

# Lateral Torsional Buckling of Steel Thin-Walled Beams with Lateral Restraints

Ivan Balázs, Jindřich Melcher

**Abstract**—Metal thin-walled members have been widely used in building industry. Usually they are utilized as purlins, girts or ceiling beams. Due to slenderness of thin-walled cross-sections these structural members are prone to stability problems (e.g. flexural buckling, lateral torsional buckling). If buckling is not constructionally prevented their resistance is limited by buckling strength. In practice planar members of roof or wall cladding can be attached to thin-walled members. These elements reduce displacement of thin-walled members and therefore increase their buckling strength. If this effect is taken into static assessment more economical sections of thin-walled members might be utilized and certain savings of material might be achieved. This paper focuses on problem of determination of critical load of steel thin-walled beams with lateral continuous restraint which is crucial for lateral torsional buckling assessment.

**Keywords**—Beam, buckling, numerical analysis, stability, steel.

## I. INTRODUCTION

RESISTANCE of steel thin-walled beams with no lateral restraints along the span is usually limited by lateral torsional buckling. In case there are planar members of cladding attached to beams, the buckling strength is higher because those members work as lateral restraint. It completely or partially prevents lateral displacement of cross-section of the beam. The paper focuses on problem of steel thin-walled beams with lateral continuous restraint against lateral torsional buckling. Theoretical background beyond lateral torsional buckling of an isolated and a laterally restrained beam is described and differences between them are highlighted. Parametric study of laterally restrained beams with various spans and load conditions is performed and results are compared with results of analysis of isolated beams (with no lateral restraints). An option of determination of critical load using numerical algorithms is outlined.

## II. LATERAL TORSIONAL BUCKLING OF AN IDEAL ISOLATED BEAM

The effect of lateral torsional buckling of an ideal beam (with no initial imperfection) is characterized by spatial displacement of the cross-section perpendicularly to the plane of bending and by rotation of the cross-section [1]. It is illustrated in Fig. 1 where  $q_z$  is vertical load,  $C_g$  is cross-section center of gravity,  $C_s$  cross-section shear center,  $a_z$  is distance of shear center and center of gravity and  $e_z$  is distance

of the point of load application and center of gravity. The compressed part of cross-section tends to buckle laterally. The resulting deformation consists of two components: lateral displacement  $v$  and angle of rotation  $\varphi$ .

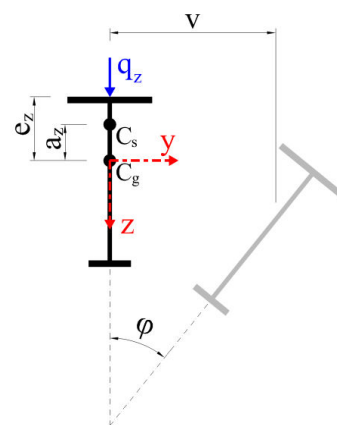


Fig. 1 Lateral torsional buckling of an ideal isolated beam

Lateral torsional buckling of an ideal beam occurs when critical load is reached. The problem was dealt with by Vlasov in form of differential equations of stability of an arbitrary thin-walled member loaded by bending and axial force [2]. After modification for a member in bending only the problem is defined by two homogenous differential equations of order four with appropriate boundary conditions. For a beam simply supported in bending as well as in torsion (1)-(4) apply.

$$EI_z v'''' + (M_y \varphi)'' = 0 \quad (1)$$

$$EI_\omega \varphi'''' - GI_t \varphi'' - 2b_z (M_y \varphi)' + q_z (e_z - a_z) \varphi + M_y v'' = 0 \quad (2)$$

$$v(0) = v(L) = 0; v''(0) = v''(L) = 0 \quad (3)$$

$$\varphi(0) = \varphi(L) = 0; \varphi''(0) = \varphi''(L) = 0 \quad (4)$$

In these expressions  $E$  is Young's modulus,  $G$  shear modulus,  $I_z$  second moment of area,  $I_\omega$  warping constant,  $I_t$  torsion constant,  $q_z$  vertical load in the XZ plane,  $M_y$  bending moment in the same plane,  $v$  and  $\varphi$  unknown functions of deformation and  $b_z$  the so called characteristic abscissa.

