

# Implementation of the Recursive Formula for Evaluation of the Strength of Daniels' Model

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**Abstract**—The paper deals with the classical fiber bundle model of equal load sharing, sometimes referred to as the Daniels' bundle or the democratic bundle. Daniels formulated a multidimensional integral and also a recursive formula for evaluation of the strength cumulative distribution function. This paper describes three algorithms for evaluation of the recursive formula and also their implementations with source codes in the Python high-level programming language. A comparison of the algorithms are provided with respect to execution time. Analysis of orders of magnitudes of addends in the recursion is also provided.

**Keywords**—Daniels bundle model, equal load sharing, Python, mpmath.

## I. INTRODUCTION

THIS PAPER deals with the classical fiber bundle model with equal load sharing, sometimes referred to as the Daniels bundle model [1] or the democratic bundle [4]. This model is significant for the strength of fibrous materials and composites, and the generally random strength of quasi-brittle structures. The model is also relevant for the analysis of the reliability of various parallel systems (computer components, infrastructure etc.). Daniels formulated a multidimensional integral and also recursive formula for the evaluation of the strength distribution function. In the same paper, he showed that the distribution of the strength of the bundle,  $G_n(x)$ , tends to Gaussian distribution under quite broad conditions and he gave closed formulas for the mean value and standard deviation of the Gaussian distribution. Sornette [4] later confirmed this result using a Kolmogorov theorem. The convergence of a random strength to Gaussian distribution is very slow in terms of the number of fibers and therefore, Smith [3] proposed a corrected term for the mean value that improves the original Daniels formula for small bundles. Even though the knowledge of the asymptotic form of  $G_n$  for the number of fibers  $n \rightarrow \infty$  is important, the normality does not hold in the tails of the distribution and it also does not hold when there is a small number of parallel components in the system (fibers). The real cumulative distribution function (CDF)  $G_n$  strongly deviates from the normal distribution for values of  $x$  far from the mean strength. Only a little is known about the behavior of the tails. The left tail ( $x \rightarrow 0$ ) is of great importance for reliability considerations, however.

In the present paper, the authors describe an analysis of the recursive formula by Daniels. The advantage of the recursive formula is that it provides exact values of the CDF  $G_n$  for

arbitrary values of  $x$  and for any number of fibers  $n$ . The disadvantage is that a naïve implementation of the function makes it usable only for bundles as small as approx 20 fibers. However, we show that the number of arithmetic operations can be significantly decreased by exploiting the fact that some terms in the naïve recursive formula are repeated. The authors also describe a computer implementation carried out in the Python high-level programming language [5] using the NumPy [7] (scientific computing with arrays) and the mpmath [6] (library for real and complex floating-point arithmetic with arbitrary precision) packages. This implementation enables the calculation of cumulative distribution function (CDF) values for large numbers (thousands) of fibers in a bundle, including values deep in the left tail of the distribution (probabilities  $10^{-600}$ ). This computer program is used to accurately calculate the distribution functions  $G_n$  for bundles with Weibull fibers with the scale parameter  $s = 1$ , the varying number of fibers  $n$  and the varying shape parameter  $m$ . The obtained results are stored in a new created database and compared to the available formulas [1]–[3]. The main motivation of the work is to formulate an improved analytical formula for the distribution function  $G_n$  that will be valid deep to the left tail where the real distribution strongly deviates from the Gaussian approximation.

## II. BUNDLE STRENGTH

The cumulative distribution function,  $G_n$ , of bundle strength formulated by Daniels [1] reads:

$$G_n(x) = \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} [F(x)]^k G_{n-k} \left( \frac{nx}{n-k} \right) \quad (1)$$

where  $x$  is a value of random strength per fiber,  $X$ , for which the probability  $P(X \leq x) = G_n(x)$  is evaluated,  $G_1(x) \equiv F(x)$ ,  $G_0(x) \equiv 1$ ,  $\binom{n}{k}$  is the standard Binomial coefficient and  $n$  is the number of fibers. The most frequently selected distribution function for a single fiber,  $F(x)$ , is Weibull distribution. With no loss of generality, let us assume the bundles contain Weibullian fibers (elements in parallel) so that the CDF of the strength of single fiber is

$$F(x) = 1 - e^{-(x/s)^m} \quad (2)$$

where  $s$  is the scale parameter and  $m$  is the shape parameter.

