International Journal of Mechanical, Industrial and Aerospace Sciences
ISSN: 2517-9950
Vol:9, No:7, 2015

Singular Value Decomposition Based Optimisation of Design Parameters of a Gearbox

Mehmet Bozca

***Abstract*—**Singular value decomposition based optimisation of geometric design parameters of a 5-speed gearbox is studied. During the optimisation, a four-degree-of freedom torsional vibration model of the pinion gear-wheel gear system is obtained and the minimum singular value of the transfer matrix is considered as the objective functions. The computational cost of the associated singular value problems is quite low for the objective function, because it is only necessary to compute the largest and smallest singular values (μ_{max} and μ_{min}) that can be achieved by using selective eigenvalue solvers; the other singular values are not needed. The design parameters are optimised under several constraints that include bending stress, contact stress and constant distance between gear centres. Thus, by optimising the geometric parameters of the gearbox such as, the module, number of teeth and face width it is possible to obtain a light-weight-gearbox structure. It is concluded that the all optimised geometric design parameters also satisfy all constraints.

***Keywords*—**Singular value, optimisation, gearbox, torsional vibration.

I. Introduction

Gears are widely used to transmit mechanical power from one shaft to another. The purpose of the gears is to couple two shafts together such that the rotation of the driven-shaft is a function of the rotation of the driving-shaft.

A four-degree-of-freedom model of the pinion gear-wheel gear system is considered for simplicity. The pinion gear body and wheel gear body are assumed to be rigid. The teeth are assumed to be elastic and parallel spring-damper combinations are assumed to exist between the teeth and the gear body.

Differential equations are obtained from equation of motion of gears system.

A dynamic gear vibration model is a useful tool to study the vibration response of a geared system with various gear parameters and operating conditions [1].

Torsional vibration models of gears system are classified according to their rigidity and elasticity such as purely torsional multi-body models, rigid multi-body models, flexible multi- body models and semi-rigid-elastic multi-body models [1].

Torsional vibration models of gears system are also classified according to time-invariant and time-variant such as linear time invariant (LTI) models with stiffness, linear time-varying (LTV) models with stiffness, time-varying models with backlash, as well as time-invariant average stiffness and time-invariant models in both backlash and stiffness simultaneously [2].

Mehmet Bozca is with the Yildiz Technical University, Mechanical Engineering Faculty, Machine Design Division Istanbul, Turkey (e-mail: mbozca@yildiz.edu.tr).

II. Torsional Vibration Model

A. Equation of Motion of Gears System

A four-degree-of-freedom model of the pinion gear–wheel gear system is considered for simplicity. The pinion gear body and the wheel gear body are assumed to be rigid. The teeth are assumed to be elastic and parallel spring–damper combinations are assumed to exist between the teeth and the gear body. A four-degree-of-freedom model is shown in Fig. 1. The equations of motion of the pinion gear–wheel gear system are written in terms of the four-degree-of-freedom model as follows [1]-[3].

S_p denotes rotational position of tooth i of a pinion gear, $Ø_p$ denotes rotational position of the pinion gear, S_w denotes rotational position of tooth i of the wheel gear and $Ø_w$ denotes rotational position of the wheel gear body.

Equation (1) is written for tooth i of a pinion gear as:

$$J_{pth}\ddot{S}_p = D_e r_{dp}{}^2\left(\dot{\phi}_p - \dot{S}_p\right) + K_e r_{dp}{}^2\left(\phi_p - S_p\right) + T_p \quad (1)$$

where J_{pth} is the moment of inertia of tooth i of the pinion gear [kg.mm^2], r_{dp} is the pitch circle radius of the pinion gear [mm], $Ø_p$ is the rotational position of the pinion gear body [rad], S_p is the rotational position of tooth i of the pinion gear [rad], $\dot{\phi}_p$ is the rotational velocity of the pinion gear body [rad/s], $\dot{S}_p$ is the rotational velocity of the tooth i of the pinion gear [rad/s], $\ddot{S}_p$ is the rotational acceleration of the tooth i of the pinion gear [rad/s^2], and T_p is the contact torque applied to tooth i [N.mm].

