously.

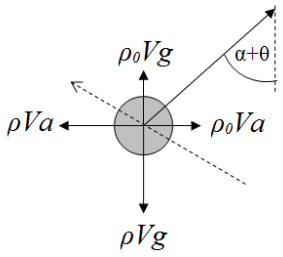

Fig. 4 A diagram of the forces that acts on the pendulum in a liquid accelerated towards the right

Taking (12) into consideration, one can see that the frequency of the pendulum free oscillations depends on the angle α and is equal to

$$\omega_0 = \sqrt{\frac{g\left(1-\frac{\rho}{\rho_0}\right)}{L\cos\alpha}} \tag{14}$$

In the opposite case of $\rho > \rho_0$ the equation of the pendulum motion looks like (13) after exchange positions of ρ and ρ_0 on its left side. Like this, the frequency is given by (14) after the same exchange.

III. Body Falling in Accelerated Liquid

The next example is a body falling in an elevator cab filled with a liquid and going down (or up) with a uniform acceleration $\vec{a} = a\hat{z}$ (Z axis is chosen in a gravity direction). In this case in non-inertial reference frame of the elevator five forces act on the body and determine its acceleration with respect to the elevator walls. They are a gravity force $\rho_0 Vg$, internal force in the accelerated liquid ρVa (both in the direction of elevator motion), inertia force $\rho_0 Va$, Archimedes force ρVg and Stokes force kv in opposite direction (v is a body velocity with respect to the elevator, k is a Stokes constant). Thus, the equation of Newton Second Law has a form:

$$(\rho_0 - \rho)V(g-a) - kv = \rho_0 V\frac{dv}{dt} \tag{15}$$

When an initial velocity is equal to zero, the solution is

$$v(t) = \frac{(\rho_0-\rho)V(g-a)}{k}\left[1-\exp\left(-\frac{kt}{\rho_0 V}\right)\right] \tag{16}$$

From (14) it is interesting to see that in a free fall of the elevator $(a = g)$ the body remains at rest without any dependence on relationship between the body and liquid densities. It is not so trivial. A relationship between gravitational and elevator cab accelerations determines the direction of a body motion relative to the cab.

IV. Body in Rotating Liquid

Now let us imagine a container with a liquid (bucket of water, for example) that rotates with uniform angular velocity ω around a fixed vertical axis and has, inside it, a small body of volume V. In this case, the pressure distribution in the liquid is

$$p(z,r) = p_0 + \rho g z + \frac{\rho\omega^2 r^2}{2}, \tag{17}$$

where r is a radial distance. From (17) one can see that the surface of equal pressure is a rotational paraboloid (Fig. 5). Then, similarly to (7), the total force that acts on every unit of the liquid volume in the non-inertial rotating reference frame is

$$\vec{f} = \rho\vec{g} + \rho\omega^2\vec{r} \tag{18}$$

So, as it derives from (4) and (16), the total force that acts on a body immersed in a liquid in a radial direction is

$$\vec{F}_r = (\rho_0 - \rho)V\omega^2\vec{r} \tag{19}$$

Here ***a centrifugal force*** [4] that acts on the body is taken into consideration as well.

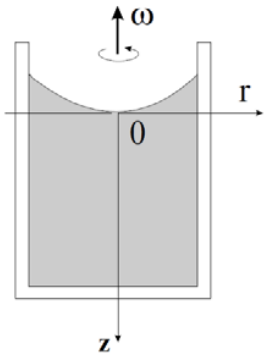

Fig. 5 A surface of a rotating liquid

From (19) one can see that in a case of $\rho > \rho_0$ the radial force carries the body towards the axis of rotation. This force plays an important role in the known tea leaf paradox [5], where tea leaves in a cup of tea migrate to the center of the cup after being stirred rather than being forced to the edges of the cup.

V. Pendulum Motion in Rotating Liquid

Combining Sections II and IV, one can consider the motion of the simple pendulum mentioned in Section II. For example, suppose a simple pendulum is suspended in a box filled with a liquid or gas and rotating with uniform angular velocity ω around a fixed vertical axis outside the box. A point of the pendulum suspension is on the distance R from the axis and $\rho < \rho_0$.

In a case of the pendulum equilibrium it is easy to see that the equations for the angle of the pendulum inclination look like (11), just now $a = \omega^2 R$. So, the angle of inclination is

$$\alpha = \arctan\frac{\omega^2 R}{g} \tag{20}$$

and again does not depend on a relationship between the body and liquid densities. In our case of $\rho < \rho_0$ the pendulum inclines away from the rotation axis (Fig. 6). The reason is that the centrifugal force is greater than the internal force in the liquid.

As before, a small deviation of the pendulum from this state causes its harmonic oscillations. The equation of the pendulum motion looks like (13) with $a = \omega^2 R$ and α from (20). The frequency of the pendulum free oscillations can be written in the form

$$\omega_0 = \sqrt{\frac{g}{L}\left(1 - \frac{\rho}{\rho_0}\right)\sqrt{1 + \left(\frac{\omega^2 R}{g}\right)^2}} \qquad (21)$$

It is seen that the frequency increases with angular velocity increasing. From (21), in the private case of $\omega = 0$, one can obtain formula for the simple pendulum oscillations in a liquid at rest.

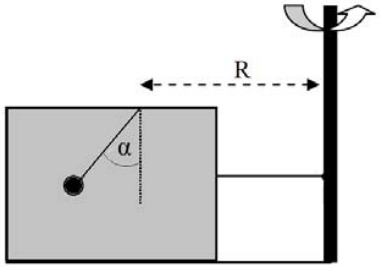

Fig. 6 Pendulum inclination in a rotating liquid in the case when the liquid density is less than that of the pendulum body

Now suppose the pendulum is attached to the bottom of a box filled with a liquid or gas and rotating with uniform angular velocity ω around a fixed vertical axis outside the box, as before. This time $\rho > \rho_0$ (it is the case of the balloon in a rotating box) and the pendulum inclines **towards** the rotation axis. The reason is that now the centrifugal force is weaker than the internal force in the liquid.

As to the equation of the pendulum motion and the oscillation frequency, they look like (13) and (20), accordingly, with exchanging positions of ρ and ρ_0, as mentioned after (14).

VI. Concluding Remarks

The above can be included into undergraduate classical mechanics course, for instance, the first part as a lecture and the next as exercises. Theoretical material can be accompanied by demonstrations, such as [6].

Essentially the nature of the internal force that arises in accelerated liquids is a mechanical one. There are quite a few students that even do not suspect the force exists and the others think that the force is a fictitious one though it is a real force. As a result, their exercise solutions are wrong. It is thought that such addition to the course can help undergraduate students to comprehend the non-inertial reference frames deeply.

Acknowledgment

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