Formula for its determination can be found in [1]. In the mathematical point of view the critical moment is defined as eigenvalue problem of above mentioned differential equations and appropriate boundary conditions. In the actual standard for

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design of steel structures [3] there is a formula for critical moment of an isolated beam of at least monosymmetric cross-section loaded by transverse load that intersects the cross-section shear center.

III. LATERAL TORSIONAL BUCKLING OF AN IDEAL BEAM WITH LATERAL CONTINUOUS RESTRAINT

Let us consider that there is a continuous lateral restraint along the span of an ideal simply supported beam (perfectly straight beam with no initial imperfections). The position of lateral restraint is  $C_{lat}$ , distance of the restraint to center of gravity is  $c_z$  (Fig. 2). Let us assume that lateral restraint is perfectly rigid so that it fully prevents lateral displacement of the cross-section. The beam span is  $L$ . Since lateral displacement is prevented the only function of deformation is angle of rotation  $\varphi$  and (4) applies.

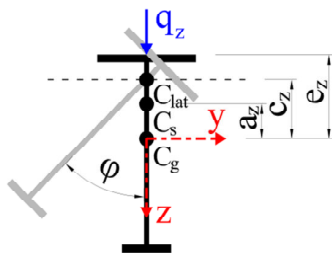


Fig. 2 Lateral torsional buckling of an ideal beam with lateral continuous restraint

Due to zero lateral displacement the fundamental differential equations of the problem are modified to (5) [1] with one unknown function of deformation  $\varphi$ ,

$$[EI_\omega + EI_z(c_z - a_z)^2]\varphi^{IV} - GI_t\varphi'' + 2(c_z - a_z - b_z)(M_y\varphi)' + q_z(e_z - c_z)\varphi = 0 \tag{5}$$

where  $q_z$  is vertical load. Its critical value (magnitude of load when lateral torsional buckling of an ideal beam occurs) is given as eigenvalue problem of (5). In this case it is the so called Sturm-Liouville eigenvalue problem of differential equation of fourth order [4]. This complex problem can be solved e.g. using selected numerical methods.

IV. SOLUTION USING FINITE ELEMENT METHOD

A. Assumptions and Process of the Analysis

As an example a double symmetric simply supported steel beam of thin-walled cross-section according to Fig. 3 is considered. There is lateral continuous restraint located 50 mm from the center of gravity. A parametric study is performed where various spans from  $L = 3$  m up to  $L = 10$  m are investigated. Vertical load is applied on the top flange of the beam. Three types of loads A, B and C according to Fig. 4 are considered – uniformly distributed load  $q_z$  (A), force  $F$  at midspan (B) and bending moments at supports (C).

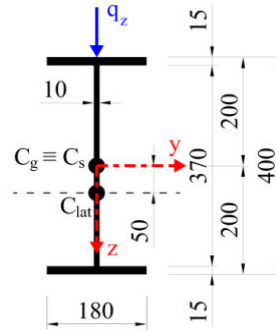


Fig. 3 Cross-section of the investigated beam

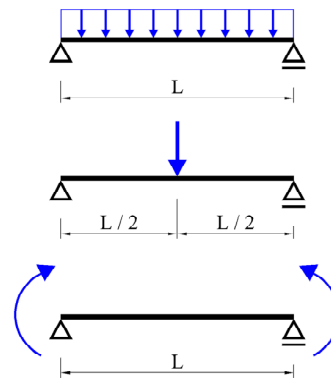


Fig. 4 Load types A, B, C considered for the analysis

This example is solved using the ANSYS 14.0 [5] code based on finite element method (FEM). The thickness of each part of the section is low in comparison with other dimensions. For this case shell elements are suitable. For the analysis the SHELL181 finite element is utilized. The finite element size is 20 mm. Part of the finite element model is in Fig. 5.