Equation (2) is written for a pinion gear body as:

$$J_{pb}\ddot{\phi}_b = 0 \quad (2)$$

where J_{pb} is the moment of inertia of the pinion gear body [kg.mm^2], $\ddot{\phi}_b$ is the rotational acceleration of the pinion gear body [rad/s^2].

Equation (3) is written for tooth i of the wheel gear as:

$$J_{wth}\ddot{S}_w = D_e r_{dw}{}^2\left(\dot{\phi}_w - \dot{S}_w\right) + K_e r_{dw}{}^2\left(\phi_w - S_w\right) + T_w \quad (3)$$

where J_{wth} is the moment of inertia of tooth i of the wheel gear [kg.mm^2], r_{dw} is the pitch circle radius of the wheel gear [mm], $Ø_w$ is the rotational position of the wheel gear body [rad], *Sw* is the rotational position of tooth i of the wheel gear [rad], $\dot{\phi}_w$ is the rotational velocity of the wheel gear body [rad/s], $\dot{S}_w$ is the rotational velocity of the tooth i of the wheel gear [rad/s], $\ddot{S}_w$ is the rotational acceleration of tooth i of the wheel gear [rad/s^2], and T_w is the contact torque applied to tooth i [N.mm].

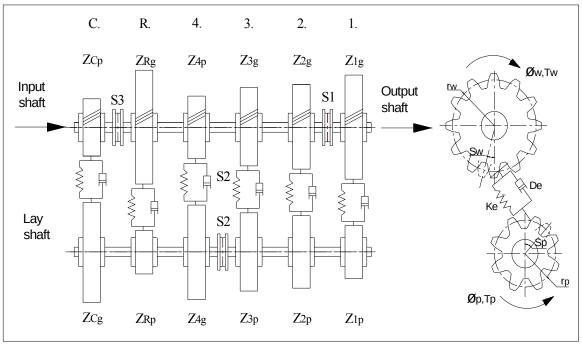

Fig. 1 A four-degree-of-freedom model

Equation (4) is written for the wheel gear body as:

$$J_{wb}\ddot{\phi}_w = 0 \tag{4}$$

where J_{wb} is the moment of inertia of the wheel gear body [kg.mm^2], $\ddot{\phi}_w$ is the rotational acceleration of the wheel gear body[rad/s^2].

K_e is the stiffness coefficient [N/mm] and is assumed to be time-invariant. The stiffness coefficients K_e resulting from the tooth surface contact are written as [3]:

$$K_e = \frac{\pi E S_0}{4\left(1-\nu^2\right)} \tag{5}$$

where E is Young's modulus [N/mm^2], υ is the Poisson's ratio [–] and S_0 is the thickness of gear [mm] which is written as [3]:

$$S_0 = \pi m_n / 2 \tag{6}$$

D_e is the viscous damping coefficient [N.s/mm] and is assumed to be time-invariant. Assuming viscous damping, Rayleigh damping is written as [3]:

$$D_e = \xi K_e \tag{7}$$

where ξ is the damping ratio [–].

The gear system equation of motion is written in matrix form as [3]:

$$\mathbf{J}\ddot{\mathbf{\Omega}} + \mathbf{D}\dot{\mathbf{\Omega}} + \mathbf{K}\mathbf{\Omega} = \mathbf{T} \tag{8}$$

where $\mathbf{J}$ is the moment of inertia matrix, $\mathbf{D}$ is the viscous damping matrix, $\mathbf{K}$ is the stiffness matrix, $\mathbf{T}$ is the vector of applied torques, $\ddot{\Omega}$ is the rotational acceleration vector, $\dot{\Omega}$ is the rotational velocity vector, and Ω is the rotational position vector.