## III. IMPLEMENTATIONS OF THE RECURSIVE FORMULA

The implementations were carried out in the Python high-level programming language. Three approaches to

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evaluate the recursive formula (1) will be described in the following text:

- recursive implementation,
- recursion with memoization, and
- loop-based implementation (no recursion).

#### A. Recursive implementation

The recursive definition in (1) can be translated directly into Python as follows:

```
import math 1
from scipy.misc import comb 2
def Gn(x, scale, shape, n): 3
    cdf = 1 - math.exp(-(x/scale)**shape) 4
    if n < 1: 5
        return 1. 6
    if n == 1.: 7
        return cdf 8
    cdf_k = 1. 9
    res = 0. 10
    for k in range(1, int(n)): 11
        cdf_k *= cdf 12
        komb = comb(n, k) #binomial coefficient 13
        if k % 2 == 0.: 14
            komb = -komb 15
        res_1 = komb * cdf_k * Gn((n / (n - k)) * x, 16
                                scale, shape, n - k) 17
        res += res_1 18
    cdf_k *= cdf 19
    if n % 2 == 0.: 20
        res -= cdf_k 21
    else: 22
        res += cdf_k 23
    return res 24
25
26
27
```

This Python implementation is limited to  $n \approx 20$  due to the execution time and precision of double-precision floating-point format of numbers. The `mpmath` (library for real and complex floating-point arithmetic with arbitrary precision) package was used to remedy the precision problem however the problem with an explosion of the number of recursive calls is still present. This naïve implementation of the recursive formula (1) for  $G_n$  requires  $2^n - 1$  function calls. For example, a bundle with  $n = 50$  fibers necessitates  $10^{15}$  calls to get the value of  $G_n$  for a single value of strength  $x$ .

#### B. Recursion with memoization

Although the straightforward implementation of the recursive formula (1) (given above) is elegant and close to the mathematical definition, it is not very practical. The time required to calculate  $G_n(x)$  is exponential in  $n$ . To remedy this, we can employ memoization to cache previous computations. A closer look at the structure of the formula (1) reveals that many identical terms for given  $n$  and  $k$  are computed repeatedly. These terms can be stored in memoization cache and accessed for repeated use. The memoization cache is an 2D array `gn_arr` where one axis is accessed over  $n$  and the other one over  $k$  and the stored value is the corresponding  $G_n(x)$ . The memoized  $G_n$  function recursively computes and stores the value of  $G_n(x)$  if it has not been previously stored in the memo array. Otherwise it returns

the memoized value of  $G_n$ . The memoized values fill the lower triangle of the array ( $\frac{1}{2}n(n+1)$  values). This implementation uses packages `NumPy` to store data in an array and `mpmath` to increase floating-point precision of numbers.

```
import numpy as np 28
import mpmath as mp 29
30
# set number of decimal places for multiprecision numbers 31
mp.mp.dps = 1000 32
33
# pre-conversion of frequently used values 34
MPF_ZERO = mp.mpf('0') 35
MPF_ONE = mp.mpf('1') 36
MPF_TWO = mp.mpf('2') 37
MPF_THREE = mp.mpf('3') 38
```

`MPF_ZERO, MPF_ONE, ...` are pre-converted values because repeated type conversions from floats, strings and integers are expensive. The precision used to evaluate  $G_n(x)$  up to  $n = 1500$  was set to 1000 decimal places (3325 bits). This value was found to be sufficient while considering demands on the execution time and to requested accuracy. Any tests of optimal precision were not performed yet.

