

# Lamb Waves in Plates Subjected to Uniaxial Stresses

Munawwar Mohabuth, Andrei Kotousov, Ching-Tai Ng

**Abstract**—On the basis of the theory of nonlinear elasticity, the effect of homogeneous stress on the propagation of Lamb waves in an initially isotropic hyperelastic plate is analysed. The equations governing the propagation of small amplitude waves in the pre-stressed plate are derived using the theory of small deformations superimposed on large deformations. By enforcing traction free boundary conditions at the upper and lower surfaces of the plate, acoustoelastic dispersion equations for Lamb wave propagation are obtained, which are solved numerically. Results are given for an aluminum plate subjected to a range of applied stresses.

**Keywords**—Acoustoelasticity, dispersion, finite deformation, lamb waves.

## I. INTRODUCTION

THE study of wave propagation problems in pre-stressed media has been the subject of much research over the past century. Early works in this area were however restricted to linear elasticity and the effect of small deformations on the propagation of small amplitude waves; see, for example, the pioneering contribution by [1], [2]. It was not until the development of the finite deformation theory by [3], [4] that the nonlinear effects of stresses were taken into account.

The acoustoelastic effect is a nonlinear phenomenon that describes the change in the speed of small amplitude waves in an elastic body due to the presence of a static pre-stress [5]. The theory of acoustoelasticity for bulk waves was initially developed by [6] who derived equations relating the wave velocity to the applied stress for isotropic materials subjected to uniaxial and hydrostatic loading. Their work was subsequently generalised by [7] and [8] to materials of arbitrary crystal symmetry.

Acoustoelasticity is now a well-established procedure utilised in the non-destructive evaluation of applied and residual stresses. Its underlying principles have been comprehensively described in the reviews [9], [10]. Ultrasonic bulk waves and the acoustoelastic effect have been used over the past sixty years for the measurement and control of residual stresses in welded structures and railroad rails, the tightening of bolts, the assessment of stress levels in bars and in multi-wire strands as well as the measurement of the stress distribution near a well bore [11].

The use of guided waves instead of bulk waves to measure

stresses has received significant attention over the last few decades due to the long propagation range associated with guided waves. In particular, guided waves in plate-like structures, also known as Lamb waves, have been found to be sensitive to changes in structural properties [12], [13], temperature and stress [14]. Despite that, there has not been much research on the theory of acoustoelasticity with regards to Lamb waves.

The paper [15] provides a fairly comprehensive acoustoelastic formulation to analyse the effect of uniaxial and biaxial loading in initially isotropic plates. However, their work is restricted to infinitesimal initial strains and small amplitude wave motion such that all the governing equations are linearised. In the current paper, the theory of acoustoelasticity is established using the theory of incremental deformations superimposed on a large deformation, which is based on the modern treatment of nonlinear elasticity by [16]. The wave propagation is considered as an infinitesimal deformation which is superimposed onto a finite static homogeneous deformation.

The paper is structured as follows. In the beginning, the constitutive equation for an isotropic hyperelastic material with initial stress and the equations governing incremental deformations superimposed on a finite deformation are recalled [17]. These equations are subsequently specialised to the case of weakly nonlinear elasticity and to uniaxial tension. The characteristic equations for symmetric and anti-symmetric Lamb wave modes are then derived by considering the propagation of homogeneous plane waves and enforcing traction free boundary conditions at the surfaces of the plate. Finally, these equations are solved numerically and results are presented for various applied stresses.

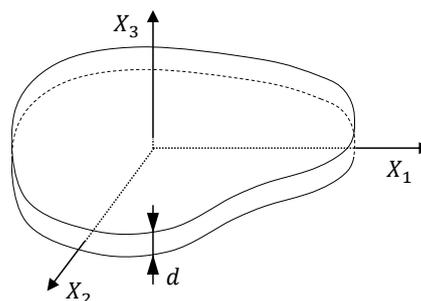


Fig. 1 Alignment of reference coordinate system

## II. PROBLEM FORMULATION

Consider an infinite plate of thickness  $d$ , composed of an isotropic hyperelastic material with density  $\rho_r$ , in some unstressed reference configuration. Material points in this configuration have position vectors  $\mathbf{X}$  relative to a Cartesian

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coordinate system  $(X_1, X_2, X_3)$  aligned as shown in Fig. 1. The origin of the coordinate system lies at the mid-plane of the plate and the normal to the surface coincides with the  $X_3$  axis.