The moment of inertia matrix $\mathbf{J}$ is written as:

$$\mathbf{J} = \begin{bmatrix} J_{pth} & 0 & 0 & 0 \\ 0 & J_{pb} & 0 & 0 \\ 0 & 0 & J_{wth} & 0 \\ 0 & 0 & 0 & J_{wb} \end{bmatrix}_{4x4} \tag{9}$$

The viscous damping matrix $\mathbf{D}$ is written as:

$$\mathbf{D} = \begin{bmatrix} -D_e r_{dp}^2 & D_e r_{dp}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -D_e r_{dw}^2 & D_e r_{dw}^2 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4x4} \tag{10}$$

The stiffness matrix $\mathbf{K}$ is written as:

$$\mathbf{K} = \begin{bmatrix} -K_e r_{dp}^2 & K_e r_{dp}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -K_e r_{dw}^2 & K_e r_{dw}^2 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4x4} \tag{11}$$

The vector of applied torque vector $\mathbf{T}$ is written as:

$$\mathbf{T} = \begin{bmatrix} T_p \\ 0 \\ T_w \\ 0 \end{bmatrix}_{4x1} \tag{12}$$

The rotational acceleration vector $\ddot{\mathbf{\Omega}}$ is written as:

$$\ddot{\mathbf{\Omega}} = \begin{bmatrix} \ddot{S}_p \\ \ddot{\phi}_p \\ \ddot{S}_w \\ \ddot{\phi}_w \end{bmatrix}_{4x1} \tag{13}$$

The rotational velocity vector $\dot{\mathbf{\Omega}}$ is written as:

$$\dot{\mathbf{\Omega}} = \begin{bmatrix} \dot{S}_p \\ \dot{\phi}_p \\ \dot{S}_w \\ \dot{\phi}_w \end{bmatrix}_{4x1} \quad (14)$$

The rotational position vector $\mathbf{\Omega}$ is written as:

$$\mathbf{\Omega} = \begin{bmatrix} S_p \\ \phi_p \\ S_w \\ \phi_w \end{bmatrix}_{4x1} \quad (15)$$

B. Singular Value Decomposition (SVD)

Some basic properties of SVD are revisited below. Let $A \in F^{mxn}$ where F is the field (the field may be real $\Re$ or complex C) [3], [4]. There exist unitary matrices:

$$\mathbf{U} = [u_1, u_2, ..., u_m] \in F^{mxm} \quad (16)$$

and

$$\mathbf{V} = [v_1, v_2, ..., v_n] \in F^{mxm} \quad (17)$$

such that

$$A = U \sum V^* \quad (18)$$

where

$$\sum = \begin{bmatrix} \sum_1 & 0 \\ 0 & 0 \end{bmatrix}, \ \sum_1 = Diag\{\mu_1, \mu_2, ... \mu_p\} \quad (19)$$

$$\mu_1 \geq \mu_2 \geq \mu_3 ... \geq \mu_p \geq 0, \ \ p = \min\{m, n\} \quad (20)$$

In the above equations, μ_i is the i^{th} singular value of A, and the vectors $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ are, respectively, the i^{th} left and right singular vectors defined by the following eigenvalue problems:

$$Av_i = \mu_i u_i \ \text{ or } \ A^* u_i = \mu_i v_i \quad (21)$$

where a superposed asterisk denotes conjugated transpose. The following notations for singular values are adopted.

$$\mu_{\max}(A) = \mu_1(A) = \text{the largest singular value of } A \quad (22)$$

$$\mu_{\min}(A) = \mu_p(A) = \text{the smallest singular value of } A \quad (23)$$

C. Structural Singular Value

Assuming zero initial conditions, one gets the following harmonic response of a structure by taking the Laplace transform of the transfer matrix (8) as [3]:

$$\mathbf{J}s^2\Theta(s) + \mathbf{D}s\Theta(s) + \mathbf{K}\Theta(s) = \mathbf{T}(s) \quad (24)$$

$$\Theta(s) = (\mathbf{J}s^2 + \mathbf{D}s + \mathbf{K})^{-1}\mathbf{T} \quad (25)$$

The complex Laplace transform variable s is substituted by $s = j\omega$, where ω is the excitation frequency, and j is the imaginary unit. Then, (26) is written in the frequency domain as [3]:

$$\mathbf{\Omega} = (-\mathbf{J}\omega^2 + \mathbf{D}j\omega + \mathbf{K})^{-1}\mathbf{T} \quad (26)$$

Note that singular values of the transfer matrix $(-\mathbf{J}\omega^2 + \mathbf{D}j\omega + \mathbf{K})^{-1}$ in (26) are called *structural singular values* in this article, and are function of ω.