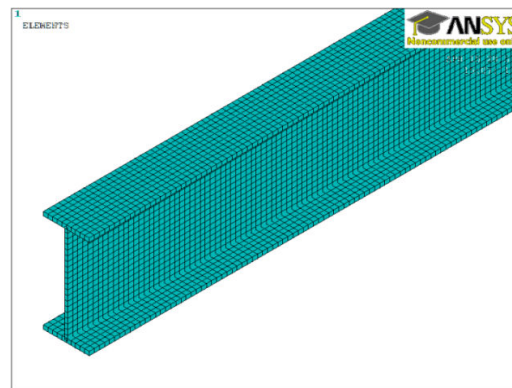


Fig. 5 Finite element model

The end supports of the beam comply with the so called fork support condition which is assumed for lateral torsional buckling analysis [6]. It prevents lateral displacement of the cross-section at supports and warping of the open thin-walled cross-section can develop freely. The analysis performed using the ANSYS code consists of linear static analysis (1<sup>st</sup>

order theory) and linear buckling analysis (LBA) that gives eigenvalues of the problem. If applied load in the finite element model is multiplied by the eigenvalue obtained from the LBA analysis, critical load is gained. Critical moment  $M_{cr}$  can be then determined. The analysis provides also the buckling shapes (eigenmodes).

For comparison, the same analysis was performed also for appropriate isolated beams to quantify the influence of lateral restraint.

*B. Results of the Analysis*

Using the numerical analysis performed in the ANSYS code the eigenvalues of the problem are obtained. Values of critical moment  $M_{cr}$  are determined using static formulae depending on type of load, e.g. for case A (uniformly distributed load) (6) applies:

$$M_{cr} = \frac{1}{8} q_{cr} L^2 \tag{6}$$

where  $q_{cr}$  is the critical load.

Charts present comparison of critical moments  $M_{cr}$  of beams with lateral restraint and of appropriate isolated beam: Fig. 6 for uniformly distributed load (case A), Fig. 7 for beam loaded by a force at midspan (case B) and a beam loaded by bending moments at supports (case C). For each case the critical moment of an isolated beam was calculated also according to the standard [3] (Czech national annex gives procedure for calculation of critical moment). The results obtained by procedure according to the standard are very close to results for isolated beams determined using ANSYS code which confirms accuracy of the finite element model.