Suppose the plate is now subjected to a finite static pure homogeneous strain so that it occupies a new configuration, referred to as the deformed configuration. The material points  $\mathbf{X}$  in the reference configuration then take up the position  $\mathbf{x}$  in the deformed configuration, given by

$$x_1 = \lambda_1 X_1, x_2 = \lambda_2 X_2, x_3 = \lambda_3 X_3, \quad (1)$$

where  $(x_1, x_2, x_3)$  is the Cartesian coordinate system in the deformed configuration, which, for convenience, is referred to the same origin as the reference coordinate system. The constants  $\lambda_1, \lambda_2, \lambda_3$  are the principal stretches of the deformation.

For isotropic hyperelastic materials, the principal Cauchy stress required to maintain the plate in its static state of finite deformation may be expressed in terms of the principal stretches as

$$\sigma_i = J^{-1} \lambda_i \bar{W}_i, \quad (2)$$

where  $\bar{W}$  is the strain energy density per unit volume which is a function of the principal stretches,  $\bar{W}_i = \frac{\partial \bar{W}}{\partial \lambda_i}$ ,  $J = \lambda_1 \lambda_2 \lambda_3$ ,  $i, j \in \{1, 2, 3\}$  and there is no sum over repeated indices.

The associated strain-induced anisotropy in the material response may also be characterised in terms of the principal stretches as

$$\begin{aligned} J \mathcal{A}_{0iij} &= \lambda_i \lambda_j \bar{W}_{ij}, \\ J \mathcal{A}_{0ijij} &= \frac{\lambda_i \bar{W}_i - \lambda_j \bar{W}_j}{\lambda_i^2 - \lambda_j^2} \lambda_i^2, & i \neq j, \lambda_i \neq \lambda_j, \\ J \mathcal{A}_{0ijji} &= \frac{\lambda_j \bar{W}_i - \lambda_i \bar{W}_j}{\lambda_i^2 - \lambda_j^2} \lambda_i \lambda_j, & i \neq j, \lambda_i \neq \lambda_j, \\ J \mathcal{A}_{0ijij} &= \frac{1}{2} (\lambda_i^2 \bar{W}_{ii} - \lambda_i \lambda_j \bar{W}_{ij} + \lambda_i \bar{W}_i), & i \neq j, \lambda_i = \lambda_j, \\ J \mathcal{A}_{0ijji} &= \frac{1}{2} (\lambda_i^2 \bar{W}_{ii} - \lambda_i \lambda_j \bar{W}_{ij} - \lambda_i \bar{W}_i), & i \neq j, \lambda_i = \lambda_j, \end{aligned} \quad (3)$$

where  $\mathcal{A}_{0ijkl}$  are the (non-zero) components of the instantaneous elasticity tensor relative to the deformed configuration,  $\bar{W}_{ij} = \frac{\partial^2 \bar{W}}{\partial \lambda_i \partial \lambda_j}$ ,  $i, j \in \{1, 2, 3\}$  and again, no summation is implied by the repetition of indices [16], [17].

The material point at  $\mathbf{x}$  in the deformed configuration is now considered to undergo a small dynamic displacement  $\mathbf{u}(\mathbf{x}, t)$ , which is superposed upon the initial finite static deformation. The material response due to this incremental deformation may then be described by the incremental constitutive relation [18]

$$\dot{S}_{0pi} = \mathcal{A}_{0piqj} \frac{\partial u_j}{\partial x_q}, \quad (4)$$

where  $\dot{S}_0$  is the incremental nominal stress tensor. The incremental equations of motion are given by

$$\mathcal{A}_{0piqj} \frac{\partial^2 u_j}{\partial x_p \partial x_q} = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad (5)$$

where  $\rho = \rho_r J^{-1}$  is the density of the material in the deformed configuration.

