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Inverse Matrix in the Theory of Dynamic Systems

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***Abstract*—**In dynamic system theory a mathematical model is often used to describe their properties. In order to find a transfer matrix of a dynamic system we need to calculate an inverse matrix. The paper contains the fusion of the classical theory and the procedures used in the theory of automated control for calculating the inverse matrix. The final part of the paper models the given problem by the Matlab.

***Keywords*—**Dynamic system, transfer matrix, inverse matrix, modeling.

I. Introduction

IN the last few decades there has been a marked increase in college studies that combine knowledge from engineering, automation and informatics (e.g. mechatronics, informatics and automation in industrial applications, etc.). A key term used in these areas is „dynamic system“ – a system that changes with regards to time. It is much more efficient to use a mathematical description of the characteristics of such a system rather than a standard input-output description.

A multidimensional linear system (either continuous or discrete) is described by a state equation. In order to calculate the transitional matrix it is necessary to calculate it´s inverse matrix. In the theory of automated control, it is calculated using algorithms (such as the algorithm for calculating the inverse matrix, Fadejev´s algorithm, Bocher´s formula for calculating the coefficient of a characteristic polynomial). But at first glance (at least for a student), the connection between the algorithms and knowledge gained during the first year math course is not evident. The goal of this paper is to remove this deficiency - to create a link between some of the basic mathematical findings (more precisely from linear algebra) and system theory.

II. Inverse Matrix

A. Classic theory

This theory is taught on most colleges in “Mathematics” or “Linear algebra”. A student learns the definition and basic properties of the inverse matrix [1], [5]. Considering that this paper is dedicated to the different methods of its computation, we will remind you of only the basic definition: Let A be a regular matrix (a square matrix with a determinant different from 0). Matrix A^{-1} is an inverse matrix to matrix A if

$$A \cdot A^{-1} = A^{-1} \cdot A = I \tag{1}$$

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where I is an identity matrix.

We can calculate it using two procedures:

a) We adjust the matrix $(A|I)$ using either column or line equivalent operations to get a resulting matrix $(I|A)$.

b) Using

$$A^{-1} = \frac{1}{|A|} \cdot \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix} \tag{2}$$

B. Inverse Matrix in the Theory of Automatized Control

While solving a state equation we need to calculate an inverse matrix to the matrix $(s \cdot I - A)$. We can use any of the procedures shown in the previous part. However, they are, especially for larger matrices, rather lengthy.

It is much more efficient to use the process shown in [2]. We will show it in a way that is understandable for a reader who doesn't possess deeper knowledge of linear algebra.

We will first calculate the characteristic polynomial of matrix A . We will get it by calculating the determinant of matrix $(s \cdot I - A)$. After adjusting it to a normalized form we end up with a polynomial

$$s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_{n-1}s + a_{0.} \tag{3}$$

Its coefficients are $1, a_{n-1}, \cdots, a_0$. Auxiliary matrices (we can denote them e.g. $R_0, R_1, \cdots, R_{n-1}$, can be calculated using the recurrent relations:

$$\begin{aligned} R_0 &= I \\ R_1 &= A \cdot R_0 + a_{n-1} \cdot I \\ &\vdots \\ R_{n-1} &= A \cdot R_{n-1} + a_1 \cdot I \end{aligned} \tag{4}$$

Then we calculate the inverse matrix

$$(sI - A)^{-1} = \frac{1}{|sI - A|}\left(R_0 s^{n-1} + R_1 s^{n-2} + \cdots + R_{n-1}\right). \tag{5}$$

We calculated the coefficients of the characteristic polynomial using the definition. Instead, we can use Bocher´s formulas. We will not mention this procedure further, but it can be found in [2].

Note: Another option for calculating the inverse matrix $(sI - A)$ is using its expansion in the Laurent series

$$(sI-A)^{-1}=\frac{1}{s}I+\frac{1}{s^2}A+\frac{1}{s^3}A^2+\cdots+\frac{1}{s^n}A^{n-1}+\cdots. \quad (6)$$

This series converges for all s, whose absolute value is grethe maximum of the eigenvalues of matrix A. Using this procedure, we can only calculate the matrix approximately, but for larger matrices it is much faster.

C. Applying Both Theories Using MATLAB

In order to obtain the inverse matrix A^{-1} to matrix A applying the classic theory, a direct command or a MATLAB function can be used [11], [12]. As regards the function, it structure is

```
Y = inv(A).
```
(7)

Under the command

```
Y = X^(-1),
```
(8)

the inverse matrix is computed the same manner, subjected to the same limitations.