The Binomial coefficients  $\binom{n}{k}$  are calculated including the sign  $(-1)^{k+1}$  and stored in the lower triangle of 2D array `binom_tab`.

```
def get_binom_tab(n): 39
    binom_tab = np.zeros((n, n), dtype=object) 40
    for i in range(1, n + 1): 41
        for j in range(1, i + 1): 42
            binom_tab[i - 1, j - 1] = (mp.binomial(i, j) * 43
                                       (-1) ** (j + 1)) 44
    return binom_tab 45
```

This array can be precalculated for greater  $n$  and stored in a file on hard-disk. It is useful in case of repeated calculations.

The CDF of Weibull distribution is implemented consistently with  $F(x)$  in (2):

```
def weib_cdf(x, shape, scale): 46
    """ Cumulative distribution function of Weibull 47
    distribution with two parameters (shape and scale). 48
    """ 49
    return MPF_ONE - mp.exp(-(x / scale) ** shape) 50
```

The following code shows the implementation of  $G_n(x)$  with memoization:

```
def gn_mp(x, scale, shape, n, binom_tab): 51
    """ 52
    Return value of CDF of a bundle strength 53
    considering Weibullian fibers . 54
    """ 55
    Parameters 56
    ----- 57
    x : mpmath 58
        Bundle strength 59
    scale : mpmath 60
        Scale parameter of Weibull distribution 61
    shape : mpmath 62
        Shape parameter of Weibull distribution 63
    n : mpmath 64
        Number of fibers 65
    Returns 66
    ----- 67
    out : mpmath 68
        CDF value for the strength x 69
    Notes 70
    ----- 71
    """ 72
    """ 73
```

Parameters can be float but it can cause inaccuracies in results .

#### Examples

```

-----
>>> mp.mp.pretty = True
>>> mp.mp.dps = 30
>>> shape = mp.mpf('6.')
>>> scale = mp.mpf('1.')
>>> n_fil = mp.mpf('10')
>>> x = mp.exp(mp.mpf('-1'))
>>> binom_tab = get_binom_tab(n_fil)
>>> gn_mp(x, scale, shape, n_fil, binom_tab)
0.000000329859130502740500994574682994
,,,
gn_arr = np.zeros((n, n), dtype=object)
gn_arr.fill(None)
cdf_arr = np.zeros(n, dtype=object)
cdf_arr.fill(None)
x_arr = np.zeros(n, dtype=object)
for i in range(1, int(n)):
    x_arr[n - i] = mp.fraction(n, n - i) * x
def recursion_gn_mp(x_val, scale, shape, n):
    index_n = int(n) - 1
    res = MPF_ZERO
    cdf = cdf_arr[index_n]
    if cdf == None:
        cdf = weib_cdf(x_val, shape, scale)
        cdf_arr[index_n] = cdf
    for k in range(1, int(n) + 1):
        gn = gn_arr[index_n, k - 1]
        if gn == None:
            cdf_k = cdf ** k
            komb = binom_tab[index_n, k - 1]
            if k != n:
                gn = (komb * cdf_k *
                    recursion_gn_mp(x_arr[index_n],
                    scale, shape, n - k))
            else:
                gn = komb * cdf_k # * G_0(x) (= 1.0)
        gn_arr[index_n, k - 1] = gn
    res += gn
    return res
#execute recursion
gn_m = recursion_gn_mp(x, scale, shape, n)

return gn_m

```

Arrays  $x\_arr$  and  $cdf\_arr$  contain precalculated values of  $x$  and  $F(x)$ .

For number of fibers  $n > 980$  the implementation will raise error "RuntimeError: maximum recursion depth exceeded in cmp". This error is caused by the default Python setting for maximum recursion depth – to increase it use `import sys; sys.setrecursionlimit(value)`. The default value can be obtained by executing `sys.getrecursionlimit()`. The disadvantage of this implementation is that the memoized values are accessed repeatedly ( $\frac{(n-2)^2(n-1)}{2} - \frac{(n-3)(n-2)(n-1)}{3}$  times).