In order to study small but finite elastic effects, the strain energy function is specialised to *weakly nonlinear elasticity* [19]. An appropriate form of the strain energy function in this case is the Murnaghan energy function [20], which is given by

$$\begin{aligned} W &= \frac{\lambda}{8} (I_1 - 3)^2 + \frac{\mu}{4} (I_1^2 - 2I_1 - 2I_2 + 3) + \\ &\frac{l}{24} (I_1 - 3)^3 + \frac{m}{12} (I_1 - 3)(I_1^2 - 3I_2) + \\ &\frac{n}{8} (I_1 - I_2 + I_3 - 1), \end{aligned} \quad (6)$$

where  $\lambda, \mu$  are the classical Lamé constants,  $l, m, n$  are the third order elastic constants and  $I_1, I_2, I_3$  are the principal invariants of the Right Cauchy-Green deformation tensor.

For definiteness, the strain energy function should be cast in terms of the principal stretches rather than in terms of the principal invariants. For this purpose, the relations  $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ ,  $I_2 = \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 + \lambda_1^2 \lambda_2^2$ ,  $I_3 = \lambda_1^2 \lambda_2^2 \lambda_3^2$  are substituted in (6), which yields

$$\begin{aligned} \bar{W} &= \frac{\lambda}{8} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)^2 + \frac{\mu}{4} (\lambda_1^4 + \lambda_2^4 + \lambda_3^4 - 2\lambda_1^2 \\ &- 2\lambda_2^2 - 2\lambda_3^2 + 3) + \frac{l}{24} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)^3 + \\ &\frac{m}{12} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)(\lambda_1^4 + \lambda_2^4 + \lambda_3^4 - \lambda_1^2 \lambda_2^2 - \\ &\lambda_1^2 \lambda_3^2 - \lambda_2^2 \lambda_3^2) + \frac{n}{8} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \lambda_2^2 \lambda_3^2 - \lambda_3^2 \lambda_1^2 \\ &- \lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_2^2 \lambda_3^2 - 1). \end{aligned} \quad (7)$$

### III. UNIAXIAL TENSION

The finite homogeneous deformation is now specialised to the case of uniaxial tension. Without loss of generality, the uniaxial Cauchy stress  $\sigma$  may be taken to be along the  $x_1$  direction, such that  $\sigma_1 = \sigma$  (with  $\sigma_2 = \sigma_3 = 0$ ) and the corresponding principal stretch is  $\lambda_1$ . Due to the Poisson effect, the plate contracts laterally in the  $x_2$  and  $x_3$  directions and by symmetry,  $\lambda_2 = \lambda_3$  [21].

In general, the uniaxial tension is specified in terms of the nominal stress tensor which relates the axial force in the current (deformed) configuration to the area in the reference configuration. The principal components of the nominal stress can be expressed in terms of the principal stretches as [22]

$$S_{ii} = \bar{W}_i = \frac{\partial \bar{W}}{\partial \lambda_i}. \quad (8)$$

For a given uniaxial nominal stress  $S_{11}$ , the principal stretches can be determined by inverting the relation in (8), and setting the lateral stresses  $S_{22}$  and  $S_{33}$  to zero. The principal Cauchy stresses and the components of the elasticity tensor can then be found using (2) and (3) respectively.

It is worth noting that, as a result of the uniaxial stress, the

elastic response of the plate becomes transversely isotropic in nature. However, the elasticity tensor does not possess the same symmetry as in the case of classical transversely isotropic linear elasticity [20].

IV. ACOUSTOELASTIC LAMB WAVES

The propagation of acoustoelastic Lamb waves in the presence of a homogeneous uniaxial stress field requires the equation governing incremental motions superimposed on a finite deformation, as given by (5), to be solved in conjunction with traction free boundary conditions at the surfaces of the plate.

Here, the analysis is restricted to the propagation of homogeneous plane waves along the direction of the applied uniaxial stress only. It is assumed that the waves are confined to the  $x_1 - x_3$  plane and propagate in the  $x_1$  direction. Following [23], the wave motion is modeled as

$$u_j = U_j e^{i\xi(x_1 + \alpha x_3 - ct)}, j = 1, 2, 3, \tag{9}$$

where  $u_j$  is the particle displacement,  $U_j$  is the amplitude of the displacement,  $\xi$  is the wavenumber along the  $x_1$  direction,  $\alpha$  is the ratio of the wavenumbers in the  $x_3$  direction to that in the  $x_1$  direction and  $c$  is the phase velocity in the  $x_1$  direction.