Using the theory of automatized control, as mentioned above, we need to calculate an inverse matrix to the matrix $(s\cdot I-A)$. It is necessary to keep in mind that *s* must be defined as a symbolic variable. The MATLAB script is then expressed as

```
function invmat(n, A)
syms('s');
I=eye(n);
invmat=inv(s*I-A)
end,
```

where *n* stands for matrix dimension and must be the same as the dimension of A matrix.

III. Dynamic System

The internal description of a dynamic system is the relation between all of the variables of the system and is defined using state equations. The external description of the dynamic system is the relation between the input and output variables. In case of a stationary linear system it is represented by a transfer matrix.

The state equations of a continuous linear system with the starting condition $x(0)=0$ are

$$\begin{aligned}\dot{x}(t)&=Ax(t)+Bu(t)\\ y(t)&=Cx(t)+Du(t)\end{aligned}. \quad (9)$$

The transfer matrix of a system can be calculated using

$$G(s)=C\cdot(sI-A)^{-1}\cdot B+D. \quad (10)$$

The most "cumbersome" part in this formula, at least as time consumption is concerned, is the calculation of the inverse matrix.

We can show the different ways of calculating the solution using an example from the textbook [2]. State equations of a continuous linear system with the starting condition $x(0)=0$ are (9).

Example 1. Matrices A, B, C, D are number matrices,

$$A=\begin{pmatrix}0&1\\0&3\end{pmatrix}, B=\begin{pmatrix}1&0\\0&1\end{pmatrix}, C=\begin{pmatrix}1&2\end{pmatrix}, D=\begin{pmatrix}0&0\end{pmatrix}.$$

The transfer matrix of a system can be calculated using the relation

$$G(s)=C\cdot(sI-A)^{-1}\cdot B+D$$
$$(sI-A)^{-1}=\frac{1}{s(s-3)}\begin{pmatrix}s-3&1\\0&s\end{pmatrix}.$$

Transfer matrix of the system

$$G(s)=\begin{pmatrix}1&2\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{s}&\frac{1}{s(s-3)}\\0&\frac{1}{s-3}\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&0\end{pmatrix}=\begin{pmatrix}\frac{1}{s}&\frac{1+2s}{s(s-3)}\end{pmatrix}.$$

Using the MATLAB code mentioned above and complemented by the matrices *B*, *C* and *D* and the formula for obtaining the transfer matrix of the system,

```
function tm(2, [0 1; 0 3], [1 0; 0 1], [1 2], [0
0])
syms('s');
I=eye(n);
invmat=inv(s*I-A);
G(s)=C*invmat*B+D
end
```

We get the result

```
G(s) = [ 1/s, 2/(s - 3) + 1/(s*(s - 3))],
```

which corresponds to the result obtained by the calculation, as

$$G(s)=\begin{pmatrix}\frac{1}{s}&\frac{2}{s-3}+\frac{1}{s(s-3)}\end{pmatrix}=\begin{pmatrix}\frac{1}{s}&\frac{1+2s}{s(s-3)}\end{pmatrix}.$$

IV. Inverse Dynamic System

A dynamic system defined by (9), whose transition matrix $G(s)$ can be calculated using (10). An inverse dynamic system exists when there exist an inverse matrix to $G(s)^{-1}$, i.e. when $G(s)$ is regular ($|G(s)|\neq 0$).

An interesting solution to this problem can be found in [3]. The author of the paper provides an algorithm that can be used to determine whether an inverse matrix exists (without calculating the determinant), and if it does, calculates the inverse dynamic system.

We will not reproduce the theory and the algorithm that the solution to the example is based on, because it is found in the

quoted work (along with the proof). We will demonstrate on an example and compare it to the traditional method of solving the problem.

Example 2. Matrices A, B, C, D are number matrices,

$$A=\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}, B=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, C=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, D=\begin{pmatrix}2 & 2\\ 1 & 1\end{pmatrix}.$$

We will create a matrix

$$M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}0 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 2 & 2\\ 1 & 1 & 1 & 1\end{pmatrix}.$$

$$\Delta_0=|M|=0$$

$$\Delta_1=\left|M\begin{pmatrix}1\\ 1\end{pmatrix}\right|+\left|M\begin{pmatrix}2\\ 2\end{pmatrix}\right|=0+0=0$$

$$\Delta_2=\left|M\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}\right|=0$$

Because $\Delta_0=\Delta_1=\Delta_2=0$, an inverse dynamic system does not exist. Let's confirm using the classic calculation.

$$(sI-A)^{-1}=\frac{1}{s^2-s-1}\begin{pmatrix}s & -1\\ -1 & s-1\end{pmatrix}.$$

The transition matrix is

$$G(s)=\frac{1}{s^2-s-1}\begin{pmatrix}2s^2-1 & 2s^2-1\\ s^2+s & s^2+s\end{pmatrix}.$$

This matrix is not regular (from the properties of its determinants it is evident that $|G(s)|=0$) and therefore an inverse matrix does not exist.