#### C. Loop-based implementation

The most efficient formulation of the algorithm, which is also a computer implementation among the three seems to be the following one. Let us divide (1) into three parts

$$G_n(x) = \sum_{k=1}^n \underbrace{(-1)^{k+1} \binom{n}{k}}_{B_{i,k}} \underbrace{[F(x)]^k}_{F_i} \underbrace{G_{n-k}\left(x \frac{n}{n-k}\right)}_{S_i}. \quad (3)$$

At the highest level of recursion, formula (3) represents a summation over  $k = 1, \dots, n$ . Each of these addends calls for

74 a recursion – an evaluation of the recursion function  $G_n(x)$   
 75 with a new parameter  $n$  renamed here as  $n_k = n - k$ .  
 76 By analyzing all arguments of random strength,  $x$ , one can  
 77 see that there are only  $n$  different values for which the  
 78 distribution function of strength is evaluated, namely  $x_i = x \frac{n}{i}$ ,  
 79  $i = 1, \dots, n$ . The values of  $x$  are stored in an array named  
 80  $x\_arr$ .  
 81  
 82

For each element of vector  $x$ , one must compute the distribution function of the strength of one fiber,  $F(x)$ . Let us now define a vector,  $F$ , that contains the values of the basic CDF evaluated at points  $x_i$ :

$$F: F_i = F(x_i) = F\left(x \frac{n}{i}\right), \quad i = 1, \dots, n \quad (4)$$

This vector is precalculated and cached in computer memory as an array named  $cdf\_arr$  at the beginning of computation. In later stages of computation, the elements of this vector are raised to integer powers  $k = 1, \dots, n$ .

The next ingredient is a lower triangular matrix  $B$ , with  $n$  rows and  $n$  columns, pre-filled with binomial coefficients multiplied by the alternating sign. Each element of the triangular matrix initially reads

$$B: B_{i,k} = (-1)^{k+1} \binom{n}{k}, \quad i = 1, \dots, n \text{ and} \\ k = 1, \dots, i. \quad (5)$$

This matrix is stored in an array named  $gn\_arr$ .

Once these two ingredients are calculated, the algorithm continues with the following two loops (the algorithm uses in-place operations and updates values of  $B$  matrix):

1. Loop over  $n$  columns of the  $B$  matrix – Starting with the first column  $k = 1$ , each column  $k = 1, \dots, n$  is multiplied by the  $k$ th power of elements  $F_i$ :  $B_{i,k} = B_{i,k} F_i^k$  for  $k = 1, \dots, n$  and  $i = k, \dots, n$ .
2. Loop over  $n - 1$  rows of the  $B$  matrix – Starting with the row  $i = 2$ , run a cycle over rows  $i = 2 \dots, n$  that (i) sums the elements in the preceding row:  $S_{i-1} = \sum_{k=1}^{i-1} B_{i-1,k}$  and (ii) use this value to update (multiply) this value with all elements of sub-diagonal number  $i$  within a cycle over columns  $j = 1, \dots, n - i + 1$ :  $B_{i+j-1,j} = S_{i-1} B_{i+j-1,j}$ .

After these two cycles are finished, the sum of  $n$  elements in the last row is the desired value of  $G_n(x)$ :

$$G_n(x) = S_n = \sum_{k=1}^n B_{n,k} \quad (6)$$

Fig. 1 shows a diagram of the triangular array for the number of fibers  $n = 4$ ; factors featured in 3 are highlighted.

We note that the sums  $S_i$  of any row correspond to the strength distribution function of a fiber bundle with  $i$  fibers evaluated at load  $x \frac{n}{i}$ :

$$S_i = G_i\left(x \frac{n}{i}\right) = \sum_{k=1}^i B_{i,k} \quad i = 1, \dots, n. \quad (7)$$

The following source code implements the described algorithm.

$$\begin{aligned}
 S_1 &= \sum \left[ \begin{array}{c} B_{1,1}F_1^1 \\ B_{2,1}F_2^1 S_1 \quad B_{2,2}F_2^2 \\ B_{3,1}F_3^1 S_2 \quad B_{3,2}F_3^2 S_1 \quad B_{3,3}F_3^3 \\ B_{4,1}F_4^1 S_3 \quad B_{4,2}F_4^2 S_2 \quad B_{4,3}F_4^3 S_1 \quad B_{4,4}F_4^4 \end{array} \right] \\
 S_2 &= \sum \\
 S_3 &= \sum \\
 G_4 &= \sum
 \end{aligned}$$