III. Calculating the Load Carrying Capacity of Helical Gears

A. Tooth Bending Stress

The real tooth-root stress, σ_F is calculated as [5]-[7]:

$$\sigma_F = \frac{F_t}{bm_n} Y_F Y_S Y_\varepsilon Y_\beta K_A K_V K_{F\beta} K_{F\alpha} \quad (27)$$

where F_t is the nominal tangential load [N], b is the face width [mm], m_n is the normal module [mm], Y_F is the form factor [–], Y_s is the stress correction factor [–], Y_ε is the contact ratio factor [–], K_A is the application factor [–], K_V is the internal dynamic factor [–], $K_{F\beta}$ is the face load factor for tooth-root stress [–] and $K_{F\alpha}$ is the transverse load factor for tooth-root stress [–].

The safety factor for bending stress S_F is calculated as [5]-[7]:

$$S_F = \frac{\sigma_{Fp}}{\sigma_F} \quad (28)$$

where σ_{Fp} is permissible bending stress.

B. Tooth Contact Stress

The real contact stress, σ_H is calculated as [5]-[7]:

$$\sigma_H = \sqrt{\frac{F_t}{bm_n}\frac{u+1}{u}} Z_H Z_E Z_\varepsilon Z_\beta \sqrt{K_A K_V K_{H\beta} K_{H\alpha}} \quad (29)$$

where u is the gear ratio [-], Z_H is the zone factor [-], Z_E is the elasticity factor [$\sqrt{N/mm^2}$], $Z\varepsilon$ is the contact ratio factor [-], Z_β is the helix angle factor [-], $K_{H\beta}$ is the face load factor for contact stress[-] and $K_{H\alpha}$ is the transverse load factor for contact stress [-].

The safety factor for contact stress, S_H is calculated as [5]-[7]:

$$S_H = \frac{\sigma_{Hp}}{\sigma_H} \quad (30)$$

where σ_{Hp} is the permissible contact stress.

IV. Optimisation of Gearbox Design Parameters

Constrained optimization is very useful tool for light-weight-structure design of machine elements with constraints such as stress, deformation and vibration.

In optimization, the goal is usually minimize the cost of a structure while satisfying the design specification. By optimizing the responsible parameters, it is possible to obtain a light-weight-gearbox structure.

The flowchart of the design parameter optimisation procedure is shown in Fig. 2.

V. Numerical Example

Constrained optimisation approaches are applied to the 5-speed gearbox. All programs are developed using MATLAB and in all optimisation studies, the sequential quadratic programming (SQP) method is employed. Torsional vibration parameters are shown in Table I.

To find the optimum design parameter, the initial design parameters of the 5-speed gearbox including m, z, β and b are varied. Twenty-four design parameters are optimized simultaneously using the programs developed. During optimisation, different initial value vectors are used to identify the global minimum solution of the objective function, $T(m,z,\beta,b)$.