Substituting (9) into the incremental equations of motion (5) gives an eigenvalue problem, which can be expressed as

$$K_{ij}(\alpha) u_j = 0, i, j = 1, 2, 3, \tag{10}$$

where the components of  $K_{ij}$  are given by

$$\begin{aligned} K_{11} &= \rho c^2 - \mathcal{A}_{01111} - \mathcal{A}_{03131} \alpha^2 - \alpha(\mathcal{A}_{01131} + \mathcal{A}_{03111}), \\ K_{12} &= -\mathcal{A}_{01112} - \mathcal{A}_{03132} \alpha^2 - \alpha(\mathcal{A}_{01132} + \mathcal{A}_{03112}), \\ K_{13} &= -\mathcal{A}_{01113} - \mathcal{A}_{03133} \alpha^2 - \alpha(\mathcal{A}_{01133} + \mathcal{A}_{03113}), \\ K_{21} &= -\mathcal{A}_{01211} - \mathcal{A}_{03231} \alpha^2 - \alpha(\mathcal{A}_{01231} + \mathcal{A}_{03211}), \\ K_{22} &= \rho c^2 - \mathcal{A}_{01212} - \mathcal{A}_{03232} \alpha^2 - \alpha(\mathcal{A}_{01232} + \mathcal{A}_{03212}), \\ K_{23} &= -\mathcal{A}_{01213} - \mathcal{A}_{03233} \alpha^2 - \alpha(\mathcal{A}_{01233} + \mathcal{A}_{03213}), \\ K_{31} &= -\mathcal{A}_{01311} - \mathcal{A}_{03331} \alpha^2 - \alpha(\mathcal{A}_{01331} + \mathcal{A}_{03311}), \\ K_{32} &= -\mathcal{A}_{01312} - \mathcal{A}_{03332} \alpha^2 - \alpha(\mathcal{A}_{01332} + \mathcal{A}_{03312}), \\ K_{33} &= \rho c^2 - \mathcal{A}_{01313} - \mathcal{A}_{03333} \alpha^2 - \alpha(\mathcal{A}_{01333} + \mathcal{A}_{03313}). \end{aligned} \tag{11}$$

Since the only non-zero components of the elasticity tensor for a pre-stressed isotropic material are  $\mathcal{A}_{0iiii}, \mathcal{A}_{0iijj}, \mathcal{A}_{0ijij}$  and  $\mathcal{A}_{0ijji}, i \neq j$  [21], (11) then reduces to

$$\begin{aligned} K_{11} &= \rho c^2 - \mathcal{A}_{01111} - \mathcal{A}_{03131} \alpha^2, \\ K_{12} &= 0, \\ K_{13} &= -\alpha(\mathcal{A}_{01133} + \mathcal{A}_{03113}), \\ K_{21} &= 0, \\ K_{22} &= \rho c^2 - \mathcal{A}_{01212} - \mathcal{A}_{03232} \alpha^2, \\ K_{23} &= 0, \\ K_{31} &= -\alpha(\mathcal{A}_{01331} + \mathcal{A}_{03311}), \\ K_{32} &= 0, \\ K_{33} &= \rho c^2 - \mathcal{A}_{01313} - \mathcal{A}_{03333} \alpha^2. \end{aligned} \tag{12}$$

The vanishing of the coefficients  $K_{12}, K_{21}, K_{23}$  and  $K_{32}$  in (12) means that the analysis can be confined to displacements in the  $x_1$  and  $x_3$  directions only as the shear horizontal wave motions uncouple from the Lamb wave motion [23]. Therefore, (10) can be re-written as

$$K_{ij} u_j = 0, i, j \in \{1, 3\}. \tag{13}$$

For non-trivial solutions to the eigenvalue problem, the determinant of the coefficient matrix in (13) must be equal to zero