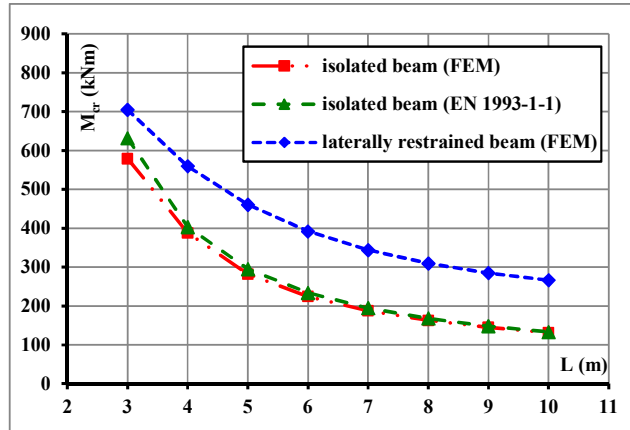


Fig. 7 Comparison of results, case B

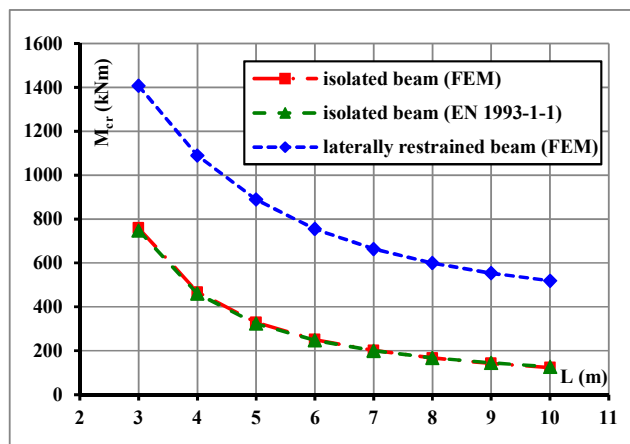


Fig. 8 Comparison of results, case C

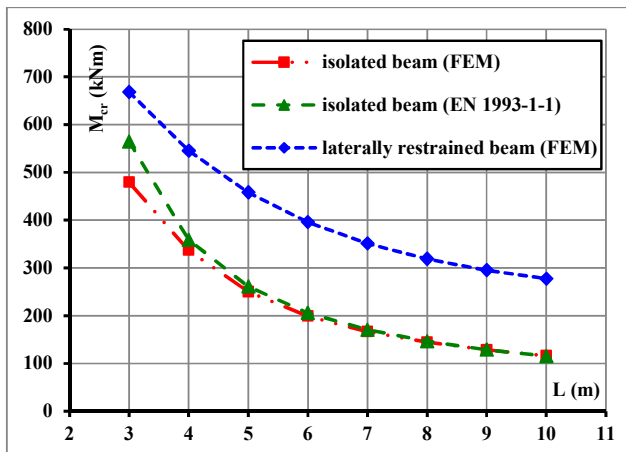


Fig. 6 Comparison of results, case A

In the following figure there is first eigenmode (buckling shape related to the lowest eigenvalue) for the load type A. First eigenmodes for other types are similar, beam buckles in one wave as well.

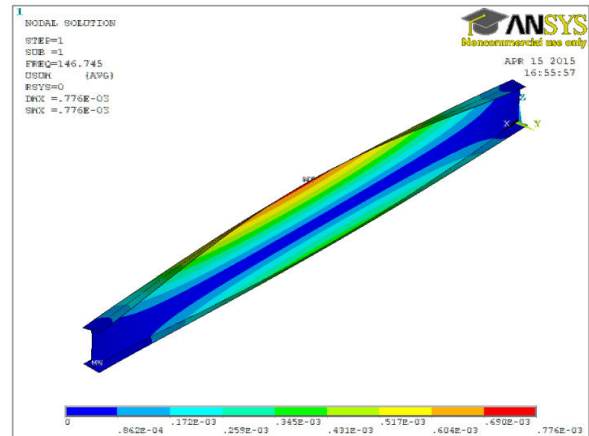


Fig. 9 First buckling shape

In Fig. 10 there is cross-section of the buckled beam at midspan (result of the numerical finite element method eigenvalue analysis using the ANSYS code). The eigenmode complies with the theoretically considered buckling shape illustrated in Fig. 2 – the cross-section rotates about the position of lateral restraint.



$$K_{i,i} = 5A + S_i \quad \text{for } i = \{1; N - 1\} \quad (20)$$

The expression  $S_i$  is given as:

$$S_i = h^2 (CM'_y - B) \quad (21)$$

*B. Eigenvalue Problem Solution*

There are several methods of numerical mathematics for the lowest eigenvalue of a matrix and appropriate eigenvector (in the physical point of view it is the buckling shape) solution. In this paper the iterative QR-algorithm [8] applied on the matrix  $G$  is utilized. This algorithm is briefly outlined [8]:

$$G = G_0 = QR \quad (22)$$

$$G_{k+1} = R_k Q_k, k \geq 1 \quad (23)$$

where matrices  $Q_k$  and  $R_k$  are products of the QR-decomposition of the matrix  $G$ . The matrix  $G$  can be decomposed to the matrices  $Q_k$  and  $R_k$  e.g. using Gram-Schmidt orthonormalisation process [9]. Expression (24) then applies:

$$\lim_{k \rightarrow \infty} G_k = D \quad (24)$$

where  $D$  is a diagonal matrix with eigenvalues on its diagonal (other members are zero). The eigenvectors are given as columns of a matrix that is product of matrices  $Q_{k+1}$  and  $Q_k$  multiplication in each step of iterative calculation.

The above described algorithms of the finite difference method and the QR-algorithm were applied to a simply supported beam of thin-walled cross-section according to Fig. 3 of a span equal to  $L = 5$  m. The beam span was divided into 50 elements (step  $h = 0,10$  m). The beam is supposed to be loaded by vertical uniformly distributed load of a magnitude of 1 kN/m at the top flange (it complies with the load type A). Differential equation of the problem was assembled for each node of the beam span and resulting system of algebraic equations with system matrix  $K$  was solved using the QR-algorithm. The resulting lowest critical load for this case is  $q_{cr} = 142660$  N/m. Normalized eigenvectors (buckling shapes) are depicted in Fig. 12. For clarity purposes, in Fig. 12 only first five buckling shapes are selected and labeled as "EV".

*C. Comparison with the Finite Element Method*

Results obtained by the QR-algorithm applied to the investigated problem are compared with results of the LBA analysis performed using the ANSYS code and summarized in Table I. The critical moment  $M_{cr}$  was calculated using the lowest critical load  $q_{cr}$  according to (6).