TABLE I
Torsional Vibration Parameters

	1st pinion	2nd pinion	3rd pinion	4th pinion	C. pinion	R. pinion
Torque T_L	392.10^3	392.10^3	316.10^3	252.10^3	200.10^3	1148.10^3
Gear ratio u	1.814	1.147	1.242	1.560	1	2.84
Young's modulus E	21.10^4	21.10^4	21.10^4	21.10^4	21.10^4	21.10^4
Poisson's ratio υ	0.3	0.3	0.3	0.3	0.3	0.3
Damping ratio ξ	0.1	0.1	0.1	0.1	0.1	0.1

A. Objective Function

The following objective function was employed:

$$F = \mu_{Min}(T) \quad (31)$$

The minimum singular values of the transfer matrix $\mu_{Min}(T)$ are defined as:

$$\mu_{Min}(T) = \mu_{Min}(-J\omega^2 + Dj\omega + K)^{-1} \quad (32)$$

The following optimization problem is solved:

$$\min T(m,z,\beta,b) \quad (33)$$

$$\text{Subject to: } LB \le m,z,\beta,b \le UB \quad (34)$$

$$G(X) \le 0 \quad (35)$$

C. Constraint Function

Tooth bending stress, contact stress and the distance between gear centres are considered to be the constraint function in the optimization.

The following constraints are considered to be constraint functions:

$$\sigma_F - \sigma_{Fp} \le 0 \quad (36)$$

$$\sigma_H - \sigma_{Hp} \le 0 \quad (37)$$

$$a_1 = a_2 = a_3 = a_4 = a_5 = a_R = constant \quad (38)$$

C. Optimisation Results

The optimisation results using objective function T are presented in Tables II and III. Because of the limited space, only the important results are presented.

Although the results given above represent the optimum solution, the standard design parameter values should be used by gear manufacturers.

TABLE II
Optimisation Results (solution no:1)

	1st pinion	2nd pinion	3rd pinion	4th pinion	C. pinion	R. pinion
Module m	2.992	3.917	3.750	3.283	4.206	4.329
Number of teeth z	19	19	19	19	19	14
Helix angle β	26.108	26	26	26	26	26.414
Face width b	28	28	28	28	28	32
Safety factor S_F	1.062	1.568	1.784	1.715	3.544	1.000
Safety factor S_H	1.3602	1.510	1.641	1.687	2.196	1.245
Centre distance a	79,999	79,903	79,891	79,85	79,922	80

TABLE III
Optimisation Results (solution no:2)

	1st pinion	2nd pinion	3rd pinion	4th pinion	C. pinion	R. pinion
Module m	4.060	3.920	3.754	3.287	4.209	4.330
Number of teeth z	14	19	19	19	19	14
Helix angle β	32	32	32	32	32	30.165
Face width b	30	30	30	30	30	32
Safety factor S_F	1.037	1.312	1.500	1.435	2.964	1.000
Safety factor S_H	1.154	1.380	1.496	1.540	2.002	1.187
Centre distance a	79,960	79,956	79,950	79,931	79,964	80

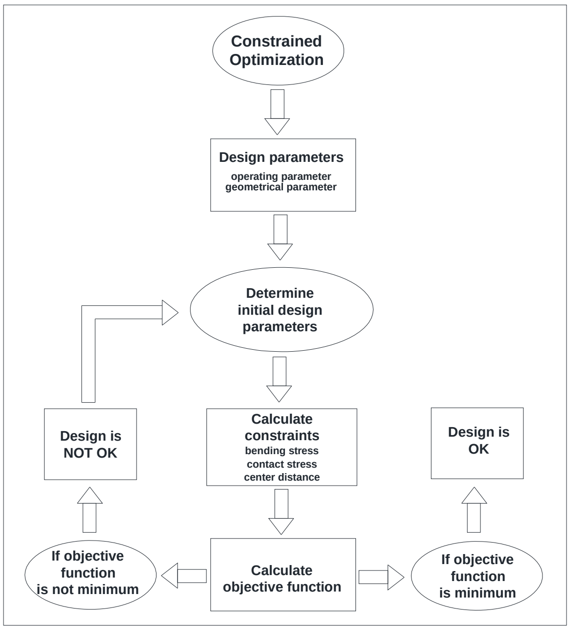

Fig. 2 Flow chart to optimize gearbox design parameters

Minimum singular values of transfer function are objective function in this study. Cost of objective functions are presented in Fig. 3.