Using MATLAB code

```
function deter(n, A, B, C, D)
syms('s');
I=eye(n);
M=[A B; C D]
invmat=inv(s*I-A);
G=C*invmat*B+D
det(G)
end,
```

We obtain the transition matrix in a form of two vectors

```
G =
[ 2 - 2/(- s^2 + s + 1) - (s - 1)/(- s^2 + s + 1)
- s/(- s^2 + s + 1), 2 - 2/(- s^2 + s + 1) - (s -
1)/(- s^2 + s + 1) - s/(- s^2 + s + 1)]
[ 1 - 2/(- s^2 + s + 1) - (s - 1)/(- s^2 + s + 1)
- s/(- s^2 + s + 1), 1 - 2/(- s^2 + s + 1) - (s -
1)/(- s^2 + s + 1) - s/(- s^2 + s + 1)],
```

which, after several modifications, corresponds to the results obtained by the classic calculation:

$$G(s)=\begin{pmatrix}2-* & 2-*\\ 1-* & 1-*\end{pmatrix}$$

where

$$*=\frac{2}{-s^2+s+1}+\frac{s-1}{-s^2+s+1}+\frac{s}{-s^2+s+1}.$$

After modification

$$G(s)=\begin{pmatrix}\frac{-2s^2+1}{-s^2+s+1} & \frac{-2s^2+1}{-s^2+s+1}\\ \frac{-s^2-1}{-s^2+s+1} & \frac{-s^2-1}{-s^2+s+1}\end{pmatrix}=\frac{1}{s^2-s-1}\begin{pmatrix}2s^2-1 & 2s^2-1\\ s^2+1 & s^2+1\end{pmatrix}.$$

Determinant is equal to 0.

Note: Matrix $M\begin{pmatrix}i_1, i_2, ..., i_k\\ j_1, j_2, ..., j_k\end{pmatrix}$ is the matrix we get when we omit lines $i_1, i_2, ..., i_k$ and columns $j_1, j_2, ..., j_k$ from matrix M.

Example 3. Matrices A, B, C, D are number matrices,

$$A=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, B=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, C=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, D=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}.$$

Matrix

$$M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}0 & 1 & 1 & 0\\ 1 & 0 & 0 & 0\\ 1 & 0 & 1 & 1\\ 0 & 1 & 1 & 1\end{pmatrix}$$

and $\Delta_0=|M|=-1\neq 0$, the inverse dynamic system exists.

Based on the algorithm shown in work [3], we get the relations for the inverse dynamic system

$$\dot{\varsigma}=\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\varsigma+\begin{pmatrix}0 & 0\\ 1 & -1\end{pmatrix}\upsilon$$

$$\eta=\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}\varsigma+\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}\upsilon$$

We will prove its existence (for comparison) using the traditional method that uses the transition matrix.

$$(sI-A)^{-1}=\frac{1}{s^2-1}\begin{pmatrix}s & 1\\ 1 & s\end{pmatrix}$$

The transition matrix is

$$G(s)=\begin{pmatrix}\frac{s^2+s-1}{s^2-1} & 1\\ \frac{s^2}{s^2-1} & 1\end{pmatrix}.$$

Its determinant is

$$|G(s)| = \frac{s-1}{s^2-1} = \frac{1}{s+1} \neq 0 ,$$

Therefore the matrix is not regular.

Using the same MATLAB code as in Example 2,

```
function deter(n, A, B, C, D)
syms('s');
I=eye(n);
M=[A B; C D]
invmat=inv(s*I-A);
G=C*invmat*B+D
det(G)
end,
```

We obtain the transition matrix in a form

```
G =
[ s/(s^2 - 1) + 1, 1]
[ 1/(s^2 - 1) + 1, 1],
```

which, after several modifications, corresponds to the results obtained by the classic calculation:

$$G = \begin{pmatrix} \frac{s}{s^2-1}+1 & 1 \\ \frac{1}{s^2-1}+1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{s^2+s-1}{s^2-1} & 1 \\ \frac{s^2}{s^2-1} & 1 \end{pmatrix}.$$

Determinant takes the form:

```
DET = 1/(s + 1).
```

V. Conclusion

The problem of finding an inverse matrix in dynamic system theory is much vaster than we have shown. Interesting results can be found in [9] and [10]. They contain finding the inverse matrix by inverting graphs.

More interesting findings can be found in [4], [6]-[8]. For example, [6] presented dynamic algorithms for computing: matrix determinant, matrix adjoint, matrix inverse, and solving linear system of equations.

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