Fig. 1: Diagram of the cached array for the number of fibers  $n = 4$ . The sum of the last line returns  $G_4(x)$ . (red: row indexes  $i$ , blue: column indexes  $k$ , sums of rows  $S_i$  as diagonal members)

```

def gn_mp(x, scale, shape, n, binom_tab):
    """ see doc string in memoized version """
    ni = int(n) # retype n from mp.mpf to int
    cdf_arr = np.zeros(ni, dtype=object)
    x_arr = np.zeros(ni, dtype=object)
    # precalculate x and F(x) vectors
    for i in range(0, ni):
        xx = mp.fraction(n, n - i) * x
        x_arr[ni - i - 1] = xx
        cdf_arr[ni - i - 1] = weib_cdf(xx, shape, scale)

    # prepare B matrix
    gn_arr = binom_tab[:ni, :ni].copy()
    for i in range(ni): # loop 1
        gn_arr[i:ni, i] *= cdf_arr[i:ni] ** (i + 1)

    for k in xrange(1, ni): # loop 2
        idx1 = np.arange(0, ni - k)
        idx2 = np.arange(k, ni)
        idx3 = np.arange(0, k)
        gn_arr[idx2, idx1] *= np.sum(gn_arr[k - 1, idx3])
    gn_m = np.sum(gn_arr[-1, :])

    return gn_m

```

#### IV. DISCUSSION

Fig. 2 shows the measured execution times of the described implementations to evaluate  $G_n(x)$  for a single value of load (random strength)  $x$  and for various numbers of fibers. The execution times for selected values of  $n$  are arranged in Tab. I. The values of execution times of the recursive implementation (A) were estimated on the basis of exponential dependency on  $n$ .

TABLE I  
CALCULATED AND ESTIMATED EXECUTION TIMES FOR  
IMPLEMENTATIONS OF THE RECURSIVE FORMULA.

$n$	A	B	C
50	8094 years	0.58 s	0.12 s
1000	$2.5 \cdot 10^{284}$ years	1 h	70 s
1500	—	3.5 h	167 s

[Dell T7610, 2x Intel Xeon E5-2687w, 192 GB, Kubuntu 14.04]

The implementations using mpmath enable to calculate values of CDF for large numbers (thousands) of fibers in bundle including values deep in the left tail of the distribution (probabilities  $10^{-600}$ ). In addition, the loop-based implementation C reduces significantly the execution time (speedup for  $n = 1000$  compared to B is  $51\times$  and for  $n = 1500$  it is  $75\times$ ).

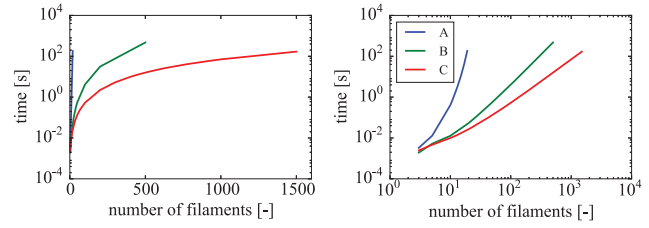


Fig. 2: Execution time of implementations A, B, and C. Left: semi-log plot shows the exponential dependence on  $n$  of the implementation A (blue linear line). Right: log-log plot.

#### V. ANALYSIS OF ADDENDS

The analysis of the recursive formula exploited in algorithm C seems to require the minimal possible number of arithmetic operations. The only option how to reduce the computing time could be to ignore elements of loop implementation that are insignificant for the resulting  $G_n(x)$ . To identify such elements, the analysis of orders  $\log_{10} G_i$  of magnitudes of  $G_i$  was performed. The following results were obtained for the number of fibers  $n = 100$  and for the shape parameter  $m = 6$ . The mean value of strength of one fiber  $\mu_x = 0.928$  (standard deviation  $\sigma_x = 0.180$ ).