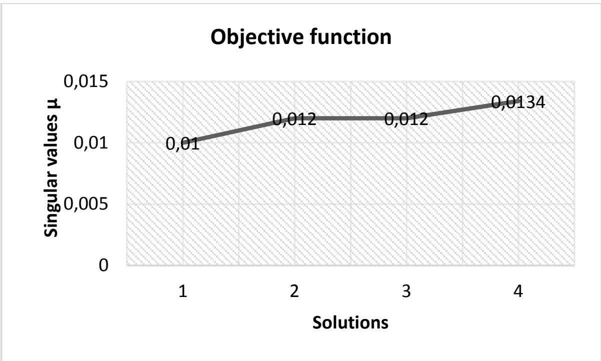

Fig. 3 Cost of Objective Function

VI. Conclusion

Singular value decomposition based optimisation of geometric design parameters of a 5-speed gearbox via torsional vibration model is studied. The following conclusions are drawn.

The computational cost of the associated singular value problems is quite low for the objective function, because it is only necessary to compute the largest and smallest singular values (μ_{max} and μ_{min}) that can be achieved by using selective eigenvalue solvers; the other singular values are not needed.

Although the results given above represent the optimum solution, the standard design parameter values should be used by gear manufacturers.

By optimising the geometric parameters of the gearbox such as, the module, number of teeth and face width it is possible to obtain a light-weight-gearbox structure and minimize the torsional vibration. It is concluded that the all optimised geometric design parameters also satisfy all constraints that include bending stress, contact stress and constant distance.

References

[1] Saeed Ebrahimi, Peter Eberhard, “Rigid-elastic Modelling of meshing gear wheels in multi-body systems”, Multi-body System Dynamics vol. 16, 2006.
[2] Kahraman, R. Singh, “Interactions between time varying mesh stiffness and clearance nonlinearities in a geared system”, Journal of Sound and Vibration vol. 146, 1990, pp. 135–156.
[3] M. Bozca, “Torsional vibration model based optimisation of gearbox geometric design parameters to reduce rattle noise in an automotive transmission”, Mechanism and Machine Theory vol. 45, 2010,pp. 1583–1598.
[4] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1995.
[5] Roloff, Matek, Maschinenelemente, Vieweg & Sohn Verlag/Fachverlage GmbH, Wiesbaden, 2005.
[6] H. Naunheimer, B. Bersche, J. Ryborz, W. Novak, Automotive Transmissions, Springer, 2011.
[7] ISO 6336–1: calculation of load capacity of spur and helical gears, part 1:basic principle, introduction and general influence factors.

Mehmet Bozca born in 1965 in Turkey. He graduated from Yildiz University, Engineering Faculty, Mechanical Engineering Department, Machine Design and Manufacturing Division, Istanbul, Turkey as a BSc. Mechanical Engineer in 1987. He graduated from Yildiz Technical University, Science Institute, Mechanical Engineering Department, Machine Design Division, Istanbul, Turkey as a MSc. Mechanical Engineer in 1996. He graduated from Istanbul Technical University, Science Institute, Mechanical Engineering Department, Machine Design and Manufacturing Division as a PhD. Mechanical Engineer in 2005. He worked respectively as a Design Engineer and Supplier Relations Department Chief at Cihan Electronik A.Ş.-Grundig TV. Company, Istanbul, Turkey between 1990 and 1993. He worked as a Method Engineer at Ford-Otosan A.Ş., Istanbul, Turkey between 1995 and 1997. He worked as a Development Engineer at Assan-Hanil A.Ş., Izmit, Turkey between 1997 and 1998. He has been as an guest researcher respectively at Technical University of Munich, University of Stuttgart, University of Applied Science Esslingen, Germany. Currently, he is working as an Associate Professor Dr. at Yıldız Technical University, Mechanical Engineering Faculty, Mechanical Engineering Department, Machine Design Division in Istanbul, Turkey. His main field of interest are Machine Elements and Machine Design. Associate Prof. Dr. Bozca is member of Chamber of Mechanical Engineering **TMMO** of Turkey since 1987.