TABLE I  
COMPARISON OF RESULTS

Method	Critical load $q_{cr}$ (kN/m)	Critical moment $M_{cr}$ (kNm)
Finite element method	146,61	458,2
Finite difference method	142,66	445,8

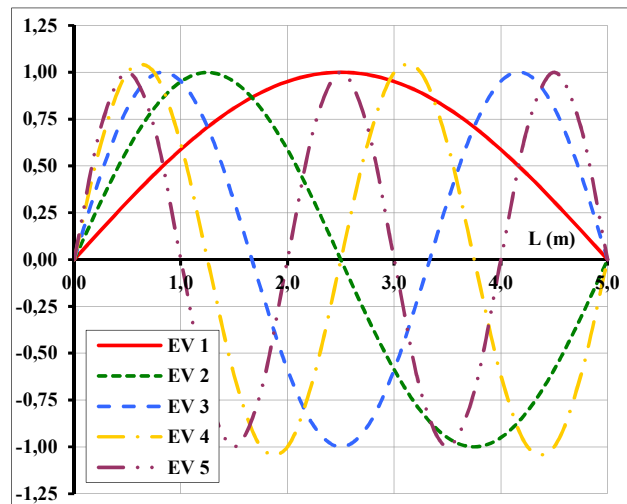


Fig. 12 Normalized eigenvectors (buckling shapes)

VI. SUMMARIZATION OF RESULTS

It implies from Figs. 6-8 that influence of lateral restraint on the critical load is significant. The lateral restraint causes increase its value of (on average) about 98 % (case A), 72 % (case B) and 211 % (case C). This notable increase of critical load results in higher buckling resistance of the laterally restrained beam. Taking it into account more economical beam cross-sections might be utilized.

Fig. 13 shows results of a parametric study of various spans of laterally restrained beams ( $c_z = 50$  mm) solved by finite element method (FEM) and finite difference method (FDM). For comparison purpose, the results of analyses of identical beams with no lateral restraints solved by FEM are showed together with results of calculation of critical load of an isolated beam according to [3].

In Fig. 13 certain differences of results for short spans occur. This inaccuracy is given by the finite element method solution. The finite element method code takes into account also local buckling effects of a short beam with slender walls that contribute to reduction of critical load. For larger spans the global effect (lateral torsional buckling) completely prevails and differences between methods are insignificant.

The fundamental equation from which the problem matrix is derived is actually the equation defining global stability problem and local effects are not included.

VII. CONCLUSION

The paper deals with lateral torsional buckling of steel thin-walled beams with lateral continuous restraint. The lateral restraint causes increase of buckling resistance of cross-sections. The paper quantifies this positive influence using example of double symmetric thin-walled beam with lateral continuous restraint. For solution finite element method and selected iterative algorithms of numerical mathematics (finite difference method, QR-algorithm) for eigenvalue solution were utilized. Results of both approaches are very close. Taking higher critical load into calculation of buckling

resistance of a beam might lead to more economical cross-sections and therefore certain material savings may be achieved.

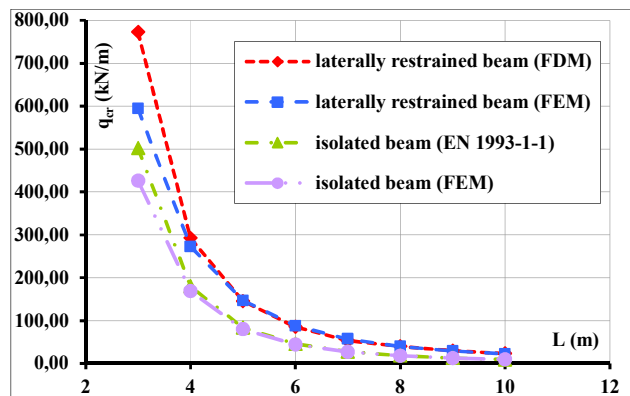


Fig. 13 Comparison of results

#### ACKNOWLEDGMENT

This paper has been elaborated within the frame of the projects of the Czech Science Foundation No. P105/12/0314 and Specific research program of the Brno University of Technology No. FAST-J-15-2804.

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