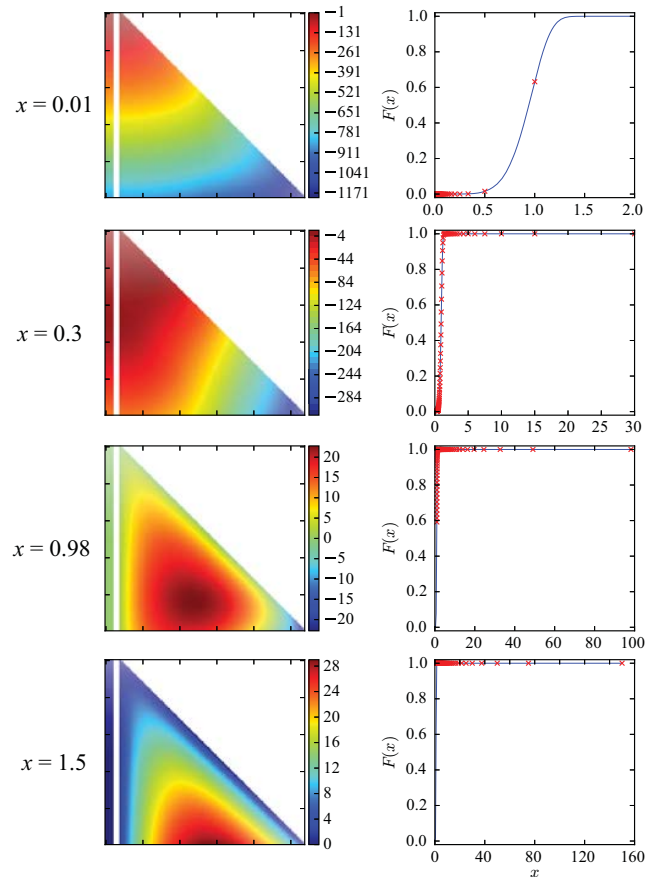


Fig. 3: Left: left column plots orders of the sum of the row and triangle contains orders of the  $B$  matrix. Right: CDF plot (blue line) and values of  $x$  and  $F(x)$  used in evaluation of  $G_n$  (red crosses)

Fig. 3 shows results of  $G_n$  for the various strengths  $x$ :

- $x = 0.01$  - the CDF value is  $G_{100} = 2.384334 \times 10^{-847}$
- $x = 0.3$  - the CDF value is  $G_{100} = 2.305565 \times 10^{-64}$
- $x = \mu_x = 0.98$  - the CDF value is very close to 1 and therefore we present the value of complement (survival probability)  $1 - G_{100} = 2 \times 10^{-39}$
- $x = 1.5$  - the complement to the CDF value is  $G_{100} = 2 \times 10^{-495}$ . In the right plot of Fig. 3, all values of  $F(x)$  are close to 1 and this implies that the orders of magnitudes in the left triangular plot are, in fact, just the orders of the Binomial coefficients  $B_{i,k} F_i^k S_i \approx B_{i,k} \cdot 1^k \cdot 1 = B_{i,k}$ .

Based on the analysis of contributions to the value of  $G_n$  from the  $B$  matrix, we conclude that no systematic way of ignoring any elements of the  $B$  matrix to further reduce the execution time can be suggested.

## VI. CONCLUSION

The paper describes three approaches to the implementation of the recursive formula for evaluation of the cumulative distribution function of random bundle strength with: A) naïve Python recursive implementation, B) recursive implementation with memoization and mpmath, and C) loop-based implementation. The last implementation (C) significantly reduces the execution times and enables to calculate  $G_n(x)$  for thousands of fibers and small probabilities. The presented algorithms are used to create a database of CDF of bundle strengths for various  $m$  and  $n$  parameters. The database will serve as a data support for newly formulated analytical approximate formulas for the CDF of Daniels bundle strength.

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