$$|K_{ij}| = 0, i, j \in \{1, 3\}. \tag{14}$$

This yields a fourth order equation in  $\alpha$  which can be written as

$$P_4 \alpha^4 + P_2 \alpha^2 + P_0 = 0, \tag{15}$$

where the coefficients  $P_4, P_2$  and  $P_0$  are given by

$$\begin{aligned} P_4 &= \mathcal{A}_{03131} \mathcal{A}_{03333}, \\ P_2 &= -\rho c^2 (\mathcal{A}_{03333} + \mathcal{A}_{03131}) + \mathcal{A}_{03333} \mathcal{A}_{01111} + \\ &\quad \mathcal{A}_{03131} \mathcal{A}_{01313} - \mathcal{A}_{01133} \mathcal{A}_{01331} - \mathcal{A}_{01133} \mathcal{A}_{03311} \\ &\quad - \mathcal{A}_{03113} \mathcal{A}_{01331} - \mathcal{A}_{03113} \mathcal{A}_{03311}, \\ P_0 &= \rho^2 c^4 - \rho c^2 (\mathcal{A}_{01313} + \mathcal{A}_{01111}) + \mathcal{A}_{01111} \mathcal{A}_{01313}. \end{aligned} \tag{16}$$

The lack of odd power coefficients in (15) means that the fourth order equation can be reduced to a quadratic equation in  $\alpha^2$ . This simplification results in four solutions for  $\alpha$ , which are denoted by  $\alpha_q, q \in \{1, 2, 3, 4\}$ , with the following properties

$$\alpha_2 = -\alpha_1, \alpha_4 = -\alpha_3. \tag{17}$$

Using the relations in (12), the displacement ratio  $U_3$  to  $U_1$  for each of the  $\alpha_q$  can be expressed as

$$W_q = \frac{(\rho c^2 - \mathcal{A}_{01111} - \mathcal{A}_{03131} \alpha_q^2)}{\alpha_q (\mathcal{A}_{01133} + \mathcal{A}_{03113})}. \tag{18}$$

The displacement field of the Lamb waves can then be written in terms of the displacement ratio (18) by using the principle of superposition

$$\begin{aligned} u_1 &= \sum_{q=1}^4 U_1(\alpha_q) e^{i\xi(x_1 + \alpha_q x_3 - ct)}, \\ u_3 &= \sum_{q=1}^4 U_1(\alpha_q) W_q e^{i\xi(x_1 + \alpha_q x_3 - ct)}. \end{aligned} \tag{19}$$

Similarly, the stress field can be found by substituting the displacement field (19) into the incremental stress-displacement relations (4)

$$\begin{aligned}\dot{S}_{033} &= \sum_{q=1}^4 i\xi D_{1q} U_1(\alpha_q) e^{i\xi(x_1 + \alpha_q x_3 - ct)}, \\ \dot{S}_{013} &= \sum_{q=1}^4 i\xi D_{2q} U_1(\alpha_q) e^{i\xi(x_1 + \alpha_q x_3 - ct)},\end{aligned}\quad (20)$$

where

$$\begin{aligned}D_{1q} &= \mathcal{A}_{03311} + \alpha_q \mathcal{A}_{03333} W_q, \\ D_{2q} &= \mathcal{A}_{01313} \alpha_q + \mathcal{A}_{01331} W_q.\end{aligned}\quad (21)$$

Incorporating the symmetries (17) into (18)-(21) results in the following restrictions

$$\begin{aligned}W_{j+1} &= -W_j, \\ D_{1j+1} &= D_{1j}, \\ D_{2j+1} &= -D_{2j}, \quad j = 1, 3.\end{aligned}\quad (22)$$

In order to satisfy the incremental traction free boundary conditions at the upper and lower surfaces of the plate, the components of the incremental nominal stress must be set to zero

$$\dot{S}_{033} = \dot{S}_{013} = 0 \text{ at } x_3 = \frac{\pm d}{2}.\quad (23)$$

This leads to four equations which can be expressed as

$$\begin{aligned}i\xi \begin{pmatrix} D_{11}E_1 & D_{12}E_2 & D_{13}E_3 & D_{14}E_4 \\ D_{21}E_1 & D_{22}E_2 & D_{23}E_3 & D_{24}E_4 \\ D_{11}\bar{E}_1 & D_{12}\bar{E}_2 & D_{13}\bar{E}_3 & D_{14}\bar{E}_4 \\ D_{21}\bar{E}_1 & D_{22}\bar{E}_2 & D_{23}\bar{E}_3 & D_{24}\bar{E}_4 \end{pmatrix} \begin{pmatrix} U_{11} \\ U_{12} \\ U_{13} \\ U_{14} \end{pmatrix} e^{i\xi(x_1 - ct)} &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\end{aligned}\quad (24)$$

where  $U_{1q} = U_1(\alpha_q)$ ,  $E_q = e^{i\xi\alpha_q \frac{d}{2}}$  and  $\bar{E}_q = e^{-i\xi\alpha_q \frac{d}{2}}$ . For non-trivial solutions, the determinant of the coefficient matrix in (24) must be equal to zero

$$\begin{vmatrix} D_{11}E_1 & D_{12}E_2 & D_{13}E_3 & D_{14}E_4 \\ D_{21}E_1 & D_{22}E_2 & D_{23}E_3 & D_{24}E_4 \\ D_{11}\bar{E}_1 & D_{12}\bar{E}_2 & D_{13}\bar{E}_3 & D_{14}\bar{E}_4 \\ D_{21}\bar{E}_1 & D_{22}\bar{E}_2 & D_{23}\bar{E}_3 & D_{24}\bar{E}_4 \end{vmatrix} = 0\quad (25)$$

Finally, using row-column operations and the symmetries in (22), (25) can be reduced to two characteristic equations

$$\begin{aligned}D_{11}D_{23} \cot(\gamma\alpha_1) - D_{13}D_{21} \cot(\gamma\alpha_3) &= 0, \\ D_{11}D_{23} \tan(\gamma\alpha_1) - D_{13}D_{21} \tan(\gamma\alpha_3) &= 0,\end{aligned}\quad (26)$$

corresponding to the symmetric and anti-symmetric Lamb wave modes respectively, with  $\gamma = \frac{\xi d}{2} = \frac{\omega d}{2c}$  and  $\omega$  being the angular frequency of the wave.

## V. SELECTED RESULTS

In this section, the characteristic equations (26) derived in Section IV are solved numerically using the algorithm developed by [24]. Dispersion results are presented in terms of the phase velocity as a function of the frequency-thickness product for an aluminum plate. The elastic properties of the plate are listed in Table I and were obtained from the experimental work of Asay and Guenther [25].

TABLE I  
ELASTIC PROPERTIES FOR 6061-T6 ALUMINUM

Material property	Value
$\lambda$	54.308 GPa
$\mu$	27.174 GPa
$l$	-281.5 GPa
$m$	-339.0 GPa
$n$	-416.0 GPa
$\rho$	2704 kg/m <sup>3</sup>

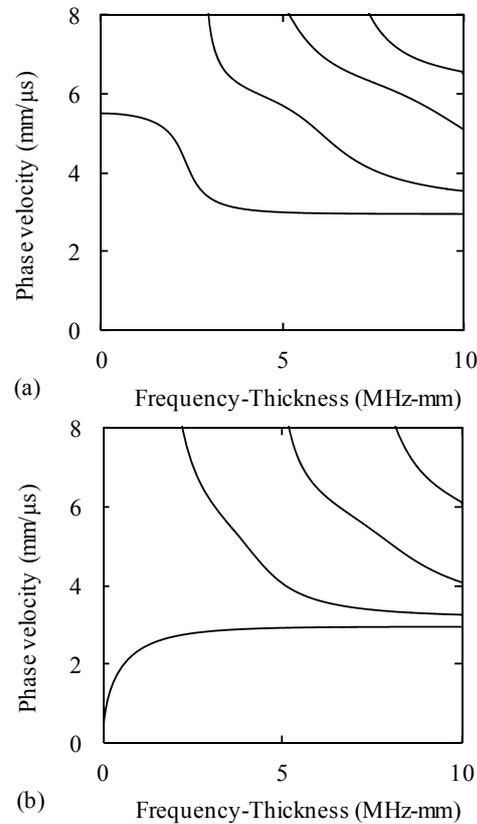


Fig. 2 (a) Symmetric modes, (b) Anti-symmetric modes, for Lamb waves propagating in an aluminum plate along the direction of a uniaxial tension of 50 MPa.

Fig. 2 (a) and (b) show the symmetric and anti-symmetric Lamb wave modes for the aluminum plate subjected to a uniaxial tension of 50 MPa. The propagation of the waves is considered to be along the direction of the applied stress. The shear horizontal modes are not shown here as they decouple from the Lamb wave modes. It can be seen that the dispersion curves obtained are very similar to the ones for an unstressed

aluminum plate. This is because the phase velocity is not significantly affected by the applied stress as the acoustoelastic effect acts only on the third-order elastic constants [11].

Fig. 3 shows the relative change of the phase velocity of the fundamental symmetric mode (S0), compared to the unstressed state, as a function of the level of applied tension. Although the relative change is quite small, it can be observed that the change in the phase velocity is negative for all the values of stress considered. This means that tensile stresses cause a decrease in the phase velocity of the S0 mode, which is consistent with the fact that the bulk wave speed along the direction of an applied tensile load is less than the unstressed wave speed [6]. Furthermore, it can be seen that higher levels of applied stress result in larger changes in the phase velocity, particularly in the lower frequency-thickness region. However, at higher frequency-thickness values, the change in the phase velocity is relatively constant but is still negative.

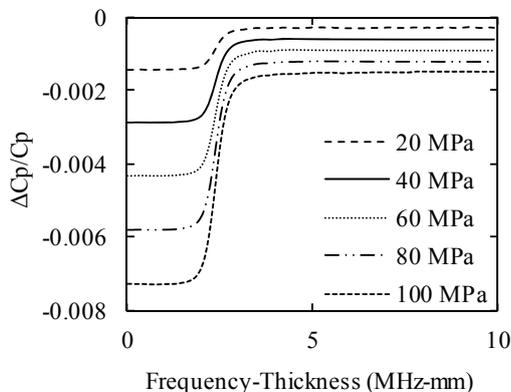


Fig. 3 Relative change of phase velocity for the S0 mode as a function of the applied tension

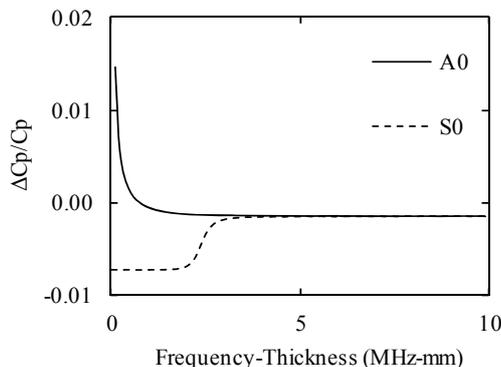


Fig. 4 Relative change of phase velocity for a uniaxial tension of 100 MPa

The sensitivity of the S0 mode and the fundamental anti-symmetric mode (A0) to an applied stress of 100 MPa is compared in Fig. 4. At low frequency-thickness values, the A0 mode shows a high sensitivity to the applied stress. However, the phase velocity decreases rapidly with increasing values of the frequency-thickness product. Thus, in practice, it would be

preferable to use the S0 mode as it maintains a higher sensitivity over a longer range of frequencies. Moreover, at higher frequency-thickness values, it can be seen that both the S0 and A0 modes converge towards the same value of the relative change of phase velocity. This is not surprising since both modes converge to the Rayleigh wave velocity at high frequencies [26].

## VI. CONCLUDING REMARKS

In this paper, the problem of Lamb wave propagation in an initially isotropic elastic plate subjected to a finite homogeneous deformation is analysed. The governing equations derived are different to previously published relations which considered the initial strains to be small.

The theoretical predictions demonstrate that the Lamb wave phase velocity generally decreases with an increase in the magnitude of tensile stress. In the low frequency-thickness region, specifically below 3 MHz-mm, the S0 mode shows a relatively high sensitivity to the applied stress. Combining that with the ability of Lamb waves to propagate over large distances, the theoretical equations could form the basis for a non-destructive stress measurement technique in plate-like